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BIFURCATIONS OF GENERIC ONE PARAMETER FAMILIES OF FUNCTIONS ON FOLIATED MANIFOLDS

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Introduction.

Let N, P, Q differentiable manifolds and φ: $N \to Q$ a fixed differentiable mapping. The mappings f, g: $N \to P$ are φ-equivalent if there exist diffeomorphisms h, k and l commuting the diagram

$$\begin{array}{ccccc} N & \xrightarrow{(f,\varphi)} & P \times Q & \xrightarrow{\pi} & Q \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle k} & & \downarrow{\scriptstyle l} \\ N & \xrightarrow{(g,\varphi)} & P \times Q & \xrightarrow{\pi} & Q \end{array}$$

where π is the projection on the second factor.

This equivalence relation gives qualitative information on the mapping f, by leaving invariant the "foliation" in N defined by the level sets φ = constant. The φ-stability of mappings f: $N \to P$ was studied by L. Favaro and C. Mendes in [6].

Allowing φ to vary in the above definition, we obtain an auxiliary equivalence relation, that we call $\mathscr{D}$-equivalence. Both equivalence relations can be considered as special cases of the equivalence of convergent diagrams, as defined by Dufour in [2].

In the present paper, we study the concept of φ-versal unfolding and its relationship with the corresponding concept for $\mathscr{D}$-equivalence. We classify one parameter φ-versal unfoldings of germs and multigerms f: $(\mathrm{R}^n, S) \to (\mathrm{R}, 0)$, where φ: $(\mathrm{R}^n, S) \to (\mathrm{R}, 0)$ is infinitesimally stable, $n \geqq 2$ (Propositions 3.1, 3.2). The generic bifurcations in this classification are associated to the $\mathscr{D}$-orbits of pairs (f, φ): $N \to \mathrm{R} \times \mathrm{R}$ of $\mathscr{D}$-codimension $\leqq 1$, with one exception: the transversal intersection of three transverse folds. This singularity presents modality. In Proposition 3.3, we show that its topological φ-codimension is one.

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Our main result, Theorem 4.2, shows that given two φ-stable functions $f, g\colon N \to \mathsf{R}$, the transition from the pair (f, φ) to the pair (g, φ) can be realized by a generic path that preserves the foliation in N defined by the level sets of a fixed Morse function $\varphi\colon N \to \mathsf{R}$.

1. Notations and basic definitions.

Let $C(n, p)$ be the space of smooth map-germs $f\colon (\mathsf{R}^n, 0) \to (\mathsf{R}^p, 0)$.

For a fixed map-germ $\varphi\colon (\mathsf{R}^n, 0) \to (\mathsf{R}^q, 0)$, we define an equivalence relation on $C(n, p)$, called φ-equivalence, as follows.

DEFINITION 1.1. Two map-germs $f, g\colon (\mathsf{R}^n, 0) \to (\mathsf{R}^p, 0)$ are *φ-equivalent* if there exist germs of diffeomorphisms h of $(\mathsf{R}^n, 0)$, k of $(\mathsf{R}^p \times \mathsf{R}^q, 0)$ and l of $(\mathsf{R}^q, 0)$ commuting the following diagram

$$\begin{array}{ccccc}
(\mathsf{R}^n,0) & \xrightarrow{(f,\varphi)} & (\mathsf{R}^p \times \mathsf{R}^q,0) & \xrightarrow{\pi_q} & (\mathsf{R}^q,0) \\
{\scriptstyle h}\downarrow & & \downarrow{\scriptstyle k} & & \downarrow{\scriptstyle e} \\
(\mathsf{R}^n,0) & \xrightarrow{(g,\varphi)} & (\mathsf{R}^p \times \mathsf{R}^q,0) & \xrightarrow{\pi_q} & (\mathsf{R}^q,0)
\end{array}$$

where π_q is the usual projection.

The group associated to this equivalence relation is the subgroup G_φ of the group $\mathscr{A} = \mathrm{Diff}(\mathsf{R}^n, 0) \times \mathrm{Diff}(\mathsf{R}^p \times \mathsf{R}^q, 0)$ consisting of pairs (h, k) such that there exists $l \in \mathrm{Diff}(\mathsf{R}^q, 0)$ satisfying the relations $l \circ \varphi \circ h^{-1} = \varphi$ and $\pi_q \circ k = l \circ \pi_q$. G_φ acts on $C(n, p)$ by $(h, k) \cdot f = \pi_p \circ (k \circ (f, \varphi) \circ h^{-1})$, where $\pi_p\colon (\mathsf{R}^p \times \mathsf{R}^q, 0) \to (\mathsf{R}^p, 0)$ is the usual projection.

Allowing the germ φ to vary in the above diagram, we obtain an equivalence relation on $C(n, p + q)$, called $\mathscr{D}$-equivalence:

DEFINITION 1.2. Two map-germs (f, φ), (g, ψ) are *$\mathscr{D}$-equivalent* if there exist germs of diffeomorphisms h of $(\mathsf{R}^n, 0)$, k of $(\mathsf{R}^p \times \mathsf{R}^q, 0)$, and l of $(\mathsf{R}^q, 0)$, with $k(y, z) = (k_1(y, z), l(z))$, such that $(g, \psi) = k \circ (f, \varphi) \circ h^{-1}$.

REMARK 1.3. These equivalence relations can be considered as special cases of the equivalence of convergent diagrams, as defined by Dufour in [2]. More precisely: f, g (resp. $(f, \varphi), (g, \psi)$) are φ-equivalent (resp. $\mathscr{D}$-equivalent) if and only if the convergent diagrams $((f, \varphi), \pi_q)$, $((g, \varphi), \pi_q)$ (resp. $((f, \varphi), \pi_q)$, $((g, \psi), \pi_q)$) are equivalent.

We need some notation to describe the tangent spaces to the G_φ and $\mathscr{D}$-orbits.

For any integer m, let C_m be the local ring of function-germs at the origin in R^m, and $\mathscr{M}_m$ the corresponding maximal ideal. Given a map-germ $f \in C(m, k)$, let θ_f denote the C_m-module of vector fields of f, and set $\theta_m = \theta_{I(\mathsf{R}^m, 0)}$.

Given $(\mathsf{R}^q, 0) \xleftarrow{\varphi} (\mathsf{R}^n, 0) \xrightarrow{f} (\mathsf{R}^p, 0)$, the tangent spaces (respectively the extended tangent spaces) to the G_φ and $\mathscr{D}$-orbits are defined as follows:

(i) T_fG_φ (resp. $T_fG^e_\varphi$) is the set of all $\sigma \in \mathcal{M}_n\theta_f$ (resp. $\sigma \in \theta_f$) such that there exist $\xi \in \mathcal{M}_n\theta_n$ (resp. $\xi \in \theta_n$), $\eta \in \mathcal{M}_{p+q}\theta_{\pi_p}$ (resp. $\eta \in \theta_{\pi_p}$) and $\mu \in \mathcal{M}_q\theta_q$ (resp. $\mu \in \theta_q$) satisfying

$$\begin{cases} \sigma = df(\xi) + \eta \circ (f, \varphi) \\ 0 = d\varphi(\xi) + \mu \circ \varphi \end{cases}$$

(ii) $T_{(f,\varphi)}\mathscr{D}$ (resp. $T_{(f,\varphi)}\mathscr{D}^e$) is the set of all $\sigma \in \mathcal{M}_n\theta_{(f,\varphi)}$ (resp. $\sigma \in \theta_{(f,\varphi)}$) such that there exist $\xi \in \mathcal{M}_n\theta_n$ (resp. $\xi \in \theta_n$), $\zeta \in \mathcal{M}_{p+q}\theta_{p+q}$ (resp. $\zeta \in \theta_{p+q}$) and $\mu \in \mathcal{M}_q\theta_q$ (resp. $\mu \in \theta_q$) satisfying

$$\begin{cases} \sigma = d(f, \varphi)(\xi) + \zeta \circ (f, \varphi) \\ 0 = d\pi_q(\zeta) + \mu \circ \pi_q \end{cases}$$

The φ^e-codimension of f and the $\mathscr{D}^e$-codimension of (f, φ) are defined by

$$\operatorname{cod}_{\varphi^e} f = \dim_{\mathrm{R}} \theta_f / T_f G^e_\varphi,$$
$$\operatorname{cod}_{\mathscr{D}^e}(f, \varphi) = \dim_{\mathrm{R}} \theta_{(f,\varphi)} / T_{(f,\varphi)}\mathscr{D}^e.$$

REMARK 1.4. The definitions of this section extend to multigerms.

2. Unfolding theory for φ-equivalence and $\mathscr{D}$-equivalence.

In this section, we consider the concepts of versality of unfoldings for $\mathscr{D}$ and φ-equivalence. When φ is infinitesimally stable, Proposition 2.4 below gives a useful relationship between these two concepts.

Let F: $(\mathrm{R}^n \times \mathrm{R}^r, 0) \to (\mathrm{R}^p \times R^r, 0)$, $F(x, u) = (\bar{f}(x, u), u)$, be an r-parameter unfolding of $f(x) = \bar{f}(x, 0)$.

Writing $u_1, \ldots, u_r$ for the standard coordinates in R^r, we define $\partial_i F \in \theta_f$ by $\partial_i F = \left.\dfrac{\partial \bar{f}}{\partial u_i}\right|_{u=0}$, $i = 1, \ldots, r$. The real vector subspace of θ_f spanned by $\{\partial_1 F, \ldots, \partial_r F\}$ will be denoted by $\langle \partial_1 F, \ldots, \partial_r F \rangle_{\mathrm{R}}$.

Given a map-germ h: $(\mathrm{R}^s, 0) \to (\mathrm{R}^r, 0)$, the unfolding induced from F by h, h^*F, is the s-parameter unfolding of f defined by $h^*F(x, v) = (\bar{f}(x, h(v)), v)$.

Let $(\mathrm{R}^q, 0) \xleftarrow{\varphi} (\mathrm{R}^n. 0) \xrightarrow{f} (\mathrm{R}^p, 0)$. We denote by (F, r) an r-parameter unfolding $F(x, u) = (\bar{f}(x, u), u)$ of f, and by (G, r) an r-parameter unfolding $G(x, u) = (\bar{f}(x, u), \bar{\varphi}(x, u), u)$ of the pair (f, φ). When $\bar{\varphi}(x, u) = \varphi(x)$, the unfolding $(x, u) \mapsto (\bar{f}(x, u), \varphi(x), u)$ will be denoted by F_φ. With these notations:

DEFINITION 2.1. (i) (F, r) is *φ-transversal* if $T_f G^e_\varphi + \langle \partial_1 F, \ldots, \partial_r F \rangle_{\mathrm{R}} = \theta_f$.

(ii) (F, r) is *φ-versal* if for any unfolding $(\tilde{F}, s)$ of f there exist a map-germ h: $(\mathrm{R}^s, 0) \to (\mathrm{R}^r, 0)$ and unfoldings (H, s), (K, s), (L, s) of the identities $I_{(\mathrm{R}^n, 0)}$, $I_{(\mathrm{R}^p \times \mathrm{R}^q, 0)}$, $I_{(\mathrm{R}^q, 0)}$, respectively, such that:

$$\begin{cases} K \circ (h^* F_\varphi) \circ H^{-1} = \tilde{F}_\varphi \\ L \circ (\pi_q \times I_{(\mathrm{R}^s,0)}) \circ K^{-1} = \pi_q \times I_{(\mathrm{R}^s,0)}, \end{cases}$$

DEFINITION 2.2. (i) (G,r) is *$\mathscr{D}$-transversal* if $T_{(f,\varphi)}\mathscr{D}^e + \langle \partial_1 G, \ldots, \partial_r G \rangle_{\mathrm{R}} = \theta_{(f,\varphi)}$.

(ii) (G,r) *is $\mathscr{D}$-versal* if for any unfolding $(\tilde{G},s)$ of (f,φ) there exist a map-germ $h: (\mathrm{R}^s,0) \to (\mathrm{R}^r,0)$ and unfoldings (H,s), (K,s), (L,s) of the identities $I_{(\mathrm{R}^n,0)}$, $I_{(\mathrm{R}^p \times \mathrm{R}^q,0)}$, $I_{(\mathrm{R}^q,0)}$, respectively, such that:

$$\begin{cases} K \circ (h^* G) \circ H^{-1} = \tilde{G} \\ L \circ (\pi_q \times I_{(\mathrm{R}^s,0)}) \circ K^{-1} = \pi_q \times I_{(\mathrm{R}^s,0)}. \end{cases}$$

The standard result establishing the equivalence between versality and transversality also holds in this context:

PROPOSITION 2.3. (a) *An unfolding (G,r) of (f,φ) is $\mathscr{D}$-versal if and only if (G,r) is $\mathscr{D}$-transversal.*

(b) *Let φ be infinitesimally stable. Then, an unfolding (F,r) of f is φ-versal if and only if (F,r) is φ-transversal.*

PROOF. As we saw in Remark 1.3, $\mathscr{D}$-equivalence and φ-equivalence can be considered as special cases of equivalence of convergent diagrams. Thus, the statements (a) and (b) follow from Theorem 2 of Dufour [2], on the equivalence between the concepts of versality and transversality for convergent diagrams.

To each $\mathscr{D}$-versal unfolding of (f,φ), with φ infinitesimally stable, we can associate a φ-versal unfolding of f, and vice-versa.

Given an unfolding (G,r) of (f,φ), $G(x,u) = (\bar{f}(x,u), \bar{\varphi}(x,u), u)$, let (F,r) and (Φ,r) be the unfoldings of f and φ defined by $F(x,u) = (\bar{f}(x,u),u)$ and $\Phi(x,u) = (\bar{\varphi}(x,u),u)$, respectively. Since φ is infinitesimally stable, there exist unfoldings (H,r) of $I_{(\mathrm{R}^n,0)}$ and (L,r) of $I_{(\mathrm{R}^q,0)}$ such that

$$L^{-1} \circ \Phi \circ H = \varphi \times I_{(\mathrm{R}^r,0)}.$$

Define $(\tilde{F},r)$ to be the unfolding of f given by $\tilde{F} = F \circ H$. Then,

PROPOSITION 2.4. *(G,r) is $\mathscr{D}$-versal if and only if $(\tilde{F},r)$ is φ-versal.*

To prove this, we need the following

LEMMA. *Let (F,r) be an unfolding of f. Then, the unfolding (F_φ,r) of (f,φ) is $\mathscr{D}$-versal if and only if (F,r) is φ-versal.*

PROOF. Necessity is clear.

To prove the sufficiency, let (G,s) be an arbitrary unfolding of (f,φ). Then, we can write

$$G = (I_{(\mathrm{R}^p,0)} \times L) \circ \tilde{F}_\varphi \circ H^{-1}. \tag{2.3.1}$$

where $(\tilde{F}, s)$ is an unfolding of f, and $(H, s), (L, s)$ are unfoldings of $I_{(\mathrm{R}^n,0)}$ and $I_{(\mathrm{R}^q,0)}$, respectively, such that $L^{-1} \circ \Phi \circ H = \varphi \times I_{(\mathrm{R}^s,0)}$.

Since (F, r) *is* φ-versal, there exist a map-germ h: $(\mathrm{R}^s, 0) \to (\mathrm{R}^r, 0)$ and unfoldings $(\bar{H}, s)$, $(\bar{K}, s)$, $(\bar{L}, s)$ of $I_{(\mathrm{R}^n,0)}$, $I_{(\mathrm{R}^p \times \mathrm{R}^q,0)}$, $I_{(\mathrm{R}^q,0)}$, respectively, such that:

$$\begin{cases} \bar{K} \circ (h^* F_\varphi) \circ \bar{H}^{-1} = \tilde{F}_\varphi \\ \bar{L} \circ (\pi_q \times I_{(\mathrm{R}^s,0)}) \circ \bar{K}^{-1} = \pi_q \times I_{(\mathrm{R}^s,0)}. \end{cases} \tag{2.3.2}$$

From (2.3.1) and (2.3.2), we get the result.

PROOF OF PROPOSITION 2.4. As above, $G = (I_{(\mathrm{R}^p,0)} \times L) \circ \tilde{F}_\varphi \circ H^{-1}$. Then, given h: $(\mathrm{R}^s, 0) \to (\mathrm{R}^r, 0)$, we have

$$h^* G = \tilde{K} \circ (h^* \tilde{F}_\varphi) \circ \tilde{H}^{-1}$$

where $(\tilde{H}, s)$, $(\tilde{K}, s)$ are, respectively, the unfoldings of $I_{(\mathrm{R}^n,0)}$ and $I_{(\mathrm{R}^p \times \mathrm{R}^q,0)}$ defined by $\tilde{H} = h^* H$ and $\tilde{K} = I_{(\mathrm{R}^p,0)} \times h^* L$. The result follows from the precedent lemma.

As a consequence of Propositions 2.4 and 2.3, we have:

COROLLARY 2.5. $\mathrm{cod}_{\varphi^e} f = \mathrm{cod}_{\mathscr{D}^e}(f, \varphi)$.

3. Classification of germs and multigerms of φ^e-codimension $\leqq 1$

In this section, we classify germs and multigerms with $\mathrm{cod}_{\varphi^e} f \leqq 1$, in the case

$$\begin{array}{ccc} & & (\mathrm{R}, 0) \\ & f\nearrow & \\ (\mathrm{R}^n, S) & & \\ & \varphi\searrow & \\ & & (\mathrm{R}, 0) \end{array}$$

where $n \geqq 2$ *and* φ *is infinitesimally stable.*

In what follows, a singular point of (f, φ) will receive the adjective transverse or tangent whether φ is regular or singular.

The following propositions summarize the results:

PROPOSITION 3.1. *Let* $(\mathrm{R}, 0) \xleftarrow{\varphi} (\mathrm{R}^n, 0) \xrightarrow{f} (\mathrm{R}, 0)$, *where* $n \geqq 2$ *and* φ *is infinitesimally stable. Then*

(i) $\mathrm{cod}_{\varphi^e} f = 0$ *if and only if the pair* (f, φ) *is of one of the types: submersion, transverse fold, tangent fold, transverse cusp.*

(ii) $\mathrm{cod}_{\varphi^e} f = 1$ *if and only if the pair* (f, φ) *is of one of the types: tangent cusp, transverse lips, transverse beak to beak, transverse swallowtail.*

The normal forms of these singularities, and their versal unfoldings are shown in Table 1.

TABLE 1.

Type (f, φ)	$\mathscr{D}$-Normal Form	$\mathrm{cod}_{\varphi^*} f$	$\mathscr{D}$-versal unfolding
1 submersion	(x, y)	0	
2 transverse fold	$(x^2 + q(z), y)$	0	
2′ tangent fold	$(x, x^2 \pm y^2 + q(z))$	0	
3 transverse cusp	$(x^3 + xy + q(z), y)$	0	
3′ tangent cusp	$(x, y^3 + xy + \lambda(x) + q(z)), \lambda'(0) = 0$	1	$(x + uy, y^3 + xy + \lambda(x) + q(z))$
4_2^+ transverse lips	$(x^3 + xy^2 + q(z), y)$	1	$(x^3 + xy^2 + q(z) + ux, y)$
4_2^- transverse beak to beak	$(x^3 - xy^2 + q(z), y)$	1	$(x^3 - xy^2 + q(z) + ux, y)$
5 transverse swallowtail	$(x^4 + xy + x^2y + q(z), y)$	1	$(x^4 + xy + x^2y + q(z) + ux^2, y)$

where $z = (z_1, \ldots, z_{n-2})$ and $q(z) = \sum_{i=1}^{n-2} \pm z_i^2$.

PROPOSITION 3.2. *Let* $(\mathrm{R}, 0) \xleftarrow{\varphi} (\mathrm{R}^n, S) \xrightarrow{f} (\mathrm{R}, 0)$, $n \geqq 2$, $\mathrm{card}(S) \geqq 2$. *If* φ *is* $\mathscr{A}$*-infinitesimally stable and* (f, φ) *is a singular multigerm,* i.e., $S \subset \Sigma(f, \varphi)$, *the singular set of* (f, φ), *then*

(i) $\mathrm{cod}_{\varphi^*} f = 0$ *if and only if* $\mathrm{card}(S) = 2$ *and* (f, φ) *is a transversal intersection of two transverse folds.*

(ii) $\mathrm{cod}_{\varphi^*} f = 1$ *if and only if* $\mathrm{card}(S) = 2$ *and* (f, φ) *is of one of the types: intersection of a transverse fold and a tangent fold, intersection of two transverse folds with second order contact, transversal intersection of a transverse fold and a transverse cusp* (*see Table* 2).

To obtain the above results, we classify the $\mathscr{D}$-orbits contained in the $\mathscr{A}$-orbits of germs and multigerms of codimension $\leqq 1$ ([5], [10]).

The main distinction between our classification and the $\mathscr{A}$-classification appears in the $\mathscr{A}$-orbit of the intersection of 3-folds. More precisely, this singularity has $\mathscr{A}^e$-codimension equal to one, but its $\mathscr{D}^e$-codimension is two. However, we have the following result:

PROPOSITION 3.3. *The topological* $\mathscr{D}^e$*-codimension of the intersection of* 3*-folds is one* (*see Table* 2).

3.4. *Classification of* φ*-orbits of germs* f: $(\mathrm{R}^n, 0) \to (\mathrm{R}, 0)$.

The proof of Proposition 3.1 will follow from the Lemmas 3.4.1 to 3.4.5. below. In Lemmas 3.4.1 and 3.4.2, (f, φ) is a fold or a cusp singularity.

LEMMA 3.4.1. $\mathrm{cod}_{\varphi^*} f = 0$ *if and only if the pair* (f, φ) *is of one of the types: transverse fold, tangent fold and transverse cusp.*

TABLE 2.

Type (f, φ)	$\mathscr{D}$-Normal Form	$\mathrm{cod}_{\varphi^*} f$	$\mathscr{D}$-versal unfolding
transversal intersection of two transverse folds	$\begin{cases} (\pm\bar{x}^2 + \bar{q}(\bar{z}), \bar{y}) \\ (y \pm x^2 + q(z), y) \end{cases}$	0	
intersection of a transverse fold and a tangent fold	$\begin{cases} (\pm\bar{x}^2 + \bar{q}(\bar{z}), \bar{y}) \\ (x, \pm x^2 \pm y^2 + q(z)) \end{cases}$	1	$\begin{cases} (\pm\bar{x}^2 + \bar{q}(\bar{z}) + u, \bar{y}) \\ (x, \pm x^2 \pm y^2 + q(z)) \end{cases}$
intersection of two transverse folds with second order contact	$\begin{cases} (\pm\bar{x}^2 + \bar{q}(\bar{z}), \bar{y}) \\ (\pm x^2 \pm y^2 + q(z). y) \end{cases}$	1	$\begin{cases} (\pm\bar{x}^2 + \bar{q}(\bar{z}) + u, \bar{y}) \\ (\pm x^2 \pm y^2 + q(z), y) \end{cases}$
transversal intersection of a transverse fold and a transverse cups	$\begin{cases} (\pm\bar{x}^2 + \bar{q}(\bar{z}), \bar{y}) \\ (x^3 + xy + y + q(z), y) \end{cases}$	1	$\begin{cases} (\pm\bar{x}^2 + \bar{q}(\bar{z}) + u, \bar{y}) \\ (x^3 + xy + y + q(z), y) \end{cases}$
intersection of 3-folds	$\begin{cases} (\pm\bar{\bar{x}}^2 + \bar{\bar{q}}(\bar{\bar{z}}), \bar{\bar{y}}) \\ (\bar{y} \pm \bar{x}^2 + \bar{q}(\bar{z}), \bar{y}) \\ (ay \pm x^2 + q(z), y), a \neq 0, 1 \end{cases}$	2	$\begin{cases} (\pm\bar{\bar{x}}^2 + \bar{\bar{q}}(\bar{\bar{z}}), \bar{\bar{y}}) \\ (\bar{y} \pm \bar{x}^2 + \bar{q}(\bar{z}), \bar{y}) \\ (ay \pm x^2 + q(z) + u_1 y + u_2, y) \end{cases}$

PROOF. See [6].

LEMMA 3.4.2. *If (f, φ) is a tangent cusp, then*
(i) *the normal form with respect to $\mathscr{D}$-equivalence is*

$$(x, y, z) \xrightarrow{(f,\varphi)} (x, y^3 + xy + \lambda(x) + q(z)), \lambda'(0) = 0$$

where $z = (z_1, \ldots, z_{n-2})$ and $q(z) = \pm z_1^2 \pm z_2^2 \pm \ldots \pm z_{n-2}^2$.
(ii) $\mathrm{cod}_{\varphi^*} f = 1$.

PROOF. The normal form in (i) is essentially obtained in [13]. Routine calculations with this normal form show that $\theta_f = T_f G_\varphi^e \oplus \langle y \rangle_{\mathbb{R}}$ and, hence, $\mathrm{cod}_{\varphi^*} f = 1$.

In the following lemmas, we classify the $\mathscr{D}$-orbits obtained by refinement of the $\mathscr{A}$-orbits of germs (f, φ) of $\mathscr{A}^e$-codimension one. These are: lips, beak to beak and swallowtail ([5], [10]).

When φ is regular, the pair (f, φ) is $\mathscr{D}$-equivalent to a germ of the form:

$$\left(\sum_{i=1}^{k} a_i x_i y \pm x_{k+1}^2 \pm \ldots \pm x_{n-1}^2 + \rho(x_1, \ldots, x_k, y), y\right),$$

$\rho \in \mathscr{M}_{k+1}^3$ ([4]).
And we have:

LEMMA 3.4.3. *If $\mathrm{cod}_{\varphi^*} f = 1$, then $k = 1$.*

PROOF. Let $g(x_1, \ldots, x_k, y) = \sum_{i=1}^{k} a_i x_i y + \rho(x_1, \ldots, x_k, y)$, $\rho \in \mathscr{M}_{k+1}^3$. Then $\mathrm{cod}_{\varphi^*} f = \mathrm{cod}_{\varphi^*} g$. Now, let $g_0(x_1, \ldots, x_k) = g(x_1, \ldots, x_k, 0) =$

$\rho(x_1, \ldots, x_k, 0)$. We can think of (g, φ) as a 1-parameter unfolding of g_0. Now, it is not hard to see that $\operatorname{cod}_{\mathscr{A}^e} g_0 \leqq \operatorname{cod}_{\varphi^e} g + 1$. Since $g_0 \in \mathscr{M}_k^3$, we have $\operatorname{cod}_{\mathscr{A}^e} g_0 \geqq (k^2 + k)/2$. From these two inegualities, it follows that $\operatorname{cod}_{\varphi^e} g = 1$ implies $k = 1$.

As a consequence of the above lemma, we can reduce the analysis to a germ $f(x, y) \in \mathscr{M}_2^2$, with $\operatorname{cod}_{\varphi^e} f < \infty$.

A normal form for (f, φ) is

$$(f, \varphi)(x, y) = (x^r + xQ_1(y) + \ldots + x^{r-2}Q_{r-2}(y), y)$$

where $Q_i(y)$ $(i = 1, \ldots, r - 2)$ is a polynomial in y without constant term ([3, Proposition (4.8)]).

Furthermore, if $\operatorname{cod}_{\varphi^e} f = 1$ then $r = 3, 4$.

LEMMA 3.4.4. *If* $\operatorname{cod}_{\varphi^e} f = 1$ *and* $r = 3$, *then f is φ-equivalent to*

(i) $x^3 + xy^2$*-transverse lips, or*

(ii) $x^3 - xy^2$*-transverse beak to beak.*

PROOF. After some coordinate changes, we may assume that f has the form:

$$x^3 \pm xy^2 + O(4).$$

Now, computing the φ-tangent space for such f, we obtain:

$$\mathscr{M}_2^4 \theta_f \subset T_f G_\varphi + \mathscr{M}_2^s \theta_f \quad \forall s \geqq 5.$$

The Infinitesimal Criterium for φ-determinacy ([9]) and Mather's Lemma ([11, Lemma 3.1]) imply that f is 3-φ-determined, hence φ-equivalent to $x^3 \pm xy^2$. Furthermore, $\theta_f = T_f G_\varphi^e \oplus \langle x \rangle_{\mathbf{R}}$.

LEMMA 3.4.5. *If* $\operatorname{cod}_{\varphi^e} f = 1$ *and* $r = 4$, *then f is φ-equivalent to* $x^4 + xy + x^2y$*-transverse swallowtail.*

PROOF. After some simple coordinate changes, we may assume that f has the form

$$x^4 + xy + ax^2y + xQ_1(y)y^2 + x^2Q_2(y)y^2.$$

Making the change of coordinates in the target

$$\begin{cases} U = u - uvQ_1(v) \\ V = v \end{cases}, \quad \text{we obtain:}$$

$$x^4 + xy + ax^2y + xP_1(y)y^3 + x^2P_2(y)y^2 + x^4P_4(y)y.$$

Again, with coordinate changes of the above type, we can eliminate the terms xy^k, $k \geqq 2$, up to terms of high order, to get:

$$x^4 + xy + ax^2y + x^2R_2(y)y^2 + x^4R_4(y)y.$$

Making the coordinate change in the source

$$\begin{cases} x = X - X^2R_2(Y)Y - X^4R_4(Y) \\ y = Y \end{cases},$$

we get

$$x^4 + xy + ax^2y + x^3L_3(y)y^2 + O(6).$$

And, repeating change of coordinate of this type, we obtain

$$x^4 + xy + ax^2y + O(6). \tag{3.4.6}$$

A direct computation shows that for any f of the form (3.4.6) $\dim_{\mathrm{R}} \dfrac{\theta_f}{T_f G_\varphi^e + \mathcal{M}_2^6 \theta_f} \geqq 1$ and $\dim_{\mathrm{R}} \dfrac{\theta_f}{T_f G_\varphi^e + \mathcal{M}_2^6 \theta_f} = 1 \Leftrightarrow a \neq 0$.

Now, we can see easily that $x^4 + xy + ax^2y + O(6)$, $a \neq 0$, is φ-equivalent to $x^4 + xy + x^2y + O(6)$.

As in the previous lemma, we show (after very tedious calculations) that for any such f

$$\mathcal{M}_2^6\theta_f \subset T_f G_\varphi + \mathcal{M}_2^r\theta_f, \quad \forall r \geqq 7.$$

Thus, f is 5-φ-determined. Furthermore,

$$\theta_f = T_f G_\varphi^e \oplus \langle x^2 \rangle_{\mathrm{R}}.$$

Proposition 3.1 now follows from the above lemmas and from the fact that $\mathrm{cod}_{\varphi^e} f > 1$ for all pairs (f, φ) such that $\mathrm{cod}_{\mathscr{A}^e}(f, \varphi) \geqq 1$ and φ is singular.

3.5. *Classification of φ-orbits of multigerms f: $(\mathrm{R}^n, S) \to (\mathrm{R}, 0)$.*

In this section, we classify multigerms

$$f\colon (\mathrm{R}^n, S) \to (\mathrm{R}, 0),$$

with φ^e-codimension $\leqq 1$, where φ is $\mathscr{A}$-infinitesimally stable and S is a finite subset of $\Sigma(\mathrm{f}, \varphi)$.

It will be useful to consider in the sequel the multigerm f on $S = \{x_1, \ldots, x_s\}$ as a convergent diagram of map-germs $f_i = f \mid (\mathrm{R}^n, x_i)\colon (\mathrm{R}^n, x_i) \to (\mathrm{R}, 0)$ $(i = 1, \ldots, s)$. This allow us to consider $x_i = 0 \in \mathrm{R}^n$ for all $i = 1, \ldots, s$.

We consider two cases:

(I) $\mathrm{cod}_{\mathscr{A}^e}(f, \varphi) = 0$

In this case, $\mathrm{card}(S) = 2$ and the singular points in S are fold points with transversal intersection ([5], [10]).

With respect to $\mathscr{D}$-equivalence, we have two possibilities:

(I1) two transverse folds intercepting transversally. In this case, $\mathrm{cod}_{\mathscr{D}^e}(f, \varphi) = 0$ and the normal form is:

$$\begin{cases}(\bar{x}, \bar{y}, \bar{z}) \to (\pm\bar{x}^2 + \bar{q}(\bar{z}), \bar{y}) \\ (x, y, z) \to (y \pm x^2 + q(z), y)\end{cases}$$

(I2) a transverse fold and a tangent fold, with $\mathrm{cod}_{\mathscr{D}^e}(f, \varphi) = 1$.
Normal form:

$$\begin{cases}(\bar{x}, \bar{y}, \bar{z}) \to (\pm\bar{x}^2 + \bar{q}(\bar{z}), \bar{y}) \\ (x, y, z) \to (x, \pm x^2 \pm y^2 + q(z))\end{cases}$$

(II) $\mathrm{cod}_{\mathscr{A}^e}(f, \varphi) = 1$
In this case, $\mathrm{card}(S) = 2$ or 3 and the corresponding $\mathscr{A}$-orbits are: two folds with non transversal intersection, transversal intersection of fold and cusp and intersection of three fold lines ([5], [10]).

Refining the above classification, we find that only the following bigerms have $\mathrm{cod}_{\varphi^e} = 1$.

(II1) two transverse folds with second order contact

$$\begin{cases}(\bar{x}, \bar{y}, \bar{z}) \to (\pm\bar{x}^2 + \bar{q}(\bar{z}), \bar{y}) \\ (x, y, z) \to (\pm x^2 \pm y^2 + q(z), y)\end{cases}$$

(II2) transversal intersection of a transverse fold and a transverse cusp

$$\begin{cases}(\bar{x}, \bar{y}, \bar{z}) \to (\pm\bar{x}^2 + \bar{q}(\bar{z}), \bar{y}) \\ (x, y, z) \to (x^3 + xy + y + q(z), y)\end{cases}$$

The above classification proves Proposition 3.2.

The classification of the $\mathscr{D}$-orbits inside the orbit corresponding to the intersection of three fold lines presents modality. In fact, the cross-ratio of the set of fold lines and the horizontal axis in the target is an invariant of $\mathscr{D}$-equivalence. Next, we show that the topological $\mathscr{D}^e$-codimension of such singularity is one.

PROOF OF PROPOSITION 3.3. φ is a trigerm of submersion and we represented it by the normal form $[\bar{\bar{y}}, \bar{y}, y]$. We denote by f_t the family

$$f_t = [\pm\bar{\bar{x}}^2 + \bar{\bar{q}}(\bar{\bar{z}}), \bar{y} \pm \bar{x}^2 + \bar{q}(\bar{z}), ty \pm x^2 + q(z)], \quad t \neq 0, 1.$$

To show that this is topologically φ-trivial we proceed as in [8], [12], [13] by constructing continuous vector fields V in the source and W in the target as follows:

$$V\colon (\mathrm{R}^n \times \mathrm{R}, S \times \mathrm{R}) \to (\mathrm{R}^n \times \mathrm{R}, S \times \mathrm{R}), \quad \pi_{\mathrm{R}} \circ V = \frac{\partial}{\partial t},$$

$\pi_{\mathsf{R}^n} \circ V(0,t) = 0$, $\pi_2 \circ V \equiv 0$ $\left(\text{here } \pi_2 \text{ denotes the projection on the directions } \left[\frac{\partial}{\partial \bar{\bar{y}}}, \frac{\partial}{\partial \bar{y}}, \frac{\partial}{\partial y}\right]\right)$.

$W\colon (\mathsf{R}^2 \times \mathsf{R}, 0) \to (\mathsf{R}^2 \times \mathsf{R}, 0)$, $W = W(u,v,t)$, $\pi_{\mathsf{R}} \circ W = \frac{\partial}{\partial t}$, $\pi_{\mathsf{R}^2} \circ W(0,\mathrm{t}) = 0$, $\pi_2 \circ W \equiv 0$ $\left(\pi_2 \text{ denotes the projection on the } \frac{\partial}{\partial v}\text{-direction}\right)$.

To show that V and W are integrable, we construct a control function ρ in the target as in [8], [12] such that

$$\|V\| \leqq c\rho \circ (f,\varphi) \quad \text{and} \quad \|W\| \leqq c'\rho$$

for some constants c and c'.

By integrating these vector fields, we get homeomorphisms in the source and target given the desired φ-equivalence.

The construction of V and W will follow from the following lemma:

LEMMA. *There exist:* (i) *a weighted homogeneous trigerm of vector fields* $[\bar{\bar{\xi}}, \bar{\xi}, \xi]$ *in* $(\mathsf{R}^n \times \mathsf{R}, S \times \mathsf{R})$, *of the type* $(1, 2, 1, \ldots, 1; 5)$, *with zero components in the directions* $\left[\frac{\partial}{\partial \bar{\bar{y}}}, \frac{\partial}{\partial \bar{y}}, \frac{\partial}{\partial y}\right]$ *and* $\frac{\partial}{\partial t}$,

(ii) *a germ of vector field* η *in* $(\mathsf{R}^2 \times \mathsf{R}, 0 \times \mathsf{R})$, *homogeneous of degree* 3, *with zero components in the directions* $\frac{\partial}{\partial v}$ *and* $\frac{\partial}{\partial t}$, *such that the following equality holds:*

$$\rho \cdot \frac{\partial f_t}{\partial t} = df_t([\bar{\bar{\xi}}, \bar{\xi}, \xi]) + \eta \circ (f_t, \varphi, t) \tag{3.5.1}$$

where $\rho(u,v) = u^2 + v^2$, *and* (u,v) *denotes the coordinates in the target.*

PROOF. Let $F = (f_t, t)$. The right-hand side of (3.5.1) contains the set of trigerms $[\bar{\bar{\sigma}}, \bar{\sigma}, \sigma]$ given by:

$$\begin{cases} \bar{\bar{\sigma}}(\bar{\bar{x}}, \bar{\bar{y}}, \bar{\bar{z}}, t) = \pm 2\bar{\bar{x}}\bar{\bar{\xi}}_0(\bar{\bar{x}}, \bar{\bar{y}}, \bar{\bar{z}}, t) \\ \qquad + \sum_{i=1}^{n-2} \frac{\partial \bar{\bar{q}}}{\partial \bar{\bar{z}}_i} \bar{\bar{\xi}}_i(\bar{\bar{x}}, \bar{\bar{y}}, \bar{\bar{z}}, t) + \eta(\pm \bar{\bar{x}}^2 + \bar{\bar{q}}(\bar{\bar{z}}), \bar{\bar{y}}, t) \\ \\ \bar{\sigma}(\bar{x}, \bar{y}, \bar{z}, t) = \pm 2\bar{x}\bar{\xi}_0(\bar{x}, \bar{y}, \bar{z}, t) + \\ \qquad \sum_{i=1}^{n-2} \frac{\partial \bar{q}}{\partial \bar{z}_i} \bar{\xi}_i(\bar{x}, \bar{y}, \bar{z}, t) + \eta(\bar{y} \pm \bar{x}^2 + \bar{q}(\bar{z}), \bar{y}, t) \\ \\ \sigma(x, y, z, t) = \pm 2x\xi_0(x, y, z, t) + \\ \qquad \sum_{i=1}^{n-2} \frac{\partial q}{\partial z_i} \xi_i(x, y, z, t) + \eta(ty \pm x^2 + q(z), y, t) \end{cases} \tag{3.5.2}$$

where all the germs $\bar{\bar{\xi}}_i$'s, $\bar{\xi}_i$'s, and ξ_i's in the source, as well the germ η in the target, are zero in their base points.

Notice that (3.5.2) contains all elements $[\bar{\bar{\sigma}}, \bar{\sigma}, \sigma]$ where

(3.5.3) $\bar{\bar{\sigma}}, \bar{\sigma}, \sigma \in \mathcal{M}_n^2 C_{n+1}$ and $\bar{\bar{\sigma}}(0, \bar{\bar{y}}, 0, t) = \bar{\sigma}(0, \bar{y}, 0, t) = \sigma(0, y, 0, t) \equiv 0.$

Thus, for $t \neq 0, 1$, we can take the germ of the vector field $\eta(u, v, t) = \dfrac{1 + t^2}{t^2 - t^3} \cdot (u^2 v - u^3)$ in the target, and from the above remark, define convenient trigerm of vector fields $[\bar{\bar{\xi}} = (\bar{\bar{\xi}}_i), \bar{\xi} = (\bar{\xi}_i), \xi = (\xi_i)]$ in the source to be able to write $[0, 0, (t^2 + 1)y^3]$ satisfying the system of equations (3.5.2). From this expression and using remark (3.5.3) again, we get the equation (3.5.1). It is not hard to see that the defined vector fields are weighted homogeneous with the specified degrees.

4. Genericity Theorems.

Let N be a compact, smooth manifold of dimension $n \geqq 2$, and let φ: $N \to \mathsf{R}$ be a fixed Morse function. The subset $S_\varphi = \{f\colon N \to \mathsf{R};\ f$ is a φ-stable Morse function$\}$ is open and dense in $C^\infty(N, \mathsf{R})$ ([6]).

The main result of this section, Theorem 4.2, shows that given f and g in S_φ, the transition from the pair (f, φ) to the pair (g, φ) can be realized by a generic path that preserves the foliation in N defined by the level set $\varphi =$ constant. (One such path will be called φ-adequate).

THEOREM 4.2. *Let f, g be in S_φ. Then, there exists a φ-adequate path F in $C^\infty(N \times [0, 1], \mathsf{R})$ such that $F_0 = f$ and $F_1 = g$. Moreover, the set C_φ of all φ-adequate paths is open and dense in $C^\infty(N \times [0, 1], \mathsf{R})$.*

4.1. *An auxiliary lemma.* Let $\Lambda(x, t) = (F(x, t), \Phi(x, t))$ denote a path in $C^\infty(N \times [0, 1], \mathsf{R}^2) \equiv C^\infty(N \times [0, 1], \mathsf{R}) \times C^\infty(N \times [0, 1], \mathsf{R})$.

We want to define a subset $\mathscr{C}$ in $C^\infty(N \times [0, 1], \mathsf{R}^2)$ of 1-parameter families $\Lambda = (F, \Phi)$ that are generic with respect to the auxiliary equivalence relation $\mathscr{D}$, and also have the property that the projection on the second factor gives a versal family (This last assumption says that the transition from the foliation defined by Φ_0 to the one defined by Φ_1 has only singularities of codimension $\leqq 1$ ([7])).

Let $\mathscr{C}$ be the subset of $C^\infty(N \times [0, 1], \mathsf{R}^2)$ consisting of paths $\Lambda = (F, \Phi)$ such that

(A_0) $\Phi_0 = \Phi(-, 0)$ and $\Phi_1 = \Phi(-, 1)$ are stable, and F_0 is Φ_0-stable and F_1 is Φ_1-stable.

(A_1) There exists a finite subset $B_1 \subset (0, 1)$ such that, for all $t \notin B_1$, Φ_t is stable.

(A_2) There exists a finite subset $B_2 \subset (0, 1)$, $B_2 \cap B_1 = \phi$, such that, for all $t \notin B_2$, F_t is Φ_t-stable.

(A_3) For each $t \in B_1$, only one of the following non-stable singularities may occur for Φ_t:
germ: birth-death singularity (normal form: $x^3 \pm y^2 + q(z)$).
multigerm: two critical points of same height.

(A_4) For each $t \in B_2$, F_t may have only one non Φ_t-stable singularity of one of the types:
germs: Φ_t-tangent cusp; Φ_t-transverse lips, beak to beak and swallowtail (cf. Prop. 3.1 (ii)).
multigerms: intersection of a Φ_t-transverse fold and a Φ_t-tangent fold, intersection of two Φ_t-transverse folds with second order contact, transversal intersection of a Φ_t-transverse fold and a Φ_t-transverse cusp, transversal intersection of three Φ_t-transverse folds (cf. Prop. 3.2 (ii)).

LEMMA. *The set $\mathscr{C}$ is open and dense in $C^\infty(N \times [0,1], \mathrm{R}^2)$.*

PROOF. That the set S of paths Φ satisfying conditions A_1 and A_3 of the definition of $\mathscr{C}$ is open and dense in $C^\infty(N \times [0,1], \mathrm{R})$, follows from Theorem 8.12 of Looijenga [7]. Furthermore, let S' be the set of paths $\Lambda = (F, \Phi)$ in $C^\infty(N \times [0,1], \mathrm{R}^2)$ such that there exists a finite subset $B \subset (0,1)$ such that, for all $t \notin B$, Λ_t is $\mathscr{A}$-stable and, for all $t \in B$, only one of the singularities of $\mathscr{A}$-codimension one occurs:

germs: lips, beak to beak, swallowtail.
multigerms: transversal intersection of a fold and a cusp, non-transversal intersection of two folds, transversal intersection of three folds.

S' is just the set of paths transverse to the stratification of $J_r^k(N, \mathrm{R}^2)$ ($r = 1,2,3$) where the strata are the $\mathscr{A}_r^k$-orbits of codimension $\leqq 1$, and its complement. Thus S' is open and dense ([5], [10]). The set $\mathscr{C}$ consists of paths in $\pi^{-1}(S) \cap S'$ that are transverse to a refinement of the above stratification. More precisely, each stratum of the new stratification is obtained as follows:

(1) $\mathscr{D}$-orbits of (f, φ) of $\mathrm{cod}_{\mathscr{D}^e} \leqq 1$, with φ stable.

(2) $\mathscr{D}$-orbits of $\mathrm{cod}_{\mathscr{D}^e} = 1$, corresponding to tangent fold singularity (f, φ), with φ non stable. There are two types of pairs (f, φ), according to the bifurcations of Φ:
germ: third order contact with the horizontal line.
multigerm: double tangency with the horizontal line.

(3) the union of $\mathscr{D}$-orbits of the moduli family corresponding to the intersection of three transverse folds.

(4) the complement of the union of the above strata. This set is a finite union of algebraic sets of cod $\geqq 2$.

4.2. *Proof of Theorem.* Let φ: $N \to \mathrm{R}$ be a fixed Morse function.

DEFINITION 4.1.1. A path F: $N \times [0, 1] \to \mathrm{R}$ *is φ-adequate* if

(i) $F_0 = f$ and $F_1 = g$, with $f, g \in S_\varphi$.

(ii) F embeds in $\mathscr{C}$ by $F \to F_\varphi$, where $F_\varphi(x, t) = (F(x, t), \varphi(x))$.

We denote by C_φ the set of φ-adequate paths.

Notice that if $F \in C_\varphi$ then F_t is φ-stable for all $t \in [0, 1]$ except for a finite number of bifurcation points corresponding to maps (F_t, φ) presenting singularities of the following types:

germs: tangent cusp; transverse lips, beak to beak and swallowtail.

multigerms: intersection of a transverse fold and a tangent fold, intersection of two transverse folds with second order contact, transversal intersection of a transverse fold and a transverse cusp, transversal intersection of three transverse folds.

PROOF OF THEOREM. Let f, g be in S_φ. Since $\mathscr{C}$ is dense, we can choose a path $\Lambda = (G, \Phi)$ in $\mathscr{C}$ joining (f, φ) to (g, φ), with Φ arbitrary close to the constant path φ. Since φ is a Morse function, there exist paths sufficiently close to the identities, $H \in C^\infty(N \times [0, 1], N)$ and $L \in C^\infty(\mathrm{R} \times [0, 1], \mathrm{R})$, such that:

$$H_0 = H_1 = I_N,\ L_0 = L_1 = I_{\mathrm{R}},\ L_t^{-1} \circ \Phi_t \circ H_t = \varphi \quad ([1])$$

The path $F = G \circ \bar{H} \in C^\infty(N \times [0, 1], \mathrm{R})$, where $\bar{H}(x, t) = (H(x, t), t)$, joins f to g; moreover, F is φ-adequate, i.e., $F_\varphi = (F, \varphi) \in \mathscr{C}$. This last assertion follows from Proposition 2.4 which establish a relationship between $\mathscr{D}$-versal unfoldings of (f, φ) and φ-versal unfoldings of f.

That C_φ is open follows easily from the openess of $\mathscr{C}$.

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