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INTERPOLATION OF NONLINEAR OPERATORS BETWEEN FAMILIES OF BANACH SPACES

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0. Introduction.

In the theory and applications of interpolation spaces we usually consider a Banach couple (A_0, A_1), i.e. A_0 and A_1 are Banach spaces, which are embedded in a Hausdorff topological vector space U. There are several studied constructions for obtaining interpolation spaces with respect to the couple (A_0, A_1) and the most well-studied and applied such spaces are the spaces $[A_0, A_1]_\theta$ and $(A_0, A_1)_{\theta,q}$ obtained by using the complex and the real methods of interpolation, respectively. See e.g. the books [2], [14] and also the Bibliography of Maligranda [16] including more than 2000 references.

Parts of the theory concerning interpolation between two Banach spaces can be generalized to cover also cases where we interpolate between finite many Banach spaces and even between general families of Banach spaces. Here we mention the following developments:

1) A theory for *complex interpolation* between families of Banach spaces was developed by Coifman-Cwikel-Rochberg-Sagher-Weiss (see [5], [6], [7]) and, independently, by Krein-Nikolova (see [12], [13]). These spaces are sometimes called the St. Louis-spaces and Voronez-spaces, respectively. Another complex interpolation method between n-tuples of Banach spaces was suggested by Lions [15] and studied in detail by Favini [9]. This method of Favini-Lions was extented by Cwikel-Janson [8] to cover also complex interpolation between very general families of spaces.

(2) A theory for *real interpolation* between n-tuples of Banach spaces was introduced and studied by Sparr [24]. A similar theory for real interpolation between 2^k-tuples of Banach spaces was studied by Fernandez [10]. In this connection we also mention early works by Yoshikawa, Kerzman and

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Foias-Lions (cf. the discussion in [24, p. 248]). Moreover, Cobos-Peetre [4] have recently developed a theory for real interpolation between finite many Banach spaces which, in particular, covers both Sparr's and Fernandez' constructions for the cases $n = 3$ and $n = 4$, respectively. The construction of Sparr was extended by Cwikel-Janson [8] to cover also real interpolation between a fairly general family $A = \{A_t\}_{t \in \Gamma}$, where A_t are Banach spaces and Γ is a general probability measure space (this idea was early suggested by J. Peetre).

Concerning the most important properties of these methods and the relations between them we refer to the extremely useful paper [8] by Cwikel-Janson.

In the theory of interpolation we usually study linear operators. However, also some results concerning interpolation of nonlinear operators are known in the case of interpolation between two spaces (see e.g. [3], [11] and [17] and the references given in these papers). In this paper we prove and discuss some theorems about interpolation of nonlinear operators acting on interpolation spaces constructed by some method for interpolation between families of spaces. In particular, we generalize and/or complement some previous results of Cobos [3], Cobos-Peetre [4], Janson [11], Maligranda [17], Nikishin [19] and Nikolova [21].

We can not give all details about the constructions of the interpolation spaces we have discussed above. However, complete definitions and related basic facts can always be found in our main reference [8] and the references given there. This paper is organized in the following way: In section 1 we present some necessary theory and notations from [8] and we also include some complementary theory and results. In section 2 we state a compactness interpolation result, which for the case with only two different spaces gives a recent of Cobos [3]. In section 3 we present an interpolation theorem for non-linear operators satisfying a condition, which may be regarded as a natural generalization of a condition introduced by Bergh in a similar context (see [1]). For the case with only two different spaces this theorem coincides with a recent result of Maligranda [17]. For a complex interpolation method for n spaces a generalization of Bergh's result was given by Stan [25]. In section 4 we state a theorem for sublinear operators, which generalizes another result of Maligranda [17] in a similar way (compare also with Janson [11] and Nikishin [19]). We also prove an estimate for the generalized K-functional for C-subadditive operators and point out some applications of this estimate. In section 5 we discuss a recent construction of Cobos-Peetre [4] and mention an extension of this method to cover also cases with interpolation between families of spaces. Section 6 is reserved for some concluding remarks.

2. Preliminaries.

Let Z be a probability measure on a σ-algebra of subsets of an abstract set Γ. We consider a family $A = \{A_t\}_{t\in\Gamma}$ of Banach spaces. We shall define the "inequality" $B \leqq C$ between two Banach spaces B and C to mean that $B \overset{1}{\subset} C$, i.e., $\|a\|_C \leqq \|a\|_B$ for all $a \in B$. We also require that A is a bounded family on Γ, i.e., that there exists a Banach space U such that $A_t \leqq U$ for all $t \in \Gamma$. In this paper we consider the "lower" spaces $L_M(A, Z)$ and the "upper" spaces $U_M(A, Z)$ by Cwikel-Janson [8]. Here M stands for any of the interpolation methods FL (Favini-Lions' method), St.L. (the St. Louis school method), $J_{,p}$ (Peetre-Sparr's J-method) or $K_{,p}$(Peetre-Sparr's K-method). If M = St.L, then we use the convention that Γ is a rectifiable simple closed curve constituting the boundary of a domain D in the complex plane and that $Z = P_z$ is the harmonic measure on Γ at some fixed point $z \in D$. We remark that if $A = \{A_t\}_{t\in\Gamma}$ is of stepfunction type (i.e. Γ is divided into $n + 1$ disjoints parts E_k of positive measure and A_t is "constant" $= A_k$ on E_k, $k = 0, 1, \ldots, n$), then we get a corresponding interpolation method for interpolation between a $n + 1$-tuple of Banach-spaces.

We introduce the following generalizations of the classical definitions:

1. The *intersection* ΔA_t of A is the set of all elements a belonging to all spaces A_t for which

$$\|a\|_{\Delta A_t} = \sup_{t\in\Gamma} \|a\|_{A_t} < \infty.$$

2. The *sum* $\sum A_t$ is defined to be the set of all elements a in U which can be represented in the form

$$a = \sum a_t,\ a_t \in A_t, \text{ where } \sum \|a_t\|_{A_t} < \infty.$$

We put $\|a\|_{\Sigma A_t} = \inf \sum \|a_t\|_{A_t}$, where infimum is taken over all representations of a of the form above. Let us also note that there are only countably many summands different from zero and, thus, that we can use a representation of the type $a = \sum a_{t_j}$, $a_{t_j} \in A_{t_j}$, where $\sum \|a_{t_j}\|_{A_{t_j}} < \infty$.
The spaces ΔA_t and $\sum A_t$ are Banach spaces. If γ is a subset of Γ and $A = (A_t)_{t\in\gamma}$, then we use the notations $\Delta_\gamma A_t$ and $\sum_\gamma A_t$.

Let $\|a\|_{A_t}$ be a measurable function for every $a \in \bigcap_{t\in I} A_t$. Moreover, let $h(t)$ be an arbitrary bounded (by positive constants) and Z-measurable function on Γ, which is constant on the measurable sets of constancy of A_t,

3. For $a \in \sum A_t$ we define *the generalized K-functional* $K(h(t), a; A)$ as

$$K(h(t), a; A) = \inf \sum h(t)\, \|a_t\|_{A_t},$$

where inf is taken over all representations $a = \sum a_t$, $a_t \in A_t$, where $\sum \|a_i\|_{A_t} < \infty$.

4. For $a \in \Delta A_t$ we define *the generalized J-functional* $J(h(t),a; A)$ as

$$J(h(t), a; A) = \sup_{t\in\Gamma} h(t)\, \|a\|_{A_t}.$$

5. The Banach space E is *of the class* $K(A, Z)$ if $E \leqq \sum A_t$ and, for all $a \in E$,

$$K(h(t), a; A) \leqq C \exp\left(\int_\Gamma \log h(t)\, dZ(t)\right) \|a\|_E.$$

6. The Banach space E is *of the class* $J(A, Z)$ if $\Delta A_t \leqq E$ and, for all $a \in \Delta A_t$,

$$\|a\|_E \leqq C \exp\left(\int_\Gamma -\log h(t)\, dZ(t)\right) J(h(t), a; A).$$

EXAMPLE 1.1. Let $A_t = A_0$ on a subset γ with $0 < Z(\gamma) = \theta < 1$ $(Z(\gamma) = \int_\gamma dZ(t))$, let $A_t = A_1$ on $\Gamma\backslash\gamma$ and let $h(t) = 1$ on γ and $h(t) = u$ on γ. Then the definitions in 1-6 coincide with the classical definitions of the intersection $A_0 \cap A_1$, the sum $A_0 + A_1$, the (Peetre) K-functional $K(u, a; A_0, A_1)$, the (Peetre) J-functional $J(u, a; A_0, A_1)$, the class $C_K(\theta; A_0, A_1)$ and the class $C_J(\theta; A_0, A_1)$, respectively (see [2]).

EXAMPLE 1.2. A Banach space E is of the class $K(A, Z)$ iff $E \leqq \sum A_t$ and for an arbitrary Banach space B and any linear operator $T\colon \sum A_t \to B$ for which $\|Ta\|_B \leqq M(t)\, \|a\|_{A_t}$, $a \in A_t$ ($M(t)$ denotes a bounded (by positive constants) function, which is measurable with respect to $dZ(t)$), it holds that

$$\|T/E\|_{E\to B} \leqq C \exp\left(\int_\Gamma \log M(t)\, dZ(t)\right).$$

EXAMPLE 1.3. Let D be an arbitrary Banach space and assume that $\Delta A_t \leqq E$. E is of the class $J(B, Z)$ if for any linear operator $T\colon D \to A_t$ with $\|Ta\|_{A_t} \leqq M(t)\, \|a\|_{B_t}$ ($M(t)$ denotes a bounded (by positive constants) function, which is measurable with respect to $dZ(t)$), it holds that

$$\|T\|_{D\to E} \leqq C \exp\left(\int_\Gamma \log M(t)\, dZ(t)\right)$$

C denotes a constant in these examples and in the sequel. Proofs of the statements in the examples 1.2–1.3 can be found in [20]. By using these examples and Theorem 2.21 in [8] we obtain in particular the following useful information:

EXAMPLE 1.4. The Cwikel-Janson interpolation spaces $L_M(U, Z)$ and $U_M(A, Z)$ are both of the classes $K(A, Z)$ and $J(A, Z)$.

2. A compactness result for Lipschitz operators,

First we present the following generalization of a recent result of Cobos [3, Theorem 2.1]) concerning interpolation of compact non-linear operators:

THEOREM 1. *Assume that γ is a subset of Γ of positive Z-measure, B is a Banach space and $A = \{A_t\}_{t\in\Gamma}$ is a family of Banach spaces.*

(a) *If $T\colon \sum A_t \to B$ is a Lipschitz operator, $T\colon \sum_\gamma A_t \to B$ is compact and the Banach space E is of the class $K(A, Z)$, then $T\colon E \to B$ is compact.*

(b) *If $T\colon B \to A_t$ is a Lipschitz operator for every $t \in \Gamma$ with the Lipschitz constants $M(t) \leqq M_0$, $T\colon B \to \Delta_\gamma A_t$ is compact and the Banach space E is of the class $J(A, Z)$, then $T\colon B \to E$ is compact.*

We consider the special case when $A = \{A_t\}$ is "constant" $= A_0$ on γ, where $0 < Z(\gamma) = \theta < 1$ and "constant" $= A_1$ on $\Gamma\backslash\gamma$ and obtain the following result of Cobos [3]:

COROLLARY 2. *Let (A_0, A_1) be a Banach couple and let B be a Banach space. Assume that T is a nonlinear operator.*

(a) *If $T\colon A_0 \to B$ is a compact Lipschitz operator, $T\colon A_1 \to B$ is Lipschitz and the Banach space E is a space of the class $C_K(\theta, A_0, A_1)$ with $A_0 \cap A_1$ dense in E, then $T\colon E \to B$ is compact.*

(b) *If $T\colon B \to A_0$ is a compact Lipschitz operator, $T\colon E \to A_1$ is Lipschitz and the Banach space E is a space of the class $C_J(\theta; A_0, A_1)$,* then *$T\colon B \to E$ is compact.*

PROOF OF THEOREM 1. (a) Choose arbitrary $m > 0$ and $\varepsilon > 0$ and let $M(t)$ denote the Lipschitz constant of the operator $T\colon A_t \to B$. We put $M_1(t) = \max(m, M(t))$ and

$$h(t) = \begin{cases} M_1(t) & , \quad t \in \gamma, \\ \dfrac{2\sup M(t)}{\varepsilon}, & \quad t \in \Gamma\backslash\gamma. \end{cases}$$

Let D be a bounded subset of E, let $a \in D$ and choose a representation $a = \sum a_{t_j}$ such that

$$\sum_{t_j\in\Gamma} h(t_j)\, \|a_{t_j}\|_{A_{t_j}} \leqq 2K(h(t), a;\ A). \tag{2.1}$$

We denote $a^0 = \sum_{t_j\in\gamma} a_{t_j}$ and $a^1 = \sum_{t_j\in\Gamma\backslash\gamma} a_{t_j}$.

Now, according to (2.1), the triangle inequality and the assumption $E \in K(A, Z)$, we find that

$$\text{(2.2)} \qquad \|a^0\|_{\Sigma_\gamma A_t} \leqq \sum_{t_j \in \gamma} \|a_{t_j}\|_{A_{t_j}} \leqq \frac{1}{m} \sum_{t_j \in \gamma} h(t_j)\, \|a_{t_j}\|_{A_{t_j}} \leqq \frac{2}{m} K(h(t), a; A) \leqq$$

$$\leqq \frac{2C}{m} \exp\left(\int_\Gamma \log h(t)\, dZ(t) \right) \|a\|_E.$$

If D_0 denotes the set of all a^0 such that $a \in D$, then this estimate implies that D_0 is bounded in $\sum_\gamma A_t$. Therefore, by our assumptions, there exists a finite subset $b_1, b_2, \dots, b_n \in B$ such that

$$T(D_0) \subset \bigcup_{k=1}^{n} b_k + U_B(\varepsilon/2) \; (b \in U_B(\varepsilon) \text{ means that } \|b\|_B < \varepsilon).$$

Now we fix b_k such that $\|Ta^0 - b_k\|_B < \varepsilon$. Then, by (2.1), (2.2) and the triangle inequality, we obtain that

$$\|Ta - b_k\|_B \leqq \|Ta - Ta^0\|_B + \|Ta^0 - b_k\|_B \leqq C_0 \|a - a^0\|_{\Sigma A_t} + \|Ta^0 - b_k\|_B \leqq$$

$$\leqq C_0 \sum_{t_j \in \Gamma \setminus \gamma} \|a_{t_j}\|_{A_{t_j}} + \varepsilon \leqq \frac{\varepsilon C_0}{2 \sup M(t)} \sum_{t_j \in \Gamma \setminus \gamma} h(t_j)\, \|a_{t_j}\|_{A_{t_j}} + \varepsilon \leqq$$

$$\leqq \frac{\varepsilon C_0}{\sup M(t)} K(h(t), a; \mathrm{A}) + \varepsilon \leqq \frac{\varepsilon C_1}{\sup M(t)} \exp\left(\int_\Gamma \log h(t)\, dZ(t) \right) + \varepsilon \leqq C_2 \varepsilon^{Z(\gamma)}.$$

Thus $T\colon E \to B$ is compact.

(b) First we note that the operator $T\colon B \to \Delta A_t$ is a Lipschitz operator because

$$\text{(2.3)} \quad \|Tx - Ty\|_{\Delta A_t} = \sup_{t \in \Gamma} \|Tx - Ty\|_{A_t} \leqq \sup_{t \in \Gamma} M(t)\, \|x - y\|_B \leqq M_0 \|x - y\|_B.$$

Let $\varepsilon > 0$ and choose $h(t) = 1$, $t \in \gamma$, and $h(t) = \varepsilon$, $t \in \Gamma \setminus \gamma$. Then the condition $E \in J(A, Z)$ implies that

$$\text{(2.4)} \qquad \|a\|_E \leqq C \exp\left(\int_\Gamma -\log h(t)\, dZ(t) \right) J(h(t), a; A) =$$

$$= C\varepsilon^{Z(\gamma) - 1} \max \left\{ \sup_{t \in \gamma} \|a\|_{A_t}, \sup_{t \in \Gamma \setminus \gamma} \varepsilon\, \|a\|_{A_t} \right\}.$$

Let now D be any bounded set in B with the diameter L and let $a \in D$. The condition that $T\colon B \to \Delta_\gamma A_t$ is compact means that there exists a finite set $b_1, b_2, \dots, b_n \in D$ such that

$$T(D) \subset \bigcup_{j=1}^{n} \{Tb_j + U_{\Delta_\gamma A_t}(\varepsilon)\}$$

and, for some b_k, $k = 1,2,\dots,n$, $\|Ta - Tb_k\|_{\Delta_\gamma A_t} < \varepsilon$. Therefore, in view of (2.3)–(2.4),

$$\|Ta - Tb_k\|_E \leqq C\varepsilon^{Z(\gamma)-1} \max\left\{\sup_{t\in\gamma} \|Ta - Tb_k\|_{A_t}, \sup_{t\in\Gamma\setminus\gamma} \varepsilon \|Ta - Tb_k\|_{A_t}\right\} \leqq$$

$$\leqq C\varepsilon^{Z(\gamma)-1} \max\{\varepsilon, \varepsilon M_0 2L\} = C_1\varepsilon^{Z(\gamma)}.$$

Hence $T\colon B \to E$ is compact and the proof is complete.

REMARK. By analyzing our proof above we find that Theorem 1 (and thus Corollary 2) holds also if we consider Hölder operators instead of Lipschitz operators.

3. Interpolation of operators endowed with a generalized Bergh-condition.

In this section we restrict ourselves to the case when Γ is the boundary of the unit disc D and consider the Voronez-spaces spaces A_z, $z \in D$ (see e.g. [13]), and the St. Louis-spaces $A[z]$, $z \in D$ (see e.g. [7]). We need the following basic definitions from the foundations of these spaces:

1. Let $F^c(\sum)$ be the set of functions with values in $\sum = \sum A_t$, $t \in \Gamma$, holomorphic in D and weakly continuous in D. Let $F(\Delta)$ be the set of strongly continuous functions on D (with values in ΔA_t) and

$$F^c = \left\{f \in F^c(\textstyle\sum)\colon f(t) \in A_t,\ t = e^{is},\ 0 \leqq s \leqq 2\pi,\ \|f\|_{F^c} = \sup\limits_{0\leqq s\leqq 2\pi} \|f(t)\|_{A_t} < \infty\right\}.$$

Let $F_1 = F_1(A)$ denote the closure of $F(\Delta)$ in the norm of F^c.

2. Let $A = \{A_t\}$, $t \in \Gamma$, be a St. Louis interpolation family and let

$$\beta = \left\{a \in \cap A_t\colon \int_\Gamma \log^+ \|a\|_{A_t}\, dP_z(t) < \infty\right\}.$$

Here $\int_\Gamma f(t)\, dP_Z(t) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{is})P(z,s)\, ds$, where $P(z,s)$ denotes the Poisson kernel. Let G denote the set of all functions of the form $a(z) = \sum \Psi_j(z)a_j$ (finite sum), where $a_j \in \beta$ and each $\Psi_j \in N^+(D)$ (the positive Nevanlinna class) such that

$$\|a\|_G = \operatorname*{esssup}_{t\in\Gamma} \|g(t)\|_{A_t} < \infty.$$

Let $F(A,\Gamma)$ denote the completion of G in this norm.

Now we consider the interpolation families $A = \{A_t\}$ and $B = \{B_t\}$ of some of the forms described above. Inspired by an idea by Bergh [1] we introduce the following condition:

(*) $\quad u \in H(A) \Rightarrow Tu \in H(B)$, where $H(A) = F_1(A)$ or $H(A) = F(A, \Gamma)$.

We state the following generalization of a result of Maligranda [17]:

THEOREM 3. *Assume that T is a non-linear operator and that there exists a measurable function $M = M(t), t \in \Gamma$, such that $\|Ta\|_{B_t} \leqq M(t)\,\|a\|_{A_t}$ for all $a \in A_t$, $t \in \Gamma$.*

(a) *If (*) holds with $H(A) = F^c(A)$, then T maps A_z into B_z and*

$$\|Ta\|_{B_z} \leqq \exp\left(\int_\Gamma \log M(t)\, dP_Z(t)\right) \|a\|_{A_z}. \tag{3.1}$$

(b) *If (*) holds with $H(A) = F(A, \Gamma)$, then T maps $A[z]$ into $B[z]$ and* (3.1) *holds with A_z and B_z replaced by $A[z]$ and $B[z]$, respectively.*

REMARK. By dividing the unit circle into two disjoint parts γ_1 and γ_2 where $\frac{1}{2\pi}\int_{\gamma_1} P(z,t)\,dt = \theta$, by letting $M(t) = M_i$ on γ_i, $A(t) = A_i$ and $B(t) = B_i$ on γ_i, $i = 0, 1$, we get Theorem 3 by Maligranda [17]. See also Bergh [1].

PROOF OF THEOREM 3. Let $a \in A_z$, $z \in D$, and consider an arbitrary $\varepsilon > 0$. We can find $u \in F_1(A)$ such that $\|u\|_{F^c} \leqq \|a\|_{A_z} + \varepsilon$ and $u(z) = a$. Therefore, by (*), $Tu(w) \in F_1(B)$. Now we use Lemma 3 by Krein-Nikolova [13] to conclude that there exists a scalar function $b(w)$ which is continuous on $\bar{D}$ and holomorphic on D, $b(z) = 1$ and such that if $h = h(w) = b(w)T(u(w))$, then

$$\log \|h\|_{F_1(B)} \leqq \int_\Gamma \log \|T(u(t))\|_{B_t}\, dP_Z(t) + \varepsilon \leqq \int_\Gamma \log M(t)\, dP_Z(t) +$$

$$+ \log \sup_{t\in\Gamma} \|u(t)\|_{A_t} + \varepsilon = \int_\Gamma \log M(t)\, dP_z(t) + \log \|u\|_{F^c(A)} + \varepsilon.$$

Therefore, since also $h(z) = Tu(z) = Ta$, we obtain that

$$\|Ta\|_{B_z} \leqq \|h\|_{F^c(B)} \leqq \exp\left(\int_\Gamma \log M(t)\, dP_Z(t) + \varepsilon\right) \|u\|_{F^c(A)} \leqq$$

$$\leqq \exp\left(\int_\Gamma \log M(t)\, dP_Z(t) + \varepsilon\right)(\|a\|_{A_z} + \varepsilon).$$

The proof of part (a) follows by letting $\varepsilon \to 0+$. The proof of part (b) is similar so we omit the details.

4. Interpolation of C-subadditive operators.

The operator $T\colon A_0 \to L^0(\mu)$, is said to be *C-subadditive* if $T(a+b)$ is defined whenever $T(a)$ and $T(b)$ are defined and $|T(a+b)(t)| \leqq C(|T(a)(t)| + |T(b)(t)|)$ a.e. (C is a positive constant independent of a and b). T is called *C-sublinear* if, in addition, $|T(\lambda a)(t)| = |\lambda|\,|Ta(t)|$ a.e. and for all $\lambda \in R_+$. First we state the following result for sublinear (= 1-sublinear) operators:

THEOREM 4. *Let $A = \{A_t\}$, $t \in \Gamma$, be a bounded family of Banach spaces and let $B = \{B_t\}$, $t \in \Gamma$, be a family of Banach lattices in $L^0(\mu)$. Assume that $T\colon U \to L^0(\mu)$ is sublinear and that the following holds: if $a \in A_t$, then $Ta \in B_t$ and $\|Ta\|_{B_t} \leqq M(t)\,\|a\|_{A_t}$ for some bounded function $M(t)$, which is measurable with respect to $dZ(t)$ on Γ. Let $F(A)$ denote any of the Cwikel-Janson interpolation spaces $L_M(A,Z)$ or $U_M(A,Z)$. Then the following holds: if $a \in F(A)$, then $Ta \in F(B)$ and*

$$\|Ta\|_{F(B)} \leqq \exp\left(\int_\Gamma \log M(t)\,dZ(t)\right)\|a\|_{F(A)}.$$

REMARK. For the case with only two spaces (see the remark after Theorem 1) we obtain a result which is completely analogous to Theorem 7 in [17].

PROOF OF THEOREM 4. The main idea here is to observe that, for any $a \in \sum A_t$, there exists a linear map $L_a\colon \sum A_t \to L^0(\mu)$ such that $|L_a(b)| \leqq |Tb|$ a.e. for any $b \in \sum A_t$ and $L_a(a) = Ta$ a.e. The proof of this statement can be carried out in a quite similar way as in [17, proof of Theorem 7] so we omit the details (the main argument is to use Hahn-Banach theorem in an appropriate way). Hence, since B_t is a Banach lattice,

$$\|L_a(b)\|_{B_t} \leqq \|Tb\|_{B_t} \leqq M(t)\,\|b\|_{A_t} \quad \text{for all} \quad b \in A_t.$$

Therefore by using the interpolation theorem 2.21 in [8] we obtain that

$$\|Ta\|_{F(B)} = \|L_a(a)\|_{F(B)} \leqq \exp\left(\int_\Gamma \log M(t)\,dZ(t)\right)\|a\|_{F(A)}$$

where $F = L_M(A,Z)$ or $F = U_M(A,Z)$ and the proof is complete.

The following estimate of the generalized K-functional implies at once interpolation results for *C-subadditive operators T from the family $\{A_t\}$ into $L^0(\mu)$*, namely for operators T satisfying $|T(\sum a_{t_j})| \leqq C \sum |Ta_{t_j}|$.

THEOREM 5. *Let T denote a C-subadditive operator from the family $\{A_t\}$ into*

$L^0(\mu)$ and let $A = \{A_t\}$ and $B = \{B_t\}$, $t \in \Gamma$, denote families of Banach spaces and families of Banach lattices, respectively. Moreover we assume that if $a \in A_t$, then $Ta \in B_t$ and $\|Ta\|_{B_t} \leqq M(t)\,\|a\|_{A_t}$, $t \in \Gamma$, where $M(t)$ is bounded by positive constants. Then

$$K(h(t), Ta; B) \leqq CK(h(t)M(t), a; A).$$

PROOF. Choose arbitrary $\varepsilon > 0$. According to the definition of the generalized K-functional we can find a representation $a = \sum a_{t_j}$ such that $a_{t_j} \in A_{t_j}$ and

$$\sum h(t_j)M(t_j)\,\|a_{t_j}\|_{A_{t_j}} \leqq (1+\varepsilon)K(h(t)M(t), a, A).$$

Let $b_{t_j} = \dfrac{T(\sum a_{t_j})}{\sum |Ta_{t_j}|}\,|Ta_{t_j}|$ on the support of of $\sum |Ta_{t_j}|$ and $b_{t_j} = 0$ elsewhere. Since T is C-subadditive we have the estimate

$$|b_{t_j}| = \frac{|T(\sum a_{t_j})|}{\sum |Ta_{t_j}|}\,|Ta_{t_j}| \leqq C\,|Ta_{t_j}|$$

and, thus, because B_{t_j} are Banach lattices,

$$\|b_{t_j}\|_{B_{t_j}} \leqq C\,\|Ta_{t_j}\|_{B_{t_j}} \leqq CM(t_j)\,\|a_{t_j}\|_{A_{t_j}}.$$

We conclude that

$$K(h(t), Ta, B) \leqq \sum h(t_j)\,\|b_{t_j}\|_{B_{t_j}} \leqq C\sum h(t_j)M(t_j)\,\|a_{t_j}\|_{A_{t_j}} \leqq$$
$$\leqq C(1+\varepsilon)K(h(t)M(t), a, A)$$

and the proof follows by letting $\varepsilon \to 0+$.

Now we consider the usual Peetre-Sparr spaces $\vec{A}_{\theta q;K}$, where $\theta = (\theta_0, \theta_1, \ldots, \theta_n)$, $\sum \theta_i = 1$, $\theta_i \geqq 0$, $1 \leqq q \leqq \infty$ for an $(n+1)$-tuple $\vec{A} = (A_0, A_1, \ldots, A_n)$ of Banach spaces (see [24] or [8]) and state the following interpolation result:

COROLLARY 6. *Let A_i and B_i, $i = 0, 1, \ldots, n$, be Banach spaces and Banach lattices in $L^0(\mu)$, respectively, let T denote a C-subadditive operator from $A_0 + A_1 + \ldots + A_n$ into $L^0(\mu)$ and assume that if $a \in A_i$, then $Ta \in B_i$ and $\|Ta\|_{B_i} \leqq M_i\,\|a\|_{A_i}$, $i = 0, 1, \ldots, n$. Then T maps $\vec{A}_{\theta q;K}$ into $\vec{B}_{\theta q;K}$ and*

$$\|Ta\|_{\vec{B}_{\theta q;K}} \leqq CM_0^{\theta_0}M_1^{\theta_1}\ldots M_n^{\theta_n}\,\|a\|_{\vec{A}_{\theta q;K}}.$$

PROOF. Divide Γ into $n+1$ disjoint parts with positive measure and let $A = A_j$, $B = B_j$, $M = M_j$ and $h(t) = t_j$ on part j, $j = 0, 1, \ldots, n$. In this (finite family) case Theorem 5 implies that $K(\vec{t}, Ta, \vec{B}) \leqq CK(\vec{t}\vec{M}, a, \vec{A})$. Therefore, by using Lemma 3.1 in [24], we obtain that

$$\Phi_{\theta q}(K(\vec{t}, Ta; \vec{B})) \leqq C\Phi_{\theta q}(K(\vec{t}\vec{M}, a; \vec{A})) = CM_0^{\theta_0}M_1^{\theta_1}\dots M_n^{\theta_n}\Phi_{\theta q}(K(\vec{t}, a; \vec{A}))$$

where

$$\Phi_{\theta q}(f) = \left(\int_{R_+^n} |t_1^{-\theta_1}t_2^{-\theta_2}\dots t_n^{-\theta_n}f(1, t_1, \dots, t_n)|^q \frac{dt_1}{t_1}\dots\frac{dt_n}{t_n}\right)^{1/q}.$$

The proof is complete.

Now we introduce the following generalization of the Peetre-Sparr spaces $A_{\theta,\infty;K}$: We say that $a \in A_{Z,\infty;K}$ if $a = a(t)$, $a(t) \in A_t$, $t \in \Gamma$, and

$$\|a\|_{A_{z,\infty;K}} = \sup\left(\int_\Gamma -\log h(t)\, dZ(t)K(h(t), a; A_t)\right) < \infty,$$

where supremum is taken over all functions $h(t)$ which are bounded by positive constants, measurable with respect to $dZ(t)$ and constant on the measurable sets of constancy of A_t.

COROLLARY 7. *Let T, $A = \{A_t\}$, $B = \{B_t\}$ and $M = M(t)$, $t \in \Gamma$, be defined as in Theorem 5 and assume that $M = M(t)$ is bounded by positive constants and measurable with respect to $dZ(t)$, Then T maps $A_{z,\infty;K}$ into $B_{z,\infty;K}$ and*

$$\|Ta\|_{B_{z,\infty;K}} \leqq C\exp\left(\int_\Gamma \log M(t)\, dZ(t)\right)\|a\|_{A_{z,\infty;K}}.$$

5. On a construction of Cobos-Peetre.

We consider the $n+1$-tuple $\vec{A} = (A_0, A_1, \dots, A_n)$ of Banach spaces and the three parameter interpolation spaces $\vec{A}_{(\alpha,\beta),q;K}$ $(1 \leqq q \leqq \infty)$ recently introduced by Cobos-Peetre [4]. Here each space A_j should be thought of as sitting in the vertex (x_j, y_j), $j = 0, 1, \dots, n$, of a convex polygon Π and let (α, β) be a point in the interior of Π. For definitions and basic properties we only refer to [4]. We state the following slight generalization of a result in [4]:

THEOREM 8. *Let $\vec{A} = (A_0, A_1, \dots, A_n)$ and $\vec{B} = (B_0, B_1, \dots, B_n)$ denote $(n+1)$-tuples of Banach spaces and Banach lattices in $L^0(\mu)$, respectively. Assume that the operator T is C-subadditive from $\sum A_k$ into $L^0(\mu)$ and assume that if $a \in A_j$, then $Ta \in B_j$ and $\|Ta\|_{B_j} \leqq M_j\|a\|_{A_j}$, $j = 0, 1, \dots, n$. Then*

$$\|Ta\|_{\vec{B}_{(\alpha,\beta),q;K}} \leqq C_1 D_{\alpha,\beta}(M_0, M_1, \dots, M_n)\|a\|_{\vec{A}_{(\alpha,\beta);q;K}},$$

where $C_1 = \max(C, C^n)$ and

$$D_{\alpha,\beta}(M_0, M_1, \ldots, M_n) = \inf_{t>0, s>0} \left\{ \max_{0 \leqq j \leqq n} \{t^{x_j - \alpha} s^{y_j - \beta} M_j\} \right\}.$$

PROOF. According to our definition of the generalized K-functional, the subadditivity of T and the assumption $\|Ta\|_{B_j} \leqq M_j \|a\|_{A_j}$ we obtain that, for all $\lambda, \mu > 0$,

$$K(t, s, |Ta|; B) \leqq K\left(t, s, a, C_1 \sum_0^n |Ta_j|, B\right) \leqq$$

$$\leqq \inf\left\{\sum_0^n t^{x_j} s^{y_j} C_1 \, \|Ta_j\|_{B_j}: a = \sum_0^n a_j, a_j \in A_j\right\} \leqq$$

$$\leqq C_1 \inf\left\{\sum_0^n \lambda^{x_j} \mu^{y_j} M_j (t/\lambda)^{x_j} (s/\mu)^{y_j} \, \|a_j\|_{A_j}: a = \sum_0^n a_j, a_j \in A_j\right\} \leqq$$

$$\leqq C_1 \max_{1 \leqq j \leqq N} \{\lambda^{x_j} \mu^{y_j} M_j\} K(t/\lambda, s/\mu, a; A).$$

Thus, by arguing as in [4], i.e. by integrating and changing variables, we find that

$$\|Ta\|_{(\alpha,\beta),q;K} \leqq \left(\int_0^\infty \int_0^\infty \left(t^{-\alpha} s^{-\beta} C_1 \max_{1 \leqq j \leqq N} \{\lambda^{x_j} \mu^{y_j} M_j\} K(t/\lambda, s/\mu, a; A) \right)^q \frac{dt}{t} \frac{ds}{s} \right)^{1/q} =$$

$$= \left(\int_0^\infty \int_0^\infty \left(t^{-\alpha} s^{-\beta} C_1 \max_{1 \leqq j \leqq N} \{\lambda^{x_j - \alpha} \mu^{y_j - \beta} M_j\} K(t, s, a; A) \right)^q \frac{dt}{t} \frac{ds}{s} \right)^{1/q} =$$

$$= C_1 \max_{1 \leqq j \leqq N} \{\lambda^{x_j - \alpha} \mu^{y_j - \beta} M_j\} \, \|a\|_{(\alpha,\beta),q;K}.$$

The proof follows by taking infimum over $\lambda > 0$, $\mu > 0$.

We finish this section by mention the following possible generalization of the construction of Cobos-Peetre to the case with interpolation between a family $A = \{A_{e^{it}}\}$, $t \in [0, 2\pi)$, of Banach spaces (cf. [22]): First we define generalized two parameter K-functionals and J-functionals as

$$K(u, s, a; K) = \inf_{t_j \in [0, 2\pi)} \{\sum u^{\cos t_j} s^{\sin t_j} \|a_j\|_{A_{e^{it_j}}}, a = \sum a_j, a_j \in A_{e^{it_j}}\},$$

$$J(u, s, a; A) = \sup_{t \in [0, 2\pi)} \{u^{\cos t} s^{\sin t} \|a\|_{A_{e^{it}}}, a \in \cap A_{e^{it}}, 0 \leqq t < 2\pi\},$$

respectively. Then for $1 \leqq z < \infty$, $|z| < 1$, $\alpha = \operatorname{Re} z$, $b = \operatorname{Im} z$, we define the space $A_{z,q;K} = A_{(\alpha,\beta),q;K}$ as the completion of all elements $a \in \sum A_{e^{it}}$ having finite norm

$$\|a\|_{A_{z,q;K}} = \left(\int_0^\infty\int_0^\infty \left(u^{-\operatorname{Re} z} s^{-\operatorname{Im} z} K(u,s,a,A)\right)^q \frac{du}{u}\frac{ds}{s}\right)^{1/q}.$$

The spaces $A_{z,q;J} = A_{(\alpha,\beta),q;J}$ and the classes $K_z(A)$ and $J_z(A)$ are defined in similar suitable ways. We shall not develop this idea further in this paper but only remark that it is possible to prove that the spaces $A_{z,q,K}$ and $A_{z,q;J}$ have appropriate interpolation properties and that they are of the classes $K_z(A)$ *and* $J_2(A)$, respectively.

6. Concluding remarks.

1. We suggest that Theorem 4 do not hold in general if T is only a C-sublinear operator ($C > 1$). We remark that even for the classical case with complex interpolation between two spaces it is *not* known if there exists a C-sublinear operator T satisfying the condition "$a \in A_i \Rightarrow Ta \in B_i$ and $\|Ta\|_{B_i} \leqq M_i \|a\|_{A_i}$, $i = 0, 1$," such that T do not map $(A_0, A_1)_\theta$ into $(B_0, B_1)_\theta$ (A_i and B_i are Banach spaces and Banach lattices in $L^0(\mu)$, respectively). However, Maligranda [17, p. 268] has presented an example of such an operator for the case when the C-subadditivity assumption holds only for functions a and b satisfying $|a + b| = |a| + |b|$ a.e.

2. In connection to our Theorem 1 we remark that for the classical case with interpolation between only two Banach spaces there exists several results concerning interpolation of various kinds of geometrical properties. See e.g. the review article [18] (including more then 100 references). For the case with interpolation between families of spaces only some such results are known.

3. In order to be able to apply or illustrate our results it is important to have some concrete descriptions of interpolation spaces between families of Banach spaces. However, such descriptions can easily be obtained by using reiteration and well-known descriptions of interpolation spaces between couples of Banach spaces; see [23].

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