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FIBRATIONS AND HOMOLOGY SPHERE BORDISM

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Introduction.

In this article we discuss the homology sphere bordism groups, $\Omega_*^{\mathrm{HS}}(-)$, of J.-C. Hausmann, with special emphasis on the properties of these groups in relation to fibrations. The interest in these groups derives from a strong relation to algebraic K-theory, and as a good realization of what P. Vogel called the homological Hurewicz theorem. The main point demonstrated here, is that these groups are quite accessible from ordinary obstruction theory, and that, if the action of a particular part of the fundamental group is not too wild, they behave quite well.

We work in the category of pointed spaces homotopy equivalent to connected CW-complexes. In [4] Hausmann and Vogel defines the homology sphere bordism groups of a (pointed) space X, $\Omega_n^{\mathrm{HS}}(X)$, $n \geqq 2$, as the abelian group of H_*-cobordism classes of maps from oriented PL n-homology spheres to X. The H_*-cobordisms are equipped with base arcs connecting the base points of the spheres, and addition is induced by the connected sum. They define the relative homology sphere groups similarly. The relative groups are abelian in dimensions > 2, and we get the obvious long exact sequence

$$\begin{aligned} \ldots &\to \Omega_{n+1}^{\mathrm{HS}}(X, A) \to \Omega_n^{\mathrm{HS}}(A) \to \Omega_n^{\mathrm{HS}}(X) \to \Omega_n^{\mathrm{HS}}(X, A) \to \ldots \\ \ldots &\to \Omega_3^{\mathrm{HS}}(X, A) \to \Omega_3^{\mathrm{HS}}(A) \to \Omega_2^{\mathrm{HS}}(X) \\ &\to \Omega_2^{\mathrm{HS}}(X, A) \to \pi_1(A) \to \pi_1(X) \to \pi_1(X, A) \to 0 \end{aligned}$$

$\Omega_*^{\mathrm{HS}}(-)$ clearly defines a functor, and we have a natural map $\pi_*(-) \to \Omega_*^{\mathrm{HS}}(-)$ by sending representatives onto representatives of a possibly larger class.

The main result on homology sphere bordism given in [4] is:

THEOREM ([4, THEOREM 4.1]). *Let $A \to X \to X^+$ be the plus sequence associated with* $\mathrm{LP}\pi_1(X)$ *(see next section for definition). Then*

(1) $\Omega_*^{\mathrm{HS}}(X, A) \xrightarrow{\cong} \pi_*(X^+)$, *and*
(2) $\Omega_q^{\mathrm{HS}}(A) = 0$ *for* $q \neq 3$.

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This result is obtained through homological surgery techniques. While this is seemingly intrinsic to (2) (one may also prove (2) directly by the methods developed in [7]), a motivation for this paper is to show that this is not the case for (1). A further motivation is to obtain tools for manipulating the homology sphere groups. One sees easily that $\Omega_*^{\mathrm{HS}}(-)$ is not stable, and a natural property to seek for is an exact sequence for fibrations. One way to do this is to use Berrick's, [2], characterization of "plus constructiveness" and the above theorem to obtain sharp results in high dimensions. However, this has the drawback of giving only partial results in dimensions $\leqq 5$, so we will follow a more direct (and admittedly less powerfull) route avoiding the use of (2) (except in two examples), and giving (1) as an easy consequence.

The main results are:

THEOREM 5. *Let $F \to E \xrightarrow{p} B$ be a fiber sequence of connected spaces with* $\mathrm{LP}\pi_1(E) \subseteq \ker\{p_\# : \pi_1(E) \to \pi_1(B)\}$. *Then* $\pi_*(B) \xleftarrow{\approx} \pi_*(E, F) \xrightarrow{\approx} \Omega_*^{\mathrm{HS}}(E, F)$

THEOREM 9. *Let $F \to E \xrightarrow{p} B$ be a fiber sequence of connected spaces with*

$$\begin{aligned} &LP\pi_1(E) \to \mathrm{Aut}\,\pi_q(F) \quad 1 < q \leqq n \\ &LP\pi_1(B) \to \mathrm{Out}\,\pi_1(F) \end{aligned}$$

trivial. Then $\Omega_q^{\mathrm{HS}}(E, F) \xrightarrow{p_*} \Omega_q^{\mathrm{HS}}(B, *)$ is an epimorphism for $q \leqq n + 1$. If in addition $\mathrm{LP}\pi_1(F) = 1$ *then $\Omega_q^{\mathrm{HS}}(E, F) \xrightarrow{p_*} \Omega_q^{\mathrm{HS}}(B, *)$ is an isomorphism for $q \leqq n$.*

After a short chapter displaying some basic facts about $\Omega_*^{\mathrm{HS}}(-)$ we give the proof of theorem 5 together with (1) of [4], along with an example of a fibration $F \to E \to B$ not having $\Omega_*^{\mathrm{HS}}(E, F) = \Omega_*^{\mathrm{HS}}(B, *)$. In the next chapter we give theorem 9 and a simple result about nilpotent fibrations followed by a more subtle example of $\Omega_*^{\mathrm{HS}}(E, F) \neq \Omega_*^{\mathrm{HS}}(B, *)$. A technical lemma used in theorem 9 is deffered until the last chapter.

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Basic results.

LEMMA 1. *If $f:(W, \Sigma) \to (X, *)$ represents an element of $\Omega_n^{\mathrm{HS}}(X, *)$ with $f_\#$: $\pi_1(W) \to \pi_1(X)$ trivial, then f is equivalent to a representative $(E^n, S^{n-1}) \to (X, *)$ via an H_*-cobordism*

$$(W \times I, \Sigma \times I \cup (W - D) \times \{1\}) \to (X, *)$$

where D is a small open disk imbedded in W.

PROOF. Let $V = \partial(W \times I) - D \times \{1\}$. Let $f' : V \to X$ be f on $W \times \{0\}$ and elsewhere constant. If we can extend f' to $W \times I$ we are done. Clearly, the only problem is with the 2-skeleton, but this problem is equivalent to the problem of finding a homomorphism $\theta: \pi_1(W \times I) \to \pi_1(X)$ extending $0 = f'_{\#}: \pi_1(V) \to \pi_1(X)$.

DEFINITION. We say that a group G is *locally perfect* if any element in G is contained in a finitely generated perfect subgroup ($H_1(-) = 0$). If G is a group, then the union of all locally perfect subgroups is again a locally perfect subgroups, naturally called the *maximal locally perfect subgroup*, $\mathrm{LP}(G)$. Dropping the finiteness condition we get the classical *maximal perfect subgroup*, denoted by $P(G)$.

The homomorphic image of a locally perfect group is locally perfect and $\mathrm{LP}(G/\mathrm{LP}G) = 1$. Some further properties are given in [7]. The fundamental groups of the representatives of the $\Omega_*^{\mathrm{HS}}(-)$ are locally perfect, hence we have:

COROLLARY 2. *Let X be a space with $\mathrm{LP}\pi_1(X) = 1$. Then $\pi_*(X) \to \Omega_*^{\mathrm{HS}}(X, *)$ is an isomorphism.*

PROOF. Surjectivity follows from the lemma. If $f: S^n \to X$ is extendable to $W \to X$, where W is an acyclic manifold with border S^n, then $\bar{H}_*(f) = 0$. As $\pi_1(W)$ is locally perfect, $W \to X$ lifts to the universal covering space of X, and by an application of the Whitehead theorem, f is nullhomotopic.

LEMMA 3. $\Omega_*^{\mathrm{HS}}(X) = \Omega_*^{\mathrm{HS}}(X, *) \oplus \Omega_*^{\mathrm{HS}}(*)$. $\Omega_q^{\mathrm{HS}}(*) = 0$ *if* $q \neq 3$.

PROOF. The last statement first: Let Σ be any PL homology $q \neq 3$ sphere. By Kervaire, [6], Σ bounds a contractible $q + 1$ PL manifold C. Cutting out a small open disk in the interior of C and choosing a base arc gives the desired cobordism to $S^q \to *$. Likewise $\Omega_4^{\mathrm{HS}}(X, *) \to \Omega_3^{\mathrm{HS}}(*)$ is trivial as any representative $(W^4, \Sigma^3) \to (X, *)$ yields an H_*-cobordism from $\Sigma \to *$ to $S^3 \to *$. Finally sending representatives $(W^3, S^2) \to (X, *)$ to $W/S^2 \to X$ clearly gives a section to $\Omega_3^{\mathrm{HS}}(X) \to \Omega_3^{\mathrm{HS}}(X, *)$.

LEMMA 4. *Let X be a space and let $\tilde{X}_N \to X$ be the covering of X associated with N, $\mathrm{LP}\,\pi_1(X) \subseteq N \lhd \pi_1(X)$. Then $\Omega_*^{\mathrm{HS}}(\tilde{X}_N) \to \Omega_*^{\mathrm{HS}}(X)$ is an isomorphism.*

PROOF. Let $\Sigma \to X$ be a representative of an element of $\Omega_*^{\mathrm{HS}}(X)$. $\mathrm{im}\,\{\pi_1(\Sigma) \to \pi_1(X)\} \subseteq \mathrm{LP}\pi_1(X) \subseteq N$, so we have a unique pointed lifting. If $\Sigma' \to X$ is another representative of the same element, let $W \to X$ be the H_*-cobordism between them. Again this may be lifted, and the base arc, being connected, ensures that the lifting restricted to Σ and Σ', equals the pointed liftings. This clearly defines an inverse to $\Omega_*^{\mathrm{HS}}(\tilde{X}_N) \to \Omega_*^{\mathrm{HS}}(X)$.

Fibrations annihilating the perfect radical.

It is now easy to show:

THEOREM 5. *Let* $F \to E \xrightarrow{p} B$ *be a fiber sequence of connected spaces with* $\mathrm{LP}\pi_1(E) \subseteq \ker\{p_\#: \pi_1(E) \to \pi_1(B)\}$. *Then* $\pi_*(B) \xleftarrow{\approx} \pi_*(E, F) \xrightarrow{\approx} \Omega_*^{\mathrm{HS}}(E, F)$

PROOF. We may assume that B is simply connected. To see this, take pullback

$$\begin{array}{ccccccc} F & \longrightarrow & \bar{E} & \longrightarrow & \tilde{B} & \longrightarrow & \mathrm{E}\pi_1(B) \\ \| & & \downarrow & & \lrcorner\downarrow & & \lrcorner\downarrow \\ F & \longrightarrow & E & \longrightarrow & B & \longrightarrow & \mathrm{B}\pi_1(B) \end{array} .$$

Now, $\pi_q(B) = \pi_q(\tilde{B})$ for $q \geqq 2$, and as $\mathrm{LP}\pi_1(E) \subseteq \ker\{\pi_1(E) \to \pi_1(B)\}$, $\mathrm{LP}\pi_1(E) \subseteq \pi_1(E)$ and so $\Omega_*^{\mathrm{HS}}(\bar{E}) \approx \Omega_*^{\mathrm{HS}}(E)$. Hence, for our purpose, we may substitute $F \to \bar{E} \to \tilde{B}$ for $F \to E \to B$. Assuming this done, injectivity is guaranteed by naturality:

$$\begin{array}{ccc} \pi_*(E, F) & \longrightarrow & \Omega_*^{\mathrm{HS}}(E, F) \\ \downarrow\approx & & \downarrow \\ \pi_*(B) & \xrightarrow{\approx} & \Omega_*^{\mathrm{HS}}(B, *) \end{array} .$$

The only thing we need to show is that $\pi_*(E, F) \to \Omega_*^{\mathrm{HS}}(E, F)$ is an epimorphism. Now, let $f: (W, \Sigma) \to (E, F)$ denote a representative of an element in $\Omega_n^{\mathrm{HS}}(E, F)$. By lemma 1 there is an H_*-cobordism $(W \times I, \Sigma \times I \cup (W - \mathrm{int}(E^n)) \times \{1\}) \to (B, *)$ from $p \circ f$ to a $\phi: (E^n, S^{n-1}) \to (B, *)$. But as p is a fibration this lifts, so f is H_*-cobordant to the lifting $\bar{\phi}: (E^n, S^{n-1}) \to (E, F)$, and f represents an element in the image of $\pi_n(E, F)$.

COROLLARY 6. *Let* $F \to E \xrightarrow{p} B$ *be a fiber sequence of connected spaces with* $\mathrm{LP}\pi_1(B) = 1$. *Then*

$$p_*: \Omega_*^{\mathrm{HS}}(E, F) \to \Omega_*^{\mathrm{HS}}(B, *)$$

$(= \pi_*(B) = \pi_*(E, F))$ *is an isomorphism.*

COROLLARY 7. *Let X be a connected space and N a normal perfect subgroup of* $\pi_1(X)$, *containing* $\mathrm{LP}\pi_1(X)$. *Let* X_N^+ *be plus with respect to N,* $\tilde{X}_N$ *the covering space associated with N, and A the acyclic functor of E. Dror. Then*

$$\Omega_*^{\mathrm{HS}}(X, A\tilde{X}_N) \to \pi_*(X_N^+)$$

is an isomorphism.

By this we have reached our first goal, that is, to show that corollary 7 is not a result of intricate nature. However, theorem 5 has a wider reach than that, among other things, it provides us with our first example.

EXAMPLE 8. From the above theorem we get an example of a fibration $F \to E \to B$ where $\Omega_*^{\mathrm{HS}}(E,F) \neq \Omega_*^{\mathrm{HS}}(B,*)$. Let

$$1 \to R \to F \to \mathrm{St}\mathbf{Z} \to 1$$

be a free presentation of $\mathrm{St}\mathbf{Z}$. As F is free, $1 = \mathrm{LP}(F) = \mathrm{LP}(R)$, and so $\Omega_*^{\mathrm{HS}}(\mathrm{B}F, \mathrm{B}R) = \pi_*(\mathrm{B}\,\mathrm{St}\mathbf{Z})$, that is, zero in the relevant dimensions. But, as we see from the last corollary, $\Omega_3^{\mathrm{HS}}(\mathrm{B}\,\mathrm{St}\mathbf{Z}, *)$ surjects onto $\pi_3(\mathrm{B}\,\mathrm{St}\mathbf{Z}^+) = H_3(\mathrm{St}\mathbf{Z}) \neq 0$, $(\Omega_2^{\mathrm{HS}}(\mathrm{A}\,\mathrm{B}\,\mathrm{St}\mathbf{Z}) = 0)$.

Fibrations with nice actions.

Given a group G, let $\mathrm{Aut}\,G$ be its group of automorphisms. The inner automorphisms given by conjugation by elements in G form a normal subgroup $\mathrm{Inn}\,G$, and we will denote the quotient $\mathrm{Aut}\,G/\mathrm{Inn}\,G$, the "outer automorphisms" by $\mathrm{Out}\,G$. Given a pointed space X let $\mathrm{aut}\,X$ denote the topological monoid of self equivalences of X. There is a map $\mathrm{aut}\,X \to X$ by sending $\omega \in \mathrm{aut}\,X$ to $\omega(*) \in X$. Call the fiber $\mathrm{aut}^0\,X$. This clearly is also a topological monoid, and the inclusion $\mathrm{aut}^0\,X \to \mathrm{aut}\,X$ is a monoid homomorphism. Taking classifying spaces we obtain by [3] section 4, the universal fibration

$$X \to \mathrm{B}\,\mathrm{aut}^0\,X \to \mathrm{B}\,\mathrm{aut}\,X$$

for Hurewicz fibrations with fiber X.

We are now ready for:

THEOREM 9. *Let $F \to E \xrightarrow{p} B$ be a fiber sequence of connected spaces with*

$$\mathrm{LP}\,\pi_1(E) \to \mathrm{Aut}\,\pi_q(F) \quad 1 < q \leqq n$$
$$\mathrm{LP}\,\pi_1(B) \to \mathrm{Out}\,\pi_1(F)$$

trivial. Then $\Omega_q^{\mathrm{HS}}(E,F) \xrightarrow{p_*} \Omega_q^{\mathrm{HS}}(B,*)$ is an epimorphism for $q \leqq n+1$. If in addition $\mathrm{LP}\,\pi_1(F) = 1$ then $\Omega_q^{\mathrm{HS}}(E,F) \xrightarrow{p_*} \Omega_q^{\mathrm{HS}}(B,*)$ is an isomorphism for $q \leqq n$.

PROOF. Let $(X,\Sigma) \to (B,*)$ be a representative of an element in $\Omega_n^{\mathrm{HS}}(B,*)$. Lift the base point of X to the basepoint of F. Apply lemma 12 to the universal fibration

$$\mathrm{B}\,\pi_1(F) \to \mathrm{B}\,\mathrm{aut}^0\,\mathrm{B}\,\pi_1(F) \to \mathrm{B}\,\mathrm{aut}\,\mathrm{B}\,\pi_1(F)$$

(see [3]) with $f'\colon X \to B \to \mathrm{B}\,\mathrm{aut}\,F \to \mathrm{B}\,\mathrm{aut}\,\mathrm{B}\pi_1(F)$ and $W = *$. Letting $Z\pi_1(F)$ be the center of $\pi_1(F)$ recall by [5], that the exact homotopy sequence of the universal fibration with fiber $\mathrm{B}\pi_1(F)$ is the obvious

$$0 \to Z\pi_1(F) \to \pi_1(F) \to \mathrm{Aut}\,\pi_1(F) \to \mathrm{Out}\,\pi_1(F) \to 1.$$

As X is acyclic and $\pi_1(W) = 1$, the conditions of the lemma is met, and we may lift the 2-skeleton of X to the first Postnikov stage $B^1 = B\,\Pi_{\mathrm{Baut}B\pi_1(F)}\,\mathrm{B\,aut}^0\,\mathrm{B}\pi_1(F)$. But as $\tilde{F} = \mathrm{fiber}\,(E \to B^1)$ is simply connected, there is no obstruction to lifting the 2-skeleton of X to E. This lifting induces an action of $\pi_1(X)$ on $\pi_q(\tilde{F})$, but as it factors through $\mathrm{LP}\pi_1(E)$ it is trivial. As X is acyclic, $H^{q+1}(X, *; \pi_q(F)) = 0$ for $1 < q \leqq n$, and we may lift all of X to E. Hence surjectivity is clear.

Now, assume $\mathrm{LP}\pi_1(F) = 1$. Let $(W_1, \Sigma_1) \xrightarrow{f_1} (E, F) \xleftarrow{f_1} (W_2, \Sigma_2)$ be represen-Ptatives of elements in $\Omega_n^{\mathrm{HS}}(E, F)$ being mapped to the same element of $\Omega_n^{\mathrm{HS}}(B, *)$. Let $(X, U) \to (B, *)$ be the cobordism between $p \circ f_1$ and $p \circ f_2$. As all maps are pointed, we may lift the base arc of X to $* \in F$. Let W be the union of W_1, W_2 and the base arc. We are again in the situation of lemma 12. As $\mathrm{LP}\pi_1(F) = 1$, and $\pi_1(W)$ is locally perfect, any $\pi_1(W) \to \pi_1(F)$ is trivial, and so the zero section will do again. The action of $\pi_1(X)$ on $\pi_q(\tilde{F})$ is trivial as above, and so $H^{q+1}(X, W; \pi_q(F)) = 0$, and there is no obstruction to lifting the cobordism in accordance with f_1 and f_2. Thus f_1 and f_2 represented the same element in $\Omega_n^{\mathrm{HS}}(E, F)$, and p_* is in injective as well.

COROLLARY 10. *Let* $F \to E \xrightarrow{p} B$ *be a nilpotent fibration. Then* $\Omega_*^{\mathrm{HS}}(E, F) \xrightarrow{p_*} \Omega_*^{\mathrm{HS}}(B, *)$ *is an isomorphism.*

PROOF. For any action of a perfect group on a nilpotent group, one sees by induction that nilpotence is equivalent to triviality. Now, p nilpotent implies that F, and in particular $\pi_1(F)$, is nilpotent. Thus $\mathrm{LP}\,\pi_1(F) \subseteq P\,\pi_1(F) = 1$, and futhermore the action of $P\,\pi_1(E)$ on $\pi_1(F)$ is trivial. But, as p is nilpotent, $P\pi_1(B) = p_\#(P\pi_1(E))$ by [2], and so $\mathrm{LP}\,\pi_1(B) \subseteq P\,\pi_1(B) \to \mathrm{Out}(\pi_1(F))$ is trivial.

The major problem with theorem 9 is the condition on the actions of $\mathrm{LP}\,\pi_1(E)$ which may seem restrictive. One would be glad to dispense with the condition altogether. However, this is impossible. Indeed, a natural (and weaker) question is whether the problem is restricted to the low degrees, that is, given a fibration behaving nice in low degrees (e.g. preserving perfect radicals and inducing nice actions on the first few $\pi_q(F)$), may we hope that this will automatically prevail in some higher degrees? That no such theorem exists is demonstrated by the next example:

EXAMPLE 11. A fibration with abitrary given connectivity, but with $\Omega_*^{\mathrm{HS}}(E, F) \neq \Omega_*^{\mathrm{HS}}(B, *)$. Let E be a space with $1 \neq \xi \in \pi_1(E)$. Let $V = E \vee S^n$ and let $k: S^n \to V$ represent $(1, 2 - \xi) \in \pi_n(V) = \pi_n(E) \oplus \mathbf{Z}[\pi_1(E)]$, and form pushout

$$\begin{array}{ccc} S^n & \xrightarrow{k} & V \\ \cap & & \cap \\ E^{n+1} & \longrightarrow & B \end{array}$$

The Hurewicz homomorphism $\rho: \pi_*(V) \to H_*(V) = H_*(E) \oplus H_*(S^n)$ maps $(1, 2 - \xi) \mapsto (0, 1)$, and so the composition $f: E \subset V \subset B$ induces isomorphism in homology. Thus, if $\pi_1(B)$ is locally perfect, $0 = \Omega_*^{HS}(B, E)$ by corollary 7 (at least for high dimensions, or better, if $n \geqq 3$ in all dimensions from [4, Theorem 4.6]). On the other hand we get that

$$f_\# : \pi_q(E) \xrightarrow{\approx} \pi_q(B), \quad q < n,$$

and

$$f_\# : \pi_n(E) \xrightarrow{\mathrm{id} \oplus 0} \pi_n(E) \oplus \mathbf{Z}[\pi_1(E)]/(2 - \xi)\cdot \mathbf{Z}[\pi_1(E)] \approx \pi_n(B).$$

Let $F = \mathrm{fiber}(f)$. $\pi_q(F) = 0$ for $q < n - 1$, so $\pi_{n-1}(F) = \Omega_{n-1}^{HS}(F, *)$. All in all we get that $\Omega_n^{HS}(B, E) = 0 \neq \mathbf{Z}[\pi_1(E)]/(2 - \xi)\cdot\mathbf{Z}[\pi_1(E)] = \Omega_{n-1}^{HS}(F, *)$, and so

$$\Omega_*^{HS}(E, F) \neq \Omega_*^{HS}(B, *).$$

Lemma 12.

We end with a lemma that has already been used. A retracking of theorem 9 will show that due to particular circumstances this lemma, as stated, is somewhat an overkill. However, as 2-skeletons are always troublesome the lemma has some independent interest, and we will list the entire result.

LEMMA 12. *Let $F \to E \to B$ be a fiber sequence of connected spaces, and let $f' : X \to B$ be a map such that*
(1) $\mathrm{Ext}(H_1(X), \pi_2 B) = 0$,
(2) $\pi_1 f' = 1 : \pi_1(X) \to \pi_1(B)$,
(3) $H_2 f' = 0 : H_2(X) \to H_2(B, \mathbf{Z}[\pi_1(B)]) \quad (= \pi_2(B))$.
Let $P = X \Pi_B E$. Then $1 \to \pi_1(F) \xrightarrow{i_\#} \pi_1(P) \xrightarrow{\bar{p}_\#} \pi_1(X) \to 1$ is a split exact sequence with a splitting $\sigma : \pi_1(X) \to C_{\pi_1(P)}(\pi_1(F))$, and hence we have an isomorphism $\pi_1(X) \times \pi_1(F) \xrightarrow{\sigma \cdot i_\#} \pi_1(P)$. Let $W \subset X$ have a partial lifting $\bar{f}' : W \to E$, and let $\psi : W \to P$ be the induced map. Then we may lift the 2-skeleton of X relative to $\bar{f}'$ if and only if there exists a homomorphism $\theta : \pi_1(X) \to \pi_1(F)$ extending

$$\begin{array}{l} \pi_1(W) \xrightarrow{\psi_\#} \pi_1(P) \xrightarrow{\approx} \pi_1(X) \times \pi_1(F) \xrightarrow{\mathrm{pr}_2} \pi_1(F) \\ \downarrow \mathrm{incl}_\# \\ \pi_1(X) \end{array}$$

PROOF. By (2) f' lifts to the universal covering space of B, $f : X \to \tilde{B}$. Let $\Phi = \mathrm{Fiber}(f)$. As X is connected and $\tilde{B}$ is simply connected, Φ is connected, and by (3), the Hurewicz theorem and the lower term sequence of the Leray-Serre spectral sequence, we get

$$\begin{array}{ccccccccc} \cdots \longrightarrow & \pi_2(X) & \xrightarrow{f_\#} & \pi_2(\tilde{B}) & \xrightarrow{\partial} & \pi_1(\Phi) & \longrightarrow & \pi_1(X) & \longrightarrow 1 \\ & \downarrow & & \approx\downarrow & & \downarrow & & \downarrow & \\ \cdots \longrightarrow & H_2(X) & \xrightarrow{f_* = 0} & H_2(\tilde{B}) & \longrightarrow & H_1(\Phi) & \longrightarrow & H_1(X) & \longrightarrow 1 \end{array} .$$

As $\operatorname{Ext}(H_1(X), H_2(\tilde{B})) = 0$ the lower sequence is a split exact sequence of abelian groups. Furthermore $f_\# = 0 : \pi_2(X) \to \pi_2(\tilde{B})$, and as $\pi_2(\tilde{B}) \to H_2(\tilde{B})$ is an isomorphism and ∂ central, the upper sequence inherits the splitting, so $1 \to \pi_2(\tilde{B}) \to \pi_1(\Phi) \to \pi_1(X) \to 1$ is a split central extension, with splitting, say σ'. Perform pullback

$$\begin{array}{ccccccc}
\Phi & \xrightarrow{\bar{\jmath}} & P & \xrightarrow{\bar{f}} & \bar{E} & \longrightarrow & E \\
\| & & \big\downarrow{\scriptstyle \bar{p}} & \lrcorner & \big\downarrow{\scriptstyle \bar{p}} & \lrcorner & \big\downarrow \\
\Phi & \xrightarrow{j} & X & \xrightarrow{f} & \tilde{B} & \longrightarrow & B
\end{array}.$$

As $\{\pi_2(X) \xrightarrow{\partial} \pi_1(F)\} = \{\pi_2(X) \xrightarrow{f_\#} \pi_2(\tilde{B}) \xrightarrow{\partial} \pi_1(F)\} = 1$,

$$1 \to \pi_1(F) \xrightarrow{\bar{\imath}_\#} \pi_1(P) \xrightarrow{\bar{p}_\#} \pi_1(X) \to 1$$

is exact. Let $\sigma : \pi_1(X) \to \pi_1(P)$ be the composition $\pi_1(X) \xrightarrow{\sigma'} \pi_1(\Phi) \xrightarrow{\bar{\jmath}_\#} \pi_1(P)$.

$$\begin{array}{ccccccc}
 & & \pi_1(F) & = & \pi_1(F) & \to & \operatorname{Inn}\pi_1(F) \\
 & & \big\downarrow{\scriptstyle \bar{\imath}_\#} & & \big\downarrow{\scriptstyle i_\#} & & \big\downarrow \\
\pi_1(\Phi) & \xrightarrow{\bar{\jmath}_\#} & \pi_1(P) & \xrightarrow{\bar{f}_\#} & \pi_1(\bar{E}) & \xrightarrow{\tau} & \operatorname{Aut}\pi_1(F) \\
\| & & {\scriptstyle \sigma}\big\uparrow\big\downarrow{\scriptstyle \bar{p}_\#} & & \big\downarrow & & \big\downarrow \\
\pi_1(\Phi) & \underset{j_\#}{\overset{\sigma'}{\leftrightarrows}} & \pi_1(X) & \longrightarrow & 1 & \longrightarrow & \operatorname{Out}\pi_1(F)
\end{array}$$

This is obviously a section to $\bar{p}_\#$, and as

$$\pi_1(P) \to \operatorname{Aut}\pi_1(F), \quad \alpha \mapsto h_\alpha = (\phi \mapsto \bar{\imath}_\#^{-1}(\alpha \cdot \bar{\imath}_\#(\phi) \cdot \alpha^{-1}))$$

factors through $\pi_1(P) \xrightarrow{\bar{f}_\#} \pi_1(E) \xrightarrow{\tau} \operatorname{Aut}\pi_1(F)$, we get that $h_{\sigma(\omega)} = \tau_{\bar{f}_\#\sigma(\omega)} = \tau_{\bar{f}_\#\bar{\jmath}\sigma'(\omega)} = \tau_1 = 1$, and so the image of σ lies in $C_{\pi_1(P)}(\pi_1(F))$, the centralizer of $\pi_1(F)$ in $\pi_1(P)$.

This last point ensures that $\pi_1(X) \times \pi_1(F) \xrightarrow{\sigma \cdot \bar{\imath}_\#} \pi_1(P)$ is an isomorphism with inverse

$$\pi_1(P) \xrightarrow{\alpha \mapsto (\bar{p}_\#(\alpha), \bar{\imath}_\#^{-1}(\alpha \cdot \bar{p}_\#(\alpha^{-1})))} \pi_1(X) \times \pi_1(F).$$

Now, if $(X, W) \to (B, E)$ is as in the lemma, we have by the criterion for extendability of 2-skeletons [1], that the lemma is true if both $\bar{\imath}_\# : \pi_1(F) \to \pi_1(P)$ is a monomorphism, and there is a section θ' to $\bar{p}_\# : \pi_1(X)$ with $\psi_\# = \theta' \circ \operatorname{incl}_\# : \pi_1(W) \to \pi_1(X) \to \pi_1(P)$. But as $\psi_\#$ must be of the form $(\operatorname{incl}_\#, \gamma)$: $\pi_1(W) \to \pi_1(W) \times \pi_1(F)$, this is equivalent to γ factoring through $\operatorname{incl}_\#$.

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