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COMPLETE DUALS OF $C^*(X)$

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Abstract.

This is a study of several spaces of continuous linear functionals on various function spaces with a natural norm inherited from a larger Banach space. The completeness and lattice structure of these dual spaces have been studied. Since these duals are inherently related to spaces of measures, their measure-theoretic counterparts are also studied.

1. Introduction.

Let $C(X)$ denote the set of all continuous real-valued functions on a completely regular Hausdorff space X. Let $C_k(X)$ denote $C(X)$ with the compact-open topology k. It is a locally convex space. We note $C^*(X)$, the collection of all bounded functions in $C(X)$, is dense in $C_k(X)$. Let $\Lambda_k(X)$ be the set of all continuous linear functionals over $C_k(X)$. Since $C^*(X)$ is dense in $C_k(X)$, $\Lambda_k(X)$ can also be considered as the set of all continuous linear functionals over $C_k^*(X)$.

The supremum norm on $C^*(X)$ generates a Banach space denoted by $C_\infty^*(X)$ which has topology finer than the compact-open topology. Let $\Lambda_\infty(X)$ be the conjugate space of $C_\infty^*(X)$, that is, the Banach space of continuous linear functionals on $C_\infty^*(X)$ with the norm given by $\|\lambda\|_* = \operatorname{Sup}\{|\lambda(f)| : f \in C^*(X)$ and $\|f\|_\infty \leqq 1\}$ where $\lambda \in \Lambda_\infty(X)$ and $\|f\|_\infty = \operatorname{Sup}\{|f(x)| : x \in X\}$. As shown in [KMO], the natural map $L: \Lambda_k(X) \to \Lambda_\infty(X)$ is a linear injection where L is defined by $L(\lambda) = \lambda \circ j \circ i$ and where $\lambda \in \Lambda_k(X)$, $j: C_k^*(X) \to C_k(X)$ is the inclusion map and $i: C_\infty^*(X) \to C_k^*(X)$ is the identity map. Thus we may consider $\Lambda_k(X)$ as a linear subspace of the Banach space $\Lambda_\infty(X)$. Under this identification, $\Lambda_k(X)$ is a normed linear space with the norm given by $\|\lambda\|_* = \|L(\lambda)\|_*$.

An element $\lambda \in \Lambda_k(X)$ is positive provided that $\lambda(f) \geqq 0$ for all $f \in C(X)$ with $f \geqq 0$. Let $\Lambda_k^+(X) = \{\lambda \in \Lambda_k(X) : \lambda$ is positive$\}$ and it is called the positive cone of $\Lambda_k(X)$. $\Lambda_k^+(X)$ is a metric space with metric d_* defined by $d_*(\lambda, \mu) = \|\lambda - \mu\|_*$ where $\lambda, \mu \in \Lambda_k^+(X)$.

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In [KMO], it has been shown that $\Lambda_k(X)$ is a Banach space if and only if $\Lambda_k^+(X)$ is complete with respect to the metric d_*. In that paper the following two theorems on the completeness of $\Lambda_k^+(X)$ are given.

(KMO1) If $\Lambda_k^+(X)$ is complete, then the closure of each countable subset of X is compact.

(KMO2) If X is sham compact, then $\Lambda_k^+(X)$ is complete. (A space X is called sham compact if every σ-compact subset of X has compact closure.)

In view of the above results, the following questions have been raised in [KMO]

QUESTION 1.1. Is the converse of either (KMO1) or (KMO2) true?

QUESTION 1.2. Is there any single condition which is necessary as well as sufficient for the completeness of $\Lambda_k^+(X)$?

After the publication of [KMO], there have been at least three more works to answer the above questions, but their approaches and directions are different. In [Ku], Kundu has pointed out that it is essentially a problem of finding a suitable topology on $C^*(X)$. Here we note that a necessary condition for the completeness of $\Lambda_k^+(X)$ is that $C(X) = C^*(X)$. So from now onwards, we consider only linear functionals over $C^*(X)$. In [O], Okuyama has discussed two Banach subspaces $\Lambda_w(X)$ and $\Lambda_\sigma(X)$ of $\Lambda_\infty(X)$ containing $\Lambda_k(X)$. In [MT], a complete answer to the Question 1.2 has been found. As for the Question 1.1, it has been shown in [MT] that the converse of (KMO2) is not true while the converse of (KMO1) is not true if the continuum hypothesis is assumed. But in the above works, the completion of $\Lambda_k(X)$ in $\Lambda_\infty(X)$ as such has not been studied. In this paper, our first concern is to find this completion and to note that the necessary and sufficient condition given in [MT] is precisely the one when the equality $\Lambda_k(X) = \Lambda_w(X)$ holds. Also in this context, we answer an interesting question which has remained unanswered for a long time in negative. It has been shown that $\Lambda_k(X) = \Lambda_w(X)$ may hold while their corresponding topologies on $C^*(X)$ may be different. The second concern is to study in detail the space $\Lambda_\sigma(X)$ together with a new dual space $\Lambda_L(X)$ and discuss Gulick's conjecture made in [Gu]. To make our works self-contained, we divide it into several sections. In Section 2, we recall briefly the lattice structure of $\Lambda_\infty(X)$ and some basic concepts from measure theory. In Section 3, we study the space $\Lambda_w(X)$ as well as the completion of $\Lambda_k(X)$. In Section 4, we pursue our second concern.

Though our main interest lies in the completeness of the duals of $C^*(X)$, we would like to make occasional brief observations on their density. The density $d(X)$ of a space X is the smallest infinite cardinal number m such that X has a dense subset which has cardinality less than or equal to m. Now a space X is separable if and only if $d(X) = \aleph_0$. If X is a subspace of a metrizable space Y, then $d(X) \leqq d(Y)$.

Throughout the rest of this paper, we use the following conventions. All spaces are completely regular and Hausdorff. If X and Y are two topological spaces with the same underlying set, then we use $X = Y$, $X \leqq Y$, $X < Y$ to indicate, respectively that X and Y have the same topology, that the topology on Y is finer than or equal to the topology on X and that the topology on Y is strictly finer than the topology on X. The symbols βX, R and N denote respectively the Stone-Čech compactification of a space X, the space of real numbers and the space of natural numbers. A space is called almost σ-compact if it contains a dense σ-compact subset. Similarly an almost Lindelöf space is defined. As usual, $\bar{A}$ denotes the closure of A, but sometimes to make a distinction of spaces we use the notation $\mathrm{cl}_X A$, instead of $\bar{A}$, to mean the closure of A in X. Finally, the constant zero function defined on X is denoted by 0.

2. Basic concepts and properties.

We study the lattice structure of $\Lambda_\infty(X)$ beginning with a vector lattice. A vector lattice X is an ordered vector space which is also a lattice. The set $X^+ = \{x \in X : 0 \leqq x\}$ is called the positive cone of X and its members are called the positive elements of X. The positive part x^+, the negative part x^- and the absolute value $|x|$ of an element $x \in X$ are defined by

$$x^+ = \mathrm{Sup}\{x, 0\},\ x^- = \mathrm{Sup}\{-x, 0\} \text{ and } |x| = \mathrm{Sup}\{x, -x\}.$$

The remaining notions related to vector lattice/normed vector lattice are found in [AB].

It is easy to see that a normed vector lattice is complete, that is, a Banach lattice if and only if its positive cone is complete with respect to the metric induced by the norm. Also its density is equal to that of its positive cone. So a normed vector lattice is separable if and only if its positive cone is separable.

A linear functional λ on a vector lattice X is order bounded provided that for every $y \in X$ with $y \geqq 0$, there exists an $M > 0$ such that $|\lambda(x)| < M$ holds for all $x \in X$ with $|x| \leqq y$. Let $X^\sim$ be the set of all order bounded linear functionals on X and it is called the order dual of X. For each $\lambda, \mu \in X^\sim$ define $\lambda \leqq \mu$ provided that $\lambda(x) \leqq \mu(x)$ for all $x \in X$ with $x \geqq 0$. Then $X^\sim$ becomes a partially ordered vector space, which is in fact a vector lattice by the Riesz Theorem (see [AB]). Moreover, for each $\lambda \in X^\sim$ and $x \in X$ with $x \geqq 0$, $\lambda^+(x) = \mathrm{Sup}\{\lambda(y) : 0 \leqq y \leqq x\}$, $\lambda^-(x) = \mathrm{Sup}\{-\lambda(y) : 0 \leqq y \leqq x\}$ and $|\lambda|(x) = \mathrm{Sup}\{\lambda(y) : |y| \leqq x\}$. Extend λ^+, λ^- and $|\lambda|$ to all of X in the usual way. For example, define $\lambda^+(x) = \lambda^+(x^+) - \lambda^+(x^-)$ for any $x \in X$.

The function space $C^*(X)$ (or $C(X)$) is a vector lattice under the ordinary partial order defined by: $f \leqq g$ provided that $f(x) \leqq g(x)$ for all $x \in X$. Here $f^+(x) = \max\{f(x), 0\}$ and $f^-(x) = \max\{-f(x), 0\}$ for each $x \in X$.

It is clear that $C^*_\infty(X)$ is a Banach lattice and $\Lambda_\infty(X)$, being its norm dual is also a Banach lattice. It is easy to see that every $\lambda \in \Lambda_\infty(X)$ is order bounded. Now suppose τ is a topology on $C^*(X)$ weaker than the supremem topology and consequently any continuous linear functional λ over $C^*_\tau(X)$ belongs to $\Lambda_\infty(X)$. Hence $\lambda = \mu^+ - \lambda^-$. The question is whether λ^+ and λ^- are also continuous over $C^*_\tau(X)$. It is obviously true for $C^*_\infty(X)$ since any positive linear functional over it is continuous. It is also true in case of other duals of $C^*(X)$ which we consider in this paper. Here we explain it for $\Lambda_k(X)$ and other cases will be disucussed in Sections 3 and 4. But we need to introduce the following key idea. If $\lambda \in \Lambda_\infty(X)$ and if $A \subseteq X$, then λ is said to be supported on A provided that whenever $f \in C^*(X)$ is such that $f|_A = 0$, then necessarily $\lambda(f) = 0$. If $\lambda \in \Lambda_k(X)$, then λ has a minimal compact support which is called *the support* of λ. Also any positive linear functional over $C^*(X)$ having a compact support belongs to $\Lambda_k(X)$ (see [KMO]). This immediately gives the following result: if $\lambda \in \Lambda_\infty(X)$, then the following are equivalent (i) λ is k-continuous (ii) $|\lambda|$ is k-continuous (iii) λ^+ and λ^- are k-continuous.

From the above result it follows that $\Lambda_k(X)$ is an (order) ideal of the Banach lattice $\Lambda_\infty(X)$ and in particular $\Lambda_k(X)$ is a normed vector lattice. Here we also note that since every order bounded linear functional on $C^*_\infty(X)$ is continuous, $\Lambda_\infty(X)$ is also its order dual.

Now we recall some basic concepts from the measure theory.

The algebra generated by the closed subsets of X is denoted by $\mathscr{A}_c$ while the σ-algebra they generate is denoted by $\mathscr{B}$, called the Borel sets. For us, a finitely additive measure (also called finitely additive signed measure) on $\mathscr{A}_c$ is a *real-valued* finitely additive set function defined on $\mathscr{A}_c$. A finitely additive measure μ is called a *measure* if $\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$ holds for all pairwise disjoint sequences $(A_n)_{n=1}^{\infty}$ such that $A_n \in \mathscr{A}_c$ and $\bigcup_{n=1}^{\infty} A_n \in \mathscr{A}_c$.

When a measure μ is defined on $\mathscr{B}$, we call it a Borel measure. A measure μ defined on $\mathscr{B}$ (or on $\mathscr{A}_c$) has suppport A, where $A \subseteq X$ and $A \in \mathscr{B}$ (or $\in \mathscr{A}_c$ respectively) if $|\mu|(X - A) = 0$. A finitely additive measure μ defined on $\mathscr{A}_c$ or $\mathscr{B}$ is closed regular or simply regular whenever A is in the domain of the definition of μ and $\varepsilon > 0$, there are closed and open sets C and U in X such that $C \subseteq A \subseteq U$ and $|\mu|(U - C) < \varepsilon$. Such a closed regular measure is called compact regular or tight if the closed set C can be replaced by a compact subset of X.

A Borel measure has finite total variation. Also a finitely additive measure defined on $\mathscr{A}_c$, if bounded, has finite total variation.

Now we fix some notations.

Let $M_b(X)$ be the set of all closed regular Borel measures on X and $M_t(X) = \{\mu \in M_b(X) : \mu \text{ is compact-regular}\}$. Let $M_c(X)$ be the set of all bounded finitely additive closed regular measures defined on $\mathscr{A}_c$ and $M_{c,\sigma}(X) = \{\mu \in M_c(X) : \mu \text{ has a support } B \subseteq X \text{ where } B = \bar{A} \text{ and } A \text{ is } \sigma\text{-compact}\}$.

Let $M_b^+(X)$, $M_t^+(X)$, $M_c^+(X)$ and $M_{c,\sigma}^+(X)$ denote the corresponding positive cones of $M_b(X)$, $M_t(X)$, $M_c(X)$ and $M_{c,\sigma}(X)$ respectively. Also note while each of $M_b(X)$, $M_t(X)$, $M_c(X)$ and $M_{c,\sigma}(X)$ equipped with their corresponding total variation norm $\|\cdot\|$ is a normed linear space (rather a normed vector lattice), the positive cone of each of these is a metric space equipped with the metric induced by the total variation norm.

3. Tight Functionals over $C^*(X)$.

Let $B_0(X)$ be the set of all real-valued bounded functions on X vanishing at infinity. A functions vanishes at infinity if for any $\varepsilon > 0$, there exists a compact set $K \subseteq X$ such that $|f(x)| < \varepsilon$ for $x \notin K$. Corresponding to each function ϕ in $B_0(X)$, we define a seminorm $p_\phi(f) = \operatorname{Sup}\{|\phi(x)f(x)| : x \in X\}$. The collection of seminorms $\{p_\phi : \phi \in B_0(X)\}$ generates a locally convex topology on $C^*(X)$ and is called the strict toplogy. This topology is denoted by β and $C^*(X)$ equipped with β by $C^*_\beta(X)$. For details see [vR], [Gi], [Gu], and [S]. Here we note that Sentilles in [S] calls this topology substrict and denotes by β_0.

A real linear functional λ on $C^*(X)$ is said to be tight if for any net $f_\alpha \in C^*(X)$ with $1 \geqq \|f_\alpha\|_\infty$ such that $f_\alpha \to 0$ uniformly on compacta in X, one has $\lambda(f_\alpha) \to 0$. The tight functionals are precisely those which are continuous over $C^*_\beta(X)$. For the proof of this result, see [vR]. Also in [S], see Theorems 4.2 and 4.3 and Varadarajan's well-known result which says that for a $\lambda \in \Lambda_\infty(X)$, the conditions of its tightness are equivalent to the same for (1) $|\lambda|$ or (2) λ^+ and λ^-.

If $\lambda \in \Lambda_\infty(X)$, then it can be proved that its tightness is equivalent to: *for every $\varepsilon > 0$, there exists a compact subset K of X such that $|\lambda(f)| < \varepsilon$ whenever $\|f\|_\infty \leqq 1$ and $f|_K = 0$.* For one direction of the proof of this result, see [Gu] and the proof of other direction is easy and straightforward. From this result, it follows that Okuyama's $\Lambda_w(X)$ in [O] is precisely the collection of all β-continuous or tight functionals over $C^*(X)$. To be consistent with the notations of other duals, we replace $\Lambda_w(X)$ by $\Lambda_\beta(X)$. On $\Lambda_w(X) = \Lambda_\beta(X)$, Okuyama assigns the conjugate norm obtained from $\Lambda_\infty(X)$ as we do in case of $\Lambda_k(X)$. In [S], Sentilles shows that $\Lambda_\beta(X)$ is a Banach space. From the discussions of the previous paragraph, it follows that $\Lambda_\beta(X)$ is a Banach sublattice of the Banach lattice $\Lambda_\infty(X)$. In [S], Sentilles denotes $\Lambda_\beta(X)$ by M_t; but by $M_t(X)$ we mean it to be the collection of all tight Borel measures on X. The justification is given by the following theorem.

THEOREM 3.1. *For a space X, the Banach lattices $(M_t(X), \|\cdot\|)$ and $(\Lambda_\beta(X), \|\cdot\|_*)$ are isomorphic while $M_t^+(X)$ is identified with $\Lambda_\beta^+(X)$ under this isomorphism.*

PROOF. Define $F: M_t(X) \to \Lambda_\beta(X)$ by $F(\mu)(f) = \int f d\mu$ for each $\mu \in M_t(X)$ and $f \in C^*_\beta(X)$) Since μ is compact-regular, for given $\varepsilon >$ o, there exists a compact set K such that $|\mu|(X - K) < \varepsilon$. Now suppose $\|f\|_\infty \leqq 1$ and $f|_K = 0$. Then $|F(\mu)(f)| = |\int_{X-K} f d\mu| \leqq |\mu|(X - K) > \varepsilon$. So $F(\mu)$ is β-continuous.

Also $\|F(\mu)\|_* \leqq \operatorname{Sup}\{|\mu|(X)\|f\|_\infty : f \in C^*(X), \|f\|_\infty \leqq 1\} = |\mu|(X) = \|\mu\|$.

To prove the reverse inequality, that is, $\|\mu\| \leqq \|F(\mu)\|_*$, we note that since μ is compact regular, the technic applied in the proof of Theorem 4.3 in [Ku] can be employed here. Consequently, $\|\mu\| = \|F(\mu)\|_*$, that is F is an isometry.

Now we need to show that F is onto. It has been shown by Gulick in [Gu]. The proof of Gulick shows that given a positive linear functional, the corresponding measure in $M_t(X)$ is also positive. Our definition of F shows that a positive element in $M_t(X)$ is mapped to a positive linear functional. This also shows that F is a lattice isometry.

From this theorem, it immediately follows that (i) $\Lambda_\beta(X)$ is complete and (ii) $d(\Lambda_\beta(X) = d(M_t(X))$. So $\Lambda_\beta(X)$ is separable if and only if X is countable since $M_t(X)$ is separable if and only if X is countable.

Now we are going to show that the completion of $\Lambda_k(X)$ in $\Lambda_\infty(X)$ is precisely $\Lambda_\beta(X)$.

THEOREM 3.2. *For any space X, $\Lambda_\beta(X)$ is the completion of $\Lambda_k(X)$ in $\Lambda_\infty(X)$.*

PROOF. All we need to show that $\Lambda_k(X)$ is dense in $\Lambda_\beta(X)$. Let $\lambda \in \Lambda_\beta(X)$. Since every element of $\Lambda_\beta(X)$ can be decomposed in positive and negative parts, we can assume $\lambda \geqq 0$. So there exists a positive compact regular Borel measure μ on X such that $\lambda(f) = \int f d\mu$ for all $f \in C^*(X)$ (see [Gu]). Since μ is compact regular, there exist compact subsets K_n of X such that $K_m \subseteq K_n$ for $n \geqq m$ and $\mu\left(\bigcup_{n=1}^{\infty} K_n\right) = \mu(X)$. For every $\varepsilon > 0$, there exists a positive integer m such that $\mu(X - K_n) < \varepsilon$ for all $n \geqq m$. Now for each n define a positive linear functional λ_n on $C^*(X)$ as follows. Define $\lambda_n(f) = \int_{K_n} f d\mu$ for all $f \in C^*(X)$. Since λ_n is supported on K_n, $\lambda_n \in \Lambda_k^+(X)$. Now $|(\lambda - \lambda_n)(f)| = |\int_{X-K_n} |f| d|\mu| \leqq \|f\|_\infty |\mu|(X - K_n) = \|f\|_\infty \mu(X - K_n)$. So $\|\lambda - \lambda_n\|_* = \operatorname{Sup}\{|(\lambda - \lambda_n)(f)| : f \in C^*(X), \|f\|_\infty \leqq 1\} \leqq \mu(X - K_n) < \varepsilon$ for all $n \geqq m$.

COROLLARY 3.3. *For any space X, $\Lambda_k(X)$ is complete if and only if $\Lambda_k(X) = \Lambda_\beta(X)$.*

COROLLARY 3.4. *For any space X, $\Lambda_k(X)$ is separable if and only if $\Lambda_\beta(X)$ is separable.*

Here we note that in [KMO], it has been shown that $\Lambda_k(X)$ is separable if and only if X is countable. So $\Lambda_\beta(X)$ is separable if and only if X is countable – a result which has already been noted.

In [KMO] and [MT], the completeness of $\Lambda_k(X)$ has been studied. So the Corollary 3.3. is related to the studies done in these two papers. In [MT], there has been found a condition which is both necessary and sufficient for the completeness of $\Lambda_k(X)$. Note $\Lambda_k(X)$ is a Banach lattice if and only if its positive cone $\Lambda_k^+(X)$ (equipped with the metric induced by the norm $\|\cdot\|_*$) is complete. But to understand the aforesaid condition for completeness, we need to bring the support sets into focus. We define a subset of X to be a *support set* in X if it is the support of some λ in $\Lambda_k^+(X)$. Then we say that X is a *support space* if it is a support set in itself. It is straightforward to check that a subspace of X (with subspace topology) is a support space if and only if it is a support set in X. By definition, a support set is compact; but a compact set may not be a support set. Another way to characterise this concept is given by the result (see [J]): *A space X is a support space if and only if X is compact on which there exists a regular positive Borel measure which is strictly positive on each non-empty open subset of X.*

Kelley gives a purely topological condition in [Ke] which characterises those compact spaces having such a measure given in the above result. Using Kelley's characterisation, it has been shown in [MT] that every compact separable space is a support space and every support space is compact having the countable chain condition (i.e., every family of pairwise disjoint nonempty open subsets is countable). We abbreviate the countable chain condition by ccc.

Now the following condition which is both necessary and sufficient for the completeness of $\Lambda_k^+(X)$ can be found in [MT].

THEOREM 3.5. *The space $\Lambda_k^+(X)$ is complete if and only if every countable union of support sets in X has compact closure in X.*

Now in view of the Corollary 3.3 and Theorem 3.5, we concentrate on the following problem. $\Lambda_k(X)$ and $\Lambda_\beta(X)$ are the duals of $C_k^*(X)$ and $C_\beta^*(X)$ respectively. If $k = \beta$, obviously the duals are equal. Note $k = \beta$ if and only if X is sham compact (see [Gu]). Also this is precisely the sufficient condition given by Wheeler in [W] for the equality of these duals. But our question is in the reverse direction, that is, whether the equality of duals $\Lambda_k(X)$ and $\Lambda_\beta(X)$) implies that $k = \beta$. In [Gu], Gulick answers this kind of question for some other duals. Also see p-126 in [W]. But apparently neither Gulick nor Wheeler does have any answer for our question. Now we answer this question in negative by finding a counter-example. For this we need to talk abut P_c-point and P-point. A P_c-point is a point that is not in the closure of any countable union of compact ccc subsets not containing the point. A P-point is a point such that every G_δ-set containing it is a neighbourhood of it. Then in a compact space, a P-point is a point that is not in the closure of any σ-compact subset which does not contain the point.

COUNTER-EXAMPLE 3.6. Let $\mathsf{N}^* = \beta\mathsf{N} - \mathsf{N}$. Then there exists a P_c-point p in

$\mathbf{N}^*$ which is not a P-point. For details on these types of points see [Kn], [vM] and [MT]. Let $X = \mathbf{N}^* - \{p\}$. Since p is a P_c-point, the closure of each countable union of compact ccc subsets of X is compact. Since every support set is compact ccc, every countable union of support sets in X has compact closure in X. Consequently $\Lambda_k(X) = \Lambda_\beta(\mathcal{S})^{®}$ But since p is not a P-point, there exists a σ-compact subset of X with non-compact closure in X. Consequently $k \neq \beta$.

4. $\Lambda_\sigma(X)$ and $\Lambda_{\mathsf{L}}(X)$: The duals of $C_\sigma^*(X)$ and $C_{\mathsf{L}}^*(X)$.

Before talking about $\Lambda_\sigma(X)$, we would like to mention a few things about the σ-compact-open topology on $C^*(X)$. In [Gu], Gulick introduced this topology on $C^*(X)$ in terms of the convergence of nets. But we do it in terms of basis. In [KM], Kundu and McCoy have introduced two topologies on $C(X)$ namely the σ-compact-open topology and the topology of uniform convergence on σ-compact subsets of X. For the σ-compact-open topology, we take as subbase, the family $\{[A, B] : A \in \sigma(X), B \in \mathscr{BO}\}$ where $\sigma(X)$ is the collection of all σ-compact subsets of X, $\mathscr{BO}$ is the collection of all bounded open intervals in R and $[A, B] = \{f \in C(X) : \overline{f(A)} \subseteq B\}$. We denote this space by $C_\sigma(X)$. Note that the same topology is obtained by using $[\bar{A}, B]$ where $A \in \sigma(X)$ and $B \in \mathscr{BO}$. It can be shown that $C_\sigma(X)$ is a Tychonoff space.

For each $f \in C(X)$, $A \in \sigma(X)$ and $\varepsilon > 0$, let $\langle f, A, \varepsilon\rangle = \{g \in C(X) : |f(x) - g(x)| < \varepsilon$ for all $x \in A\}$. Then for each $f \in C(X)$, the collection $\{\langle f, A, \varepsilon\rangle : A \in \sigma(X), \varepsilon > 0\}$ forms a neighbourhood base at f for a topology on $C(X)$. This topology is called the topology of uniform convergence on σ-compact subsets of X and we denote this space by $C_{\sigma,u}(X)$. Again we note that $C_{\sigma,u}(X)$ is a Tychonoff space and as in case of $C_\sigma(X)$, one can replace A in $\langle f, A, \varepsilon\rangle$ by $\bar{A}$ where $A \in \sigma(X)$. In general, the σ-compact-open topology is weaker than or equal to the topology of uniform convergence on σ-compact subsets of X. But it has been shown in [KM] that on $C^*(X)$, these two topologies coincide and $C_\sigma(X) = C_{\sigma,u}(X)$ if and only if X is pseudocompact. Also $C_{\sigma,u}^*(X) = C_\infty^*(X)$ if and only if X is almost σ-compact. For more details on these topologies see [KM].

For each $A \in \sigma(X)$, define a seminorm p_A on $C^*(X)$ by $p_A(f) = \operatorname{Sup}\{|f(x)| : x \in A\}$. Then the σ-compact-open topology on $C^*(X)$ can be generated by the collection of seminorms $\{p_A : A \in \sigma(X)\}$. So $C_\sigma^*(X)$ is a locally convex space.

Now let $\Lambda_\sigma(X)$ be the collection of all continuous linear functionals over $C_\sigma^*(X)$. As before, on $\Lambda_\sigma(X)$ we assign the conjugate norm of $\Lambda_\infty(X)$, that is, for each $\lambda \in \Lambda_\sigma(X)$, let $\|\lambda\|_* = \operatorname{Sup}\{|\lambda(f)| : f \in C^*(X)$ and $\|f\|_\infty \leqq 1\}$. Then $\Lambda_\sigma(X)$ is a normed linear space and its positive cone is denoted by $\Lambda_\sigma^+(X)$. But $\Lambda_\sigma(X)$ is actually a Banach lattice. To show this, we need the following lemmas.

LEMMA 4.1. *For each $\lambda \in \Lambda_\sigma(X)$, there exists an $A \in \sigma(X)$ such that λ is supported* *on A. Conversely, if λ is a positive linear functional on $C^*(X)$ which is supported on some $A \in \sigma(X)$, then $\lambda \in \Lambda_\sigma^+(X)$.*

PROOF. If $\lambda \in \Lambda_\sigma(X)$, then since $\lambda : C_\sigma^*(X) \to \mathsf{R}$ is continuous at 0, there exist an $A \in \sigma(X)$ and a $\delta > 0$ such that $\lambda(\langle 0, A, \delta \rangle) \subseteq (-1, 1)$. Let $f \in C^*(X)$ with $f|_A = 0$. Now for every $\varepsilon > 0$, $\lambda\left(\frac{1}{\varepsilon} f\right) \in (-1, 1)$ which in turn implies $|\lambda(f)| < \varepsilon$.

For the converse, let λ be a positive linear functional on $C^*(X)$ which is supported on some $A \in \sigma(X)$. It suffices to check the continuity of λ at 0; so let $\varepsilon > 0$. Define $\delta = \dfrac{\varepsilon}{2\lambda(1) + 1}$ and let $f \in \langle 0, A, \delta \rangle$. Let $\hat{f}$ be the unique extension of f to βX. It is clear that $\hat{f}$ maps $\mathrm{cl}_{\beta X} A$ into $[-\delta, \delta]$. Consequently $\hat{f}|_{\mathrm{cl}_{\beta X} A}$ has an extension $\hat{g} \in C(\beta X)$ which maps into $[-\delta, \delta]$. Let $\hat{g}|_X = g$. Note on A, $g = f$, that is, $(g - f)|_A = 0$ and so $\lambda(g - f) = 0$ implying $\lambda(g) = \lambda(f)$. Therefore $|\lambda(f)| = |\lambda(g)| = |\lambda(g^+ - g^-)| \leqq |\lambda(g^+)| + |\lambda(g^-)| \leqq \lambda(g^+) + \lambda(g^-) \leqq \lambda(\delta) + \lambda(\delta) = 2\delta\lambda(1) < \varepsilon$.

LEMMA 4.2. *If $\lambda \in \Lambda_\infty(X)$, then the following are equivalent.*

i) $\lambda \in \Lambda_\sigma(X)$.
ii) *Both λ^+ and λ^- are in $\Lambda_\sigma(X)$.*
iii) $|\lambda| \in \Lambda_\sigma(X)$.

PROOF. (i) $\Rightarrow$ (ii). Since $\lambda \in \Lambda_\sigma(X)$, there exists an $A \in \sigma(X)$ such that λ is supported on A. Now for each $f \in C^*(X)$ with $f \geqq 0$; $\lambda^+(f) = \operatorname{Sup}\{\lambda(g) : 0 \leqq g \leqq f\}$. Clearly λ^+ is also supported on A and consequently by the previous lemma $\lambda^+ \in \Lambda_\sigma(X)$. The proof is similar for λ^-.

(ii) $\Rightarrow$ (iii) is obvious since $|\lambda| = \lambda^+ + \lambda^-$.

(iii) $\Rightarrow$ (i) For $f \in C^*(X)$ with $f \geqq 0$, $|\lambda|(f) = \operatorname{Sup}\{\lambda(g) : |g| \leqq f\}$. If $|\lambda|$ is supported on some $A \in \sigma(X)$, then both λ^+ and λ^- are also supported on A. Consequently both λ^+ and λ^- are continuous over $C_\sigma^*(X)$. Since $\lambda = \lambda^+ - \lambda^-$, λ is in $\Lambda_\sigma(X)$.

From the above lemmas, it is clear that $\Lambda_\sigma(X)$ as mentioned in [O] is same as the present one.

Now if $\lambda, \mu \in \Lambda_\infty(X)$, then $\lambda \vee \mu = \frac{1}{2}(\lambda + \mu + |\lambda - \mu|)$ and $\lambda \wedge \mu = \frac{1}{2}(\lambda + \mu - |\lambda - \mu|)$. So from the Lemma 4.2, it is clear that $\Lambda_\sigma(X)$ is a normed vector lattice and so $\Lambda_\sigma^+(X)$ is closed in $\Lambda_\sigma(X)$. Furthermore if $\lambda, \mu \in \Lambda_\sigma(X)$ then $\|\lambda^+ - \mu^+\|_* \leqq \|\lambda - \mu\|_*$ and $\|\lambda^- - \mu^-\|_* \leqq \|\lambda - \mu\|_*$. Consequently $(\Lambda_\sigma(X), \|\cdot\|_*)$ is complete if and only if the metric space $(\Lambda_\sigma^+(X), d_*)$ is complete where $d_*(\lambda, \mu) = \|\lambda - \mu\|_*$. Now suppose $\{\lambda_n\}$ is a Cauchy sequence in $\Lambda_\sigma^+(X)$ and each λ_n is supported on some $A_n \in \sigma(X)$. Since $\{\lambda_n\}$ is Cauchy in $\Lambda_\infty^+(X)$, there exists a $\lambda \in \Lambda_{\infty}^{+}(X)$ such that $\lambda_n \to \lambda$. It is easy to see that λ is supported on $\bigcup_{n=1}^{\infty} A_n$ which is in $\sigma(X)$. Consequently by Lemma 4.1, $\lambda \in \Lambda_{\sigma}^{+}(X)$ and so $\Lambda_{\sigma}^{+}(X)$ is complete. Thus finally we have established that $\Lambda_{\sigma}(X)$ is a Banach lattice.

$\Lambda_{\infty}(X)$ is separable if and only if X is countable, compact and metrizable. In fact, a countable compact space is always metrizable. If X is countable and compact, then $\Lambda_{\infty}(X) = \Lambda_k(X)$ is separable. Conversely, if $\Lambda_{\infty}(X)$ is separable, then $C_{\infty}^{*}(X)$ is separable which in turn implies that X is compact and metrizable (see [MN]). But the separability of $\Lambda_{\infty}(X)$ implies that of $\Lambda_k(X)$ and consequently X is countable. Now we have the following resut on the separability of $\Lambda_{\sigma}(X)$.

THEOREM 4.3. *For any space X, $\Lambda_{\sigma}(X)$ is separable if and only if X is compact and countable.*

PROOF. If $\Lambda_{\sigma}(X)$ is separable, then $\Lambda_k(X)$ is separable and so X is countable. But X being countable, $C_{\sigma}^{*}(X) = C_{\infty}^{*}(X)$ which means $\Lambda_{\infty}(X) = \Lambda_{\sigma}(X)$. But then, as argued previously, X is compact. If X is compact and countable, then $\Lambda_{\sigma}(X) = \Lambda_k(X)$ is separable.

Now we would like to concentrate on measure-theoretic counterpart of $\Lambda_{\sigma}(X)$. In [Gu], Gulick does it in terms of measures on βX. His precise result is as follows.

THEOREM 4.4. *For all X, $\Lambda_{\sigma}(X) = \cup M_t(\mathrm{cl}_{\beta X} A)$ where the union is taken over all σ-compact A (in X).*

In view of the above result, a natural question can be asked if it is possible to find a measure-theoretic counterpart of $\Lambda_{\sigma}(X)$ in terms of measures on X. The answer is yes if X is normal and if we do not insist on countable additivity of the measures. To have this, we need the following theorem which can be found either in [BNS] or in [Ku].

THEOREM 4.5. *If X is a normal Hausdorff space, then the Banach lattices $(M_c(X), \|\cdot\|)$ and $(\Lambda_{\infty}(X), \|\cdot\|_{*})$ are isomorphic while $M_c^{+}(X)$ is identified with $\Lambda_{\infty}^{+}(X)$ under this isomorphism.*

The isometry in the above theorem is given by the map $F: M_c(X) \to \Lambda_{\infty}(X)$ where $F(\mu)(f) = \int f d\mu$ for each $\mu \in M_c(X)$ and $f \in C^{*}(X)$. The fact that F is onto can be proved in the following way. Since each $\lambda \in \Lambda_{\infty}(X)$ can be decomposed into positive and negative parts, we can assume $\lambda \geqq 0$. Define a real-valued set function μ on the class of all subsets of X, that is, on $\mathscr{P}(X)$ as follows. If U is an open subset of X, we define $\mu(U) = \mathrm{Sup}\{\lambda(f): f \in C^{*}(X); 0 \leqq f \leqq \chi_U\}$ where χ_U is the characteristic function of U. If A is an arbitrary subset of X, we define

$\mu(A) = \mathrm{Inf}\{\mu(U): U$ is open in X and $A \subseteq U\}$. This set function when restricted to $\mathscr{A}_c$ is closed regular, finitely additive and $\lambda(f) = \int f d\mu$ for all $f \in C^*(X)$.

Now suppose $\mu \in M_{c,\sigma}(X)$ with a support $\bar{A}$ where $A \in \sigma(X)$. $|F(\mu)(f)| = |\int f d\mu| \leqq \int |f| d|\mu| \leqq |\mu|(\bar{A}) p_{\bar{A}}(f)$ and so $F(\mu)$ is in $\Lambda_\sigma(X)$. Conversely, let $\lambda \in \Lambda_\sigma^+(X)$ with a support A in $\sigma(X)$. Because of Theorem 4.5, we get a $\mu \in M_c^+(X)$ such that $\lambda(f) = \int f d\mu$ for $f \in C^*(X)$. Now $\mu(X - \bar{A}) = \mathrm{Sup}\{\lambda(f): f \in C^*(X); 0 \leqq f \leqq \chi_{X-\bar{A}}\} = 0$ since for $0 \leqq f \leqq \chi_{X-\bar{A}}$, $\lambda(f) = 0$. So $\mu \in M_{c,\sigma}^+(X)$. This establishes our desired result.

THEOREM 4.6. *Suppose X is a normal Hausdorff space. Then the Banach lattices $(M_{c,\sigma}(X), \|\cdot\|)$ and $(\Lambda_\sigma(X), \|\cdot\|_*)$ are isomorphic while $M_{c,\sigma}^+(X)$ is identified with $\Lambda_\sigma^+(X)$ under this isormorphism.*

It is clear that $\Lambda_\beta(X) \subseteq \Lambda_\sigma(X) \subseteq \Lambda_\infty(X)$. It is interesting to know when they are equal. In [Gu], Gulick shows that $\Lambda_\beta(X) = \Lambda_\sigma(X)$ if and only if $C_\beta^*(X) = C_\sigma^*(X)$. If $C_\sigma^*(X) = C_\infty^*(X)$ then clearly $\Lambda_\sigma(X) = \Lambda_\infty(X)$. But its converse is not necessarily true. In [Gu], Gulick shows that the converse is also true if X is locally compact and paracompact (Theorem 5.5 in [Gu]). By citing a counter-example, Gulick shows that if the local compactness is dropped, the converse may, no longer, hold. His counter-example is as follows.

COUNTER-EXAMPLE 4.7. Let $X =$ the ordinals less than or equal to the first uncountable ordinal ω_1 less the non-discrete ordinals. X is Lindelöf, normal and not locally compact. The compact subsets of X are finite. Since X is not almost σ-compact, $C_\sigma^*(X) < C_\infty^*(X)$. To show that $\Lambda_\sigma(X) = \Lambda_\infty(X)$ Gulick uses Theorem 4.4 and so he needs to move to βX. But since X is normal, without moving to βX we can prove it by using Theorem 4.6. Note that any subset of $X - \{\omega_1\}$ is closed if and only if it is countable. Since each μ in $M_c(X)$ is closed regular, there exists an F_σ-subset A of $X - \{\omega_1\}$ such that $|\mu|(X - \{\omega_1\}) = |\mu|(A)$. Note A is a closed countable set and so μ has a closed countable support. In particular, μ has a closed σ-compact support. Consequently $M_c(X) = M_{c,\sigma}(X)$ and hence $\Lambda_\sigma(X) = \Lambda_\infty(X)$.

It is interesting to know that Gulick's example is a particular case of a more general type of space. Let X be an uncountable space in which all points are isolated except for a distinguished point p, a neighbourhood of p being any set containing p whose complement is countable. X is a P-space which is Lindelöf and normal. X is not locally compact. In fact, a P-space is locally compact only when it is discrete. See 4N in [GJ] for the details on this space. Note even for this P-space, we can use Theorem 4.6 while we may not be able to use Gulick's result Theorem 4.4 because the Stone-Čech compactification of such a space may be quite difficult to tackle with.

But Gulick conjectures that the paracompactness may be dropped from his result, that is, he believes that for a locally compact space X, the statement $\Lambda_\sigma(X) = \Lambda_\infty(X)$ if and only if $C^*_\sigma(X) = C^*_\infty(X)$ is true. But we suspect otherwise, that is, his conjecture may be false. In view of Theorems 4.4 and 4.6 it will be so if we can have one of the following problems solved in affirmative.

PROBLEM 4.8. Find out a normal space X which is locally compact, not almost σ-compact, not paracompact and which satisfies the following condition: there exists a σ-compact subset A of X such that every closed subset of $X - \bar{A}$ is of the form $\bar{B}$ where B is σ-compact.

PROBLEM 4.9. Find out a Tychonoff space which is locally compact, not almost σ-compact, not paracompact and which satisfies the following condition: there exists a Borel subset A of βX such that $\beta X - A \subseteq X$ and there exists a σ-compact subset B of X such that $A \subseteq \mathrm{cl}_{\beta X} B$.

Even though we are yet to solve these problems, we can still improve a little on Gulick's result. In fact, we can obtain his result as a corollary to a more general result. For this we need to introduce a new subspace of $\Lambda_\infty(X)$. Let $\Lambda_{\mathrm{L}}(X) = \{\lambda \in \Lambda_\infty(X)\colon \lambda$ is supported on a Lindelöf subset of $X\}$ and $\Lambda^+_{\mathrm{L}}(X) = \{\lambda \in \Lambda_{\mathrm{L}}(X)\colon \lambda \geqq 0\}$. Again, as in case of $\Lambda_\sigma(X)$, it can be shown that $\Lambda_{\mathrm{L}}(X)$ is precisely the set of all continuous linear functionals over $C^*_{\mathrm{L}}(X)$ where $C^*_{\mathrm{L}}(X)$ is the function space equipped with the topology of uniform convergence on Lindelöf subsets of X. This topology can be generated by the collection of seminorms $\{p_{\mathrm{L}} : \mathscr{L}$ is a Lindelöf subset of $X\}$ where $p_{\mathrm{L}}(f) = \mathrm{Sup}\{|f(x)| : x \in L\}$ for $f \in C^*(X)$.

Let $\mathscr{L}(X)$ be the collection of all Lindelöf subsets of X. Then for each $f \in C^*(X)$, the collection $\{\langle f, L, \varepsilon\rangle\colon L \in \mathscr{L}(X), \varepsilon > 0\}$ forms a neighbourhood base at f for the topology of uniform convergence on Lindelöf subsets of X where $\langle f, \mathscr{L}, \varepsilon\rangle = [g \in C^*(X)\colon |f(x) - g(x)| < \varepsilon$ for all $x \in \mathscr{L}\}$. In view of this, we note that the Lemmas 4.1 and 4.2 remain valid if one replaces $\Lambda_\sigma(X)$ by $\Lambda_{\mathrm{L}}(X)$ and the σ-compact subsets by Lindelöf subsets. Since the union of a countable family of Lindelöf subsets of X is again Lindelöf, it follows that $\Lambda_{\mathrm{L}}(X)$ is also a Banach lattice.

It is clear that $C^*_\sigma(X) \leqq C^*_{\mathrm{L}}(X) \leqq C^*_\infty(X)$ and so $\Lambda_\sigma(X) \subseteq \Lambda_{\mathrm{L}}(X) \subseteq \Lambda_\infty(X)$. Note that $C^*_\sigma(X) = C^*_{\mathrm{L}}(X)$ if and only if given any Lindelöf subset of A of X, there exists a σ-compact subset B of X such that $A \subseteq \bar{B}$. On the other hand, $C^*_{\mathrm{L}}(X) = C^*_\infty(X)$ if and only if X is almost Lindelöf. If X is almost σ-compact, then $C^*_{\mathrm{L}}(X) = C^*_\sigma(X) = C^*_\infty(\mathrm{X})$ and consequently $\Lambda_{\mathrm{L}}(X) = \Lambda_\sigma(X) = \Lambda_\infty(X)$. Again if X is paracompact and locally compact, then $C^*_{\mathrm{L}}(X) = C^*_\sigma(X)$. This follows from the following fact. If A is a Lindelöf subset of a locally compact paracompact space, then $\bar{A}$ is σ-compact. See p-382 in [E]. So for a locally compact paracompact space, $\Lambda_{\mathrm{L}}(X) = \Lambda_\sigma(X)$. But the local compactness is not necessary in order to have $\Lambda_\sigma(X) = \Lambda_{\mathrm{L}}(X)$. The Counter-example 4.7 may be used here again to note that $\Lambda_\sigma(X) = \Lambda_{\mathrm{L}}(X) = \Lambda_\infty(X)$. But this counter-example also shows that we may have $\Lambda_\sigma(X) = \Lambda_{\mathrm{L}}(X)$ without having $C^*_\sigma(X) = C^*_{\mathrm{L}}(X)$. Note for this example $C^*_\sigma(X) < C^*_{\mathrm{L}}(X) = C^*_\infty(X)$. These facts motivate us to study $\Lambda_{\mathrm{L}}(X)$ in a little more detail. Let X be a non-compact space and $p \in \beta X - X$. For each $f \in C^*(X)$, let $\lambda_p(f) = \hat{f}(p)$ where $\hat{f}$ is the unique continuous extension of f to βX. Then $\lambda_p \in \Lambda_\infty(X)$. Let $\Lambda_0(X) = \{\lambda_p \in \Lambda_\infty(X) : p \in \beta X - X\}$. Now we have the following results.

THEOREM 4.10. *Let X be a paracompact Hausdorff space. If X is not Lindelöf then there exists a $\lambda \in \Lambda_0(X) - \Lambda_{\mathrm{L}}(X)$.*

PROOF. By the assumption, there exists a discrete collection $\{Z_\alpha : \alpha < \omega_1\}$ of zero-sets in X. Put $\mathscr{A} = \{A \subseteq \omega_1$: cardinality of $\omega_1 - A \leqq \omega_0\}$ and for each $A \in \mathscr{A}$, let $Z(A) = \cup\{Z_\alpha : \alpha \in A\}$. Then $\{Z(A) : A \in \mathscr{A}\}$ forms a filter base. Let $\mathscr{F}$ be a z-ultrafilter on X containing $\{Z(A) : A \in \mathscr{A}\}$ and let p be a point of $\beta X - X$ corresponding to $\mathscr{F}$. Then λ_p is a desired one. Because, if possible, suppose λ_p is supported on a Lindelöf subset S of X. Then we have $p \in \mathrm{cl}_{\beta X} S$. Since X is paracompact and S is Lindelöf, $\{\alpha < \omega_1 : \mathrm{cl}_X S \cap Z_\alpha \neq \phi\}$ is a countable set. Hence, there exists $A \in \mathscr{A}$ such that $\mathrm{cl}_X S \cap Z(A) = \phi$. This implies $\mathrm{cl}_{\beta X} S \cap \mathrm{cl}_{\beta X} Z(A) = \phi$. This contradicts the fact that $p \in \mathrm{cl}_{\beta X} Z(A)$ and $p \in \mathrm{cl}_{\beta X} S$, as well. This completes the proof.

COROLLARY 4.11. *Let X be a paracompact Hausdorff space. If $\Lambda_0(X) \subseteq \Lambda_{\mathrm{L}}(X)$, then X is Lindelöf.*

COROLLARY 4.12. *Let X be a paracompact Hausdorff space. Then $\Lambda_{\mathrm{L}}(X) = \Lambda_\infty(X)$ if and only if X is Lindelöf.*

Now Gulick's result Theorem 5.5 in [Gu] immediately follows from the Corollary 4.12.

COROLLARY 4.13 (Gulick). *Let X be a locally compact, paracompact Hausdorff space. Then $\Lambda_\sigma(X) = \Lambda_\infty(X)$ if and only if $C^*_\sigma(X) = C^*_\infty(X)$.*

PROOF. $\Lambda_\sigma(X) = \Lambda_\infty(X)$ implies $\Lambda_{\mathrm{L}}(X) = \Lambda_\infty(X)$ and consequently X is Lindelöf. But a Lindelöf locally compact space is σ-compact and so $C^*_\sigma(X) = C^*_\infty(X)$.

Now we make two observations on the measure-theoretic counterpart and the separability of $\Lambda_{\mathrm{L}}(X)$. Let $m_{c,\mathrm{L}}(X) = \{\mu \in M_c(X) : \mu$ has a support $B \subseteq X$ where $B = \bar{A}$ and A is Lindelöf$\}$ and $M^+_{c,\mathrm{L}}(X) = \{\mu \in M_{c,\mathrm{L}}(X) : \mu \geqq 0\}$. Now the proofs of the Theorems 4.3 and 4.6 can be modified to obtain the following results respectively.

THEOREM 4.14. *For any space X, $\Lambda_{\mathrm{L}}(X)$ is separable if and only if X is compact and countable.*

THEOREM 4.15. *Suppose X is a normal Hausdorff space. Then the Banach lattices* $(M_{c,L}(X), \|\cdot\|$ *and* $(\Lambda_L(X), \|\cdot\|_*)$ *are isomorphic while* $M^+_{c,L}(X)$ *is identified with* $\Lambda^+_L(X)$ *under this isomorphism.*

We conclude this work by citing the following problem.

PROBLEM 4.16. What are the necessary conditions in order to have $\Lambda_\sigma(X) = \Lambda_L(X)$ and $\Lambda_L(X) = \Lambda_\infty(X)$?

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