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REGULARLY RELATED LATTICES IN LOCALLY COMPACT GROUPS

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Abstract.

Let G be a second countable locally compact topological group. Let H and K be lattices in G. Suppose there is a measurable subset of G/H of strictly positive measure which meets each K-orbit in at most one point. We describe the structure of the group G.

Marce A. Rieffel proves the following theorem ([1]).

THEOREM 1. *Let G be a connected Lie group, and let H and K be lattices in G. Suppose that any one of the following conditions holds:*

1. *The set of points of G/H whose K-orbits are open in their closure,* i.e., *relatively discrete, (which is a measurable set) has strictly positive measure.*
2. *There is a K-invariant measurable subset, F, of strictly positive measure in G/H such that the quotient Borel structure in K/F is countably separated.*
3. *There is a measurable subset of G/H of strictly positive measure which meets each K-orbit in at most one point.*

Then $G \cong \mathsf{R}^n \times C$ for some compact group C. Furthermore, $H \cap K$ contains a subgroup which is central in G and of finite index in both H and K. In particular, every K-orbit in G/H is finite, (as will be every H-orbit in G/K).

The purpose of the present note is to give a version of above theorem when G is a second countable locally compact group. (This is a question raised in [1]). Precisely, we show the following condition holds.

THEOREM 2. *Let G be a second countable locally compact topological group. Let H and K be lattices with countable elements in G. Assume K is finitely generated. Suppose that any of the following conditions holds.*

1. *The set of points of G/H whose K-orbits are open in their closure,* i.e., *relatively discrete, (which is measurable set) has strictly positive measure.*

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2. *There is a K-invariant measurable subset, F, of strictly positive measure in G/H such that the quotient Borel structure in K/F is countably separated.*

3. *There is a measurable subset F of G/H of strictly positive measure which meets each K-orbit in at most one point.*

Then there exists a subgroup K_1 of K with finite index such that the centralizer $Z_G(K_1)$ of K_1 in G is an open subgroup of G. Let E be an almost connected open subgroup of $Z_G(K_1)$. Let N be the maximal compact normal subgroup of E. Then E/N is isomorphic with $\mathsf{R}^m \times \mathsf{Z}^n$ i.e. *it is a compactly generated locally compact abelian group without non-trivial compact elements.*

We are very grateful to professor S. P. Wang who provides us a proof which greatly strengthens our original theorem. Previously, we had to assume that the family of subgroups $\{gKg^{-1} \cap H : g \in G\}$ is countable.

Basic references of present work and related topics, see [1, 2, 3]. The reference [3] covers a very broad ground in the area of locally compact groups.

We start with some general observations. Notice that every lattice in an analytic group is finitely generated (so countable). Therefore, it is reasonable to impose this condition in our theorem. Also, it was proved in [1] that condition (3) in Rieffel theorem is a consequence of either conditions (1) or (2) where G is only assumed to be a second countable locally compact group with H and K countable lattices. Therefore we shall prove Theorem 1 under the condition (3).

Recall a locally compact group G is *almost connected* if G/G_0 is compact where G_0 is the identity component of G. When G is almost connected locally compact group, it is a pro-Lie group, i.e. G is inverse limit of Lie groups with fnitely many components, $G = \varprojlim G/M_\lambda$ where M_λ is compact normal subgroup of G. By means of standard arguments on inverse limit, one can generalize Rieffel's theorem easily to this case so our study should be beyond the almost connected locally compact group. Now, any open subgroup G' of a unimodular group G is unimodular. Let G' be an open almost connected subgroup of the locally compact group G satisfying one of the conditions of Rieffel's theorem. Clearly, $H \cap G'$ and $K \cap G'$ are lattices in G'. And one may obtain results from the triple $(G', H \cap G', K \cap G')$. However, from the global point of view, this is unsatisfactory, because we should be able to choose G' which is an almost "normal" subgroup of G, in the sense there exists a subgroup G'' of finite index in G such that $G' \subset G'' \subset G$ and G' is a normal subgroup of G''. But such choice is not always possible as following example shows.

EXAMPLE. Let A_i be any discrete group and B_i be a finite group of automorphism of A_i; $i = 1, 2, \ldots$. Let $B = \prod_{i=1}^{\infty} B_i$, direct produce with product topology. So B is a compact group. Let $A = \sum_{i=1}^{\infty} A_i$, weak direct product with discrete topology. Let $G = B \cdot A$ semi-direct product. Then G is a locally compact group.

Let $H = K = A$. Then (G, H, K) satisfies the conditions of Rieffel theorem. In particular, observe that K acts on G/H trivially. Since G is totally disconnected, almost connected open subgroups of G are compact-open subgroups. In this case, there is no such subgroups which is "almost" normal in G.

Now, we start the proof of the Theorem

Assume that F is a measurable subset of G/H of positive measure such that

(1) $$F \text{ meets each } K\text{-orbit in at most one point.}$$

Since G/H is of finite volume, there exist $k_1, \ldots, K_l \in K$ such that

(2) $$\operatorname{vol}\left(\bigcup_{i=1}^{l} k_i F\right) > \operatorname{vol}(kF) - \operatorname{vol}(F).$$

First we show that for any $k \in K$, there exist k_i and open subgroup M, depending on k, such that

(3) $$k \in k_i Z_K(M).$$

From (2), $\operatorname{vol}((\bigcup_{i=1}^{l} k_i F) \cap kF) > 0$ and so for some i,

(4) $$\operatorname{vol}(k_i^{-1} kF \cap F) > 0.$$

From (1) and (4),

$$\operatorname{vol}\{g \in G \mid g^{-1} k_i^{-1} k g \in H\} > 0.$$

Note that H is countable. The above condition implies that $Z_G(k_i^{-1} k)$ is open. Hence our assertion follows. Let V_n be a decreasing sequence of neighborhoods of the identity e such that $\bigcap V_n = \{e\}$. It follows that for any $k \in K$, there exists n such that

$$k \in \bigcup_{i=1}^{l} k_i Z_K(V_n).$$

Consider the group $K_1 = \bigcup_{n=1}^{\infty} Z_K(V_n)$. Then

$$K = \bigcup_{i=1}^{j} k_i K_1,$$

and $[K : K_1] \leqq l$. Since K is finitely generated, so is K_1. Hence

$$K_1 = Z_K(V_n)$$

for some n and $Z_G(K_1)$, containing V_n, is open. This shows that there exists a subgroup K_1 of K of finite index such that $Z_G(K_1)$ is open.

Since K_1 has finite index in K, K_1 is a lattice of G. Let E be an almost connected open subgroup of $Z_G(K_1)$. Then $K_1 \cap E$ is a lattice of E. Because $E/K_1 \cap E$ is a locally compact group with finite-invariant measure, $E/K_1 \cap E$ is a compact group. So $K_1 \cap E$ is a uniform lattice of E. Recall that $K_1 \cap E$ is central in E. Let N be the maximal compact normal subgroup of E. Then E/N is isomorphic with $\mathbf{R}^m \times \mathbf{Z}^n$, i.e. it is a compactly generated locally compact abelian group without non-trivial compact elements (cf. [4]). Now, the proof is complete.

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