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PUSHING DOWN LOEB MEASURES

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0. Introduction.

Representations of measures via nonstandard techniques were studied by various authors [4, 8, 12, 15, 16, 24] and these techniques play an important role in nonstandard stochastics. The aim of this paper is to develop new methods of representing measures. But before let us give a short outline of the known theory: Let $\nu: \mathscr{A} \to {}^*[0, \infty)$ be an internal finite content on the algebra $\mathscr{A}$ over an internal set Z. Peter Loeb recognized in his fundamental paper [16] that the set function $\mathrm{st}\,\nu: \mathscr{A} \to [0, \infty)$ defined by $(\mathrm{st}\,\nu)(A) := \mathrm{st}_{\mathbf{R}}(\nu(A))$ is always a premeasure. The *Loeb measure* is the measure extension of $\mathrm{st}\,\nu$ on the Loeb σ-algebra $L(\nu)$ consisting of all subsets $Q \subset Z$ such that for all $\varepsilon > 0$ there exists $A_1, A_2 \in \mathscr{A}$ with $A_1 \subset Q \subset A_2$ and $\nu(A_2) - \nu(A_1) < \varepsilon$. It is a well-known fact that some important subsets, e.g. the set of all nearstandard points, are generally not Loeb measurable. This measurability problem can be circumvented if we are looking at reasonable extensions of $\mathrm{st}\,\nu$ to the power set of Z which is denoted by $\mathscr{P}(Z)$. The *Loeb outer measure* $\bar{\nu}: \mathscr{P}(Z) \to [0, \infty)$ and the *Loeb inner measure* $\underline{\nu}: \mathscr{P}(Z) \to [0, \infty)$ respectively are defined by

$$\bar{\nu}(Q) := \inf\{\mathrm{st}\,\nu(A) : A \in \mathscr{A}, Q \subset A\}$$
$$\underline{\nu}(Q) := \sup\{\mathrm{st}\,\nu(A) : A \in \mathscr{A}, A \subset Q\}$$

respectively. If we restrict $\bar{\nu}$ on the Loeb σ-algebra $L(\nu)$ we obtain the Loeb measure. It is an important fact that the restriction of $\bar{\nu}$ and $\underline{\nu}$ on the σ-algebra $L(\nu) \cap Y := \{A \cap Y : A \in L(\nu)\}$ are measures for *every* subset $Y \subset Z$.

Assume now that ν is an internal content on a standard set $Z := {}^*X$. It is a very important problem to develop methods for pushing down the Loeb measure $\bar{\nu}$ to a measure on the space X and it is this point where usually topological methods come in. Recall that the *monad* of a point x in a topological space (X, τ) is just the set $m(x) := \bigcap_{U \in \tau, x \in U} {}^*U$ and $\mathrm{ns}\,{}^*X := \bigcup_{x \in X} m(x)$ is the set of all *nearstandard*

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points. Instead of $y \in m(x)$ we use also the more suggestive notion $y \approx x$. We call the map

$$\text{(1)} \qquad \mathrm{st}^{-1}: \mathscr{P}(X) \to \mathscr{P}({}^*X), \quad \mathrm{st}^{-1}(Q) := \cup_{x \in Q} m(x)$$

the *inverse standard part map*. Note that this definition does not require any assumption on the topological space. Landers and Rogge proved in [12] that the set functions $\bar{\nu} \circ \mathrm{st}^{-1}$ and $\underline{\nu} \circ \mathrm{st}^{-1}$ restricted to the *Borel σ-algebra* $\sigma[\tau]$ (the σ-algebra generated by τ) are measures provided that X is a regular Hausdorff space. An analysis of the proofs shows that the only important property is the fact that st^{-1} is a reasonable extension of the following set function:

$$\text{(2)} \qquad m_\tau^Y: \mathscr{F} \to \mathscr{P}({}^*X) \cap Y, \quad m_\tau^Y(F) := \cap_{U \in \tau, F \subset U} {}^*U \cap Y,$$

where $\mathscr{F}$ is the set of all closed subsets of X and Y is equal to ns ${}^*X$. The equation (2) is the key for a more general approach since it makes sense for rather general systems of subsets. An extension theorem of R. Sikorski allows us to extend m_τ^Y to a σ-homomorphism from $\sigma[\mathscr{F}]$, the σ-algebra generated by $\mathscr{F}$, into $L(\nu) \cap Y$ if we impose some conditions on $\mathscr{F}$ and τ which are dependent on the size of Y. A *σ-homomorphism* ϕ from a σ-algebra $\mathscr{B}$ over X into a σ-algebra $\tilde{\mathscr{B}}$ over $\tilde{X}$ is a function preserving all the natural operations on a σ-algebra; more precisely, it suffices to require that $\phi(X) = \tilde{X}$ and $\phi(X \backslash B) = \tilde{X} \backslash \phi(B)$ and $\phi(\cup_{n=1}^\infty B_n) = \cup_{n=1}^\infty \phi(B_n)$ for all $B, B_n \in \mathscr{B}$.

In the first section we show that $m_\tau := m_\tau^Y$ with $Y := {}^*X$ defined on the set of all closed subsets is extendible to a σ-homomorphism iff X is normal and *countably compact* (every countable open covering has a finite subcovering). The system of all *zerosets*, i.e. the preimages of zero under continuous real-valued functions, is denoted by $\mathscr{Z}$ and $\mathscr{Z}^c$ is the set of all complements Z^c of $Z \in \mathscr{Z}$. We show that $m_{\mathscr{Z}^c}$ defined on $\mathscr{Z}$ can be extended to a σ-homomorphism on the *Baire σ-algebra* $\sigma[\mathscr{Z}]$ (the σ-algebra generated by $\mathscr{Z}$) if and only if X is *pseudocompact*, i.e., that every continuous real-valued function on X is bounded. For the extension of $m_{\mathscr{Z}^c}$ we can give an explicit formula using countable open coverings. We define so-called covering functions which may also be useful for other applications.

In the second section we discuss several properties of the inverse standard part map. A byproduct of our investigations is that most of the results in [12] carry over to the class of all regular spaces; hence the Hausdorff axiom can be omitted.

In the third section we want to represent measures by internal measures. Here we assume that ϕ is a σ-homomorphism extending the set function $m_\tau^Y: \mathscr{F} \to L(\nu) \cap Y$ where ν is an internal content and $\tau^c \subset \mathscr{F}$. Roughly speaking we prove that $\bar{\nu} \circ \phi$ is equal to a measure μ on $\sigma[\tau]$ iff μ is regular with respect to τ^c and $\bar{\nu}(Y) = \bar{\nu}({}^*X)$ and $\nu({}^*F) \lesssim \mu(F)$ for all $F \in \tau^c$. Here $a \lesssim b$ for two hyperreal numbers a, b means that either $a \leq b$ or a is infinitesimal near to b, and a measure

$\mu: \mathscr{B} \to [0, \infty)$ is *regular with respect to a system* $\mathscr{F}$ of subsets if for every $B \in \mathscr{B}$ the equality $\mu(B) = \sup_{F \in \mathscr{F}, F \subset B} \mu(F)$ holds. Recall that a *Borel measure* [*Baire measure* resp.] is just a finite (non-negative) measure on $\sigma[\tau]$ [on $\sigma[\mathscr{Z}]$ resp.] where (X, τ) is a topological space. A Borel measure is *Radon* [*regular* respectively] if it is regular with respect to the system of all compact [closed respectively] subsets; it is τ-*smooth* if for every upward directed system $\mathscr{S} \subset \tau$ the equality $\mu(\sup_{S \in \mathscr{S}}(S)) = \sup_{S \in \mathscr{S}} \mu(S)$ holds. As a consequence we obtain nonstandard representations of Borel measures and Baire measures resp. on normal countably compact and pseudocompact spaces respectively. Applying the results to the case of the inverse standard part map we can give a unified treatment of representations of measures and the characterizations of the monads of the weak topology on the set of all τ-smooth measures over a regular space.

The fourth section is devoted to the question of representability of a measure $\mu: \mathscr{B} \to [0, \infty)$ with respect to a *partial map* $s: Y \to X$, i.e. a function satisfying $Y \supset {}^{\sigma}X$ and $s({}^*x) = x$ for all $x \in X$. We show that some measures μ can not be represented by a partial map: there does not exist a partial map s with $\overline{{}^*\mu}(s^{-1}(B)) = \mu(B)$ for all $B \in \mathscr{B}$.

In the last section we investigate rather general set functions. The most surprising result may be that for every internal, subadditive and additive set function defined on a *lattice* of subsets the set function $\bar{\nu} \circ \mathrm{st}^{-1}$ is still a τ-smooth Borel measure provided that X is a regular space. Using this result we can answer a question in [2]: the set ns ${}^*X$ is Loeb measurable with respect to all internal, finite, subadditive and additive set functions ν if and only if X is *pre-radon*, i.e., that every τ-smooth Borel measure is Radon.

We always assume a κ-saturated nonstandard model where κ is a cardinal larger than the cardinality of every set in the standard universe. A family of internal sets is called of *admissible cardinality* if the cardinality of the index set is smaller than κ. In the later sections we use frequently the following continuity properties of the Loeb inner and outer measure: if $\mathscr{S} \subset \mathscr{A}$ is a downward directed system of admissible cardinality then an internal content ν on $\mathscr{A}$ satisfies for every subset Q the nice formula (Theorem 2 in [12]):

(3) $$\bar{\nu}(Q \cap \inf \mathscr{S}) = \inf_{S \in \mathscr{S}} \bar{\nu}(Q \cap S).$$

We can replace in (3) the infimum by a supremum if $\mathscr{S}$ is an upward directed system. The same equations hold for the Loeb inner measure. Moreover the Loeb σ-algebra can be characterized in the following way:

(4) $$L(\nu) = \{Q \subset Z: \bar{\nu}(A \cap Q) + \bar{\nu}(A \cap Q^c) = \bar{\nu}(A) \text{ for all } A \subset Z\}.$$

For unexplained topological notions we refer to [30].

1. Covering functions and σ-homomorphisms.

In applications of measure theory the following situation often arises: assume that $\mathscr{F}$ is a system of subsets of X and $\tilde{\mathscr{B}}$ is a σ-algebra on a set $\tilde{X}$ and let $\phi: \mathscr{F} \to \tilde{\mathscr{B}}$ be a function. Is it possible to extend ϕ to a σ-homomorphism on the σ-algebra $\sigma[\mathscr{F}]$ generated by $\mathscr{F}$? Suppose there exists such an extension: The following condition is obviously necessary: for every sequence $(F_n)_{n\in\mathbb{N}}$ and $(E_m)_{m\in\mathbb{N}}$ in $\mathscr{F}$

$$(5) \qquad \bigcap_{n=1}^{\infty} F_n \cap \bigcap_{m=1}^{\infty} E_m^c = \emptyset \Rightarrow \bigcap_{n=1}^{\infty} \phi(F_n) \cap \bigcap_{m=1}^{\infty} \phi(E_m)^c = \emptyset.$$

R. Sikorski has shown [25, p. 144] that this necessary condition is also sufficient. Unfortunately this construction has the disadvantage that we do not have a simple formula for ϕ.

We restrict ourself to the following assumptions (*): let $v: \mathscr{A} \to {}^*[0, \infty)$ be an internal finite content on ${}^*X$ and let τ_0 be a base of a topology τ closed under finite unions and intersections on X such that ${}^\sigma\tau_0 \subset \mathscr{A}$ (this means that ${}^*U \in \mathscr{A}$ for all $U \in \tau_0$) and assume further that $\mathscr{F}$ be a system of subsets closed under finite intersections. Consider now the mapping

$$(6) \qquad m_{\tau_0}: \mathscr{F} \to \mathscr{P}({}^*X) \quad m_{\tau_0}(F) := \bigcap_{U\in\tau_0, F\subset U} {}^*U.$$

LEMMA 1.1. *Let $\tau_0, \mathscr{F}, v$ satisfy the condition (*). Then $m_{\tau_0}(F)$ is Loeb measurable for each $F \in \mathscr{F}$. If $m_{\tau_0}: \mathscr{F} \to L(v)$ can be extended to a σ-homomorphism on $\sigma[\mathscr{F}]$ then the following two conditions are satisfied:*

(I) *Disjoint sets F_1 and F_2 in $\mathscr{F}$ can be separated by τ_0-open sets.*

(II) *If $(F_n)_{n\in\mathbb{N}} \subset \mathscr{F}$ has the finite intersection property then $\cap_{n=1}^{\infty} F_n \neq \emptyset$.*

PROOF. Landers and Rogge have proved in [12] that the intersection of a subfamily of $\mathscr{A}$ of admissible cardinality is still Loeb measurable. Hence it is clear that every set $m_{\tau_0}(F)$ is Loeb measurable. Let F_1, F_2 be disjoint. Then $\emptyset = m_{\tau_0}(F_1) \cap m_{\tau_0}(F_2)$. A saturation argument yields $U_1, \ldots, U_n \in \tau_0$ and $V_1, \ldots V_m \in \tau_0$ such that $U := U_1 \cap \ldots \cap U_n$ and $V := V_1 \cap \ldots \cap V_m$ are disjoint and satisfy $F_1 \subset U$ and $F_2 \subset V$.

For (II) suppose that $\cap_{n=1}^{\infty} F_n = \emptyset$. Then $\cap_{n=1}^{\infty} {}^*F_n \subset \cap_{n=1}^{\infty} m_{\tau_0}(F_n) = \emptyset$ and hence $\cap_{n=1}^{m} {}^*F_n = \emptyset$ for some $m \in \mathbb{N}$, a contradiction to the finite intersection property.

THEOREM 1.2. *Let τ_0, $\mathscr{F}$, v satisfy the condition (*) and assume that $\tau_0^c \subset \mathscr{F}$. Then $m_{\tau_0}: \mathscr{F} \to L(v)$ can be extended to a σ-homomorphism on $\sigma[\mathscr{F}]$ iff (I) and (II) are satisfied.*

Proof. The necessity has been proved. We show (5). We prove at first the identity $m_{\tau_0}(F_1 \cap F_2) = m_{\tau_0}(F_1) \cap m_{\tau_0}(F_2)$ for all $F_1, F_2 \in \mathscr{F}$; the inclusion is trivial. Let $y \in m_{\tau_0}(F_1) \cap m_{\tau_0}(F_2)$ and let $V \in \tau_0$ with $F_1 \cap F_2 \subset V$. Then $F_1 \backslash V$ and $F_2 \backslash V$ are disjoint sets in $\mathscr{F}$ and can be separated by sets $U_1, U_2 \in \tau_0$. Then $F_i \subset U_i \cup V$ and consequently $y \in {}^*U_i \cup {}^*V$ and $y \in {}^*V$ by disjointness of U_1 and U_2. This shows the identity.

Now let $\cap_{n=1}^{\infty} F_n \cap \cap_{m=1}^{\infty} E_m^c = \emptyset$ with $F_n, E_m \in \mathscr{F}$. Suppose that there exists $y \in \cap_{n=1}^{\infty} m_{\tau_0}(F_n) \cap \cap_{m=1}^{\infty} {}^*X \backslash m_{\tau_0}(E_m)$. We can assume that $F_{n+1} \subset F_n$ for all $n \in \mathbb{N}$. For every $m \in \mathbb{N}$ there exists $V_m \in \tau_0$ with $y \notin {}^*V_m$ and $E_m \subset V_m$. Then $\cap_{n=1}^{\infty} F_n \cap \cap_{m=1}^{\infty} V_m^c = \emptyset$ and since $V_m^c \in \mathscr{F}$ condition (II) yields $n \in \mathbb{N}$ such that $F_n \cap \cap_{m=1}^{n} V_m^c = \emptyset$. This means that $F_n \subset \cup_{m=1}^{n} V_m$. Now $y \in m_{\tau_0}(F_n)$ implies $y \in \cup_{m=1}^{n} {}^*V_m$, a contradiction.

Corollary 1.3. *Let (X, τ) be a topological space and $\mathscr{F}$ be the set of all closed subsets. Then $m_\tau: \mathscr{F} \to L(v)$ is extendible to a σ-homomorphism on $\sigma[\tau]$ if and only if (X, τ) is normal and countably compact.*

In the following we want to obtain an explicit description of the extension of $m_{\mathscr{Z}^c}$ defined on $\mathscr{Z}$. For our applications such a description is not needed since the extension is unique (cf. [25] or Theorem 1.11) and an explicit formula for the σ-homomorphism for closed subsets is given. On the other side an explicit formula for a mathematical subject should merit a presentation and we believe that our techniques may also be useful for other applications. For our approach we need the following.

Definition 1.4. Let X be a set and $\mathscr{U}$ be a set of families of subsets of X. For every $\alpha \in \mathscr{U}$ we define $\alpha \operatorname{pts} {}^*X := \cup_{A \in \alpha} {}^*A$. We call the function $\phi_{\mathscr{U}}: \mathscr{P}(X) \to \mathscr{P}({}^*X)$ defined by

$$\phi_{\mathscr{U}}(Q) := \bigcap_{\alpha \in \mathscr{U},\, Q \subset \cup \alpha} \alpha \operatorname{pts} {}^*X \tag{7}$$

the covering function with respect to $\mathscr{U}$. If $\mathscr{U}$ is the system of all countable families of cozero-sets $\phi_{\mathscr{U}}$ is denoted by ϕ_{cc}. If $\mathscr{U}$ is the system of all countable families of open sets we denote $\phi_{\mathscr{U}}$ by ϕ_c.

In order to avoid pathologies in our last definition we always assume that $\{\emptyset\}$ and $\{X\}$ are in $\mathscr{U}$. The proof of the following proposition uses an easy saturation argument and is omitted.

Proposition 1.5. *Let $\phi_{\mathscr{U}}: \mathscr{P}(X) \to \mathscr{P}({}^*X)$ be a covering function. Then the equation $\phi_{\mathscr{U}}(X) = {}^*X$ holds if and only if every covering $\alpha \in \mathscr{U}$ of the set X has a finite subcovering.*

Our original motivation for definition 1.4 based on the following simple observation whose proof is left to the reader.

PROPOSITION 1.6. *Let (X, τ) be a topological space and $\tilde{\tau}$ be the system of all families of open sets. Then we have $\phi_{\tilde{\tau}}(Q) = \mathrm{st}^{-1}(Q)$ for every subset $Q \subset X$.*

THEOREM 1.7. *Let X be a topological space and $m_{\mathscr{Z}^c}: \mathscr{Z} \to L(v)$ as in (6). Then the following statements are equivalent:*
1. *$m_{\mathscr{Z}^c}$ can be extended to a σ-homomorphism $\bar{m}: \sigma[\mathscr{Z}] \to L(v)$.*
2. *The system $\mathscr{Z}$ has the countable intersection property.*
3. *The covering function ϕ_{cc} is a σ-homomorphism on $\sigma[\mathscr{Z}]$ extending $m_{\mathscr{Z}^c}$.*

PROOF. 1) $\Rightarrow$ 2) is clear by condition (II). For the implication 2) $\Rightarrow$ 3) we need some general properties of covering functions and the proof will be given after Theorem 1.11. Note that 3) $\Rightarrow$ 1) is trivial.

It is well-known that condition 2) of Theorem 1.7 is equivalent to the pseudocompactness of X, cf. [30].

PROPOSITION 1.8. *Let $\mathscr{U}$ be a set of families of subsets of X such that for every sequence $(Q_n)_{n \in \mathbb{N}}$ of subsets and for every covering $\alpha_n \in \mathscr{U}$ of Q_n there exists a covering $\alpha \in \mathscr{U}$ of the set $\cup_{n=1}^{\infty} Q_n$ with $\alpha \subset \cup_{n=1}^{\infty} \alpha_n$. Then the equation $\phi_{\mathscr{U}}(\cup_{n=1}^{\infty} Q_n) = \cup_{n=1}^{\infty} \phi_{\mathscr{U}}(Q_n)$ holds.*

PROOF. Obviously $\phi_{\mathscr{U}}$ is a monotone function, i.e., that $Q_1 \subset Q_0$ implies $\phi_{\mathscr{U}}(Q_1) \subset \phi_{\mathscr{U}}(Q_0)$. Hence we only need to show the inclusion part. Let $y \in \phi_{\mathscr{U}}(\cup_{n=1}^{\infty} Q_n)$ and suppose $y \notin \phi_{\mathscr{U}}(Q_n)$ for each $n \in \mathbb{N}$. Then there exists a covering $\alpha_n \in \mathscr{U}$ of Q_n with $y \notin \alpha_n \operatorname{pts} {}^*X$. Choose a covering $\alpha \in \mathscr{U}$ of $\cup_{n=1}^{\infty} Q_n$ with $\alpha \subset \cup_{n=1}^{\infty} \alpha_n$. Then $y \in \alpha \operatorname{pts} {}^*X \subset \cup_{n=1}^{\infty} \alpha_n \operatorname{pts} {}^*X$, a contradiction.

The following easy example shows that we must impose some conditions in order to obtain nice covering functions.

EXAMPLE 1.9. Let (X, τ) be a topological space and $\tilde{\tau}_f$ be the set of all finite families of open sets. Then $\phi_{\tilde{\tau}_f}(Q)$ is exactly $\cap_{U \in \tau, Q \subset U} {}^*U = m_\tau(Q)$. Choose now $X = \mathbb{R}$ with the discrete topology and $Q_n := \{n\}$. Then $\phi_{\tilde{\tau}_f}(\mathbb{N}) = {}^*\mathbb{N}$, but $\phi_{\tilde{\tau}_f}(Q_n) = \{n\}$.

The proof of the following Theorem is straightforward.

THEOREM 1.10. *Let $\mathscr{B}$ and $\tilde{\mathscr{B}}$ be σ-algebras on X and $\tilde{X}$ respectively. Assume that $T: \mathscr{B} \to \mathscr{P}(\tilde{X})$ satisfies the following three conditions:* (i) $T(\emptyset) = \emptyset$, (ii) *T is monotone and* (iii) $T(\cup_{n=1}^{\infty} A_n) = \cup_{n=1}^{\infty} T(A_n)$. *Then the set*

(8) $$\sigma_T := \{A \in \mathscr{B}: T(A) \in \tilde{\mathscr{B}} \cap T(X) \quad \text{and} \quad T(A) \cap T(A^c) = \emptyset\}$$

is a σ-algebra and $T: \sigma_T \to \tilde{\mathscr{B}} \cap T(X)$ is a σ-homomorphism.

THEOREM 1.11. *Suppose $S, T: \mathscr{B} \to \mathscr{P}(\tilde{X})$ satisfy the conditions* (i)–(iii) *of Theorem* 1.10. *If $\mathscr{F} \subset \sigma_T \cap \sigma_S$ is a generator of $\mathscr{B}$ closed under finite intersections and $S(F) = T(F)$ for all $F \in \mathscr{F}$ and $S(X) = T(X)$ then $S = T$.*

PROOF. Show that $\mathscr{A} := \{A \in \sigma_S \cap \sigma_T : S(A) = T(A)\}$ is a σ-algebra containing $\mathscr{F}$.

PROOF OF THEOREM 1.7 2) $\Rightarrow$ 3): Trivially we have $\phi_{cc}(Z) \subset m_{\mathscr{Z}^c}(Z)$ for every $Z \in \mathscr{Z}$. But every countable cover of cozero-sets can be reduced to a finite subcover by 2). Hence ϕ_{cc} is an extension. By Proposition 1.8 $T := \phi_{cc}$ satisfies the assumption of Theorem 1.10 with $\tilde{\mathscr{B}} := L(\nu)$. Hence it suffices to show the relation $\mathscr{Z} \subset \sigma_T$. By our previous remarks we have always $\phi_{cc}(Z) = m_{\mathscr{Z}^c}(Z) \in L(\nu)$. Hence it suffices to prove the relation $\phi_{cc}(Z) \cap \phi_{cc}(Z^c) = \emptyset$. Choose a sequence $(G_n)_{n \in \mathbb{N}}$ of cozero-sets and a sequence $(F_n)_{n \in \mathbb{N}}$ of zero-sets such that $G_{n+1} \subset F_n \subset G_n$ and $Z = \bigcap_{n=1}^{\infty} G_n$. If $y \in \phi_{cc}(Z) \cap \phi_{cc}(Z^c)$ then $y \in {}^*G_n$ for all $n \in \mathbb{N}$. Obviously $(F_n^c)_{n \in \mathbb{N}}$ is a countable covering of cozero-sets of Z^c. Hence $y \in \bigcup_{n=1}^{\infty} {}^*F_n^c$, i.e., that there exists $n \in \mathbb{N}$ with $y \in {}^*F_n^c \subset {}^*G_{n+1}^c$, a contradiction.

The equivalence of 1) and 2) in Theorem 1.7 can also be derived from Theorem 1.2. For the extension of m_τ in Corollary 1.3 we have no explicit description: the next example shows that the functions ϕ_c and ϕ_{cc} are in general *not* σ-homomorphisms on the Borel σ-algebra. Note also that the inverse standard part map is only an extension if $\mathrm{st}^{-1}(X) = m_{\tau_0}(X) = {}^*X$ holds, i.e., that X is necessarily compact.

EXAMPLE 1.12. Let ω_1 be the first uncountable ordinal number and let X be the set of all ordinals smaller than ω_1 with the usual topology, cf. [30]. Then X is normal and countably compact and $X_\infty := X \cup \{\omega_1\}$ is the one-point compactification of X. Then $\phi_{cc}(X) = \phi_c(X) = {}^*X$ and $\phi_{cc}(\omega_1) = \phi_c(\omega_1) \neq \{\omega_1\}$ since ω_1 is not an isolated point. Hence ϕ_c and ϕ_{cc} are not σ-homomorphisms on the Borel σ-algebra of X_∞.

In the sequel we want to show that the covering function ϕ_c coincides with the inverse standard part map for a rather general class of topological spaces, cf. Corollary 1.15.

PROPOSITION 1.13. *Let (X, τ) be a topological space. If $Q \subset X$ is Lindelöf then $\phi_c(Q) = \mathrm{st}^{-1}(Q)$.*

PROOF. Let $(U_i)_{i \in I}$ be an open covering of Q. By the *Lindelöf property* there exists a countable subcovering $(U_n)_{n \in \mathbb{N}}$. For $y \in \phi_c(Q)$ we obtain $y \in \bigcup_{n=1}^{\infty} {}^*U_n \subset \bigcup_{i \in I} {}^*U_i$. Proposition 1.6 yields now $y \in \phi_\tau(Q) = \mathrm{st}^{-1}(Q)$. The other inclusion $\mathrm{st}^{-1}(Q) \subset \phi_c(Q)$ is always valid.

THEOREM 1.14. *Let (X, τ) be a topological space and let $Q \subset X$ be a subset. Then*

$\phi_c(Q) = \mathrm{st}^{-1}(Q)$ provided that there exists a weaker regular topology ρ such that Q is Lindelöf and every point of Q is a G_δ-set in X both with respect to ρ.

PROOF. The last proposition yields $\phi_c^\tau(Q) \subset \phi_c^\rho(Q) = \mathrm{st}_\rho^{-1}(Q)$. Let $y \in \phi_c^\tau(Q)$. Then there exists $x \in Q$ with $y \approx_\rho x$. Suppose that $y \not\approx_\tau x$. Then there exists $U \in \tau$ with $x \in U$ and $y \notin {}^*U$. Choose a sequence of $U_n \in \rho$ such that $\{x\} = \cap_{n=1}^\infty U_n$. By regularity we can assume that $\overline{U_{n+1}} \subset U_n$. Then $\{U\} \cup \{X\backslash\overline{U_n}: n \in \mathrm{N}\}$ is a countable τ-open covering of Q. Since $y \in \phi_c^\tau(Q)$ there exists $n \in \mathrm{N}$ with $y \in {}^*(X\backslash\overline{U_n})$. But $y \approx_\rho x$ implies $y \in {}^*U_m$ for all $m \in \mathrm{N}$, a contradiction.

COROLLARY 1.15. *Let τ be a topology on a set X finer than a metrizable separable topology. Then $\phi_c(Q) = \mathrm{st}^{-1}(Q)$ for every subset $Q \subset X$.*

The next result shows that in some cases $\phi_{\mathscr{U}}(F)$ is determined by $\phi_{\mathscr{U}}(X)$.

PROPOSITION 1.16. *Let τ_0 be a base of a topology τ on a set X. Let $\mathscr{U}$ be a set of families of sets in τ_0 and let $F \subset X$. Then $\phi_{\mathscr{U}}(F) = m_{\tau_0}(F) \cap \phi_{\mathscr{U}}(X)$ provided that the following two conditions are satisfied:*

(i) For every $\alpha \in \mathscr{U}$ with $F \subset \cup_{U\in\alpha} U$ there exists $U_1 \in \tau_0$ and $F_1 \in \tau_0^c$ with $F \subset U_1 \subset F_1 \subset \cup_{U\in\alpha} U$.

(ii) If $U \in \tau_0$ and $\alpha \in \mathscr{U}$ then $\alpha \cup \{U\} \in \mathscr{U}$.

PROOF. Observe that by (ii) every finite family of τ_0-sets is in $\mathscr{U}$. This implies $\phi_{\mathscr{U}}(F) \subset m_{\tau_0}(F)$ and $\phi_{\mathscr{U}}(F) \subset \phi_{\mathscr{U}}(X)$ is trivial. Now let $y \in m_{\tau_0}(F) \cap \phi_{\mathscr{U}}(X)$. Let $\alpha \in \mathscr{U}$ be a covering of F. Choose U_1, F_1 as in (i). Then $\beta := \alpha \cup \{F_1^c\} \in \mathscr{U}$ is a covering of X. Hence $y \in \cup_{U\in\alpha} {}^*U \cup {}^*F_1^c$. Now $y \in m_{\tau_0}(F)$ implies $y \in {}^*U_1 \subset {}^*F_1$. Hence $y \in \alpha$ pts ${}^*X$ and therefore $y \in \phi_{\mathscr{U}}(F)$.

THEOREM 1.17. *Let X be completely regular and $v: \mathscr{A} \to {}^*[0, \infty)$ be a finite internal content with ${}^\sigma\tau_0 \subset \mathscr{A}$. Then $\phi_{cc}: \sigma[\mathscr{Z}] \to L(v) \cap \phi_{cc}(X)$ is a σ-homomorphism extending $m_{\mathscr{Z}^c}^{\phi_{cc}(X)}$ defined on $\mathscr{Z}$.*

PROOF. Let $F \in \mathscr{Z}$. It is easy to see that condition (i) of Proposition 1.16 is satisfied since the countable union of cozero-sets is again a cozero-set. Hence $\phi_{cc}(F) = m_{\tau_0}(F) \cap \phi_{cc}(X) \in L(v) \cap \phi_{cc}(X)$. We apply now Theorem 1.10: the relation $\mathscr{Z} \subset \sigma_{\phi_{cc}}$ follows as in the proof of Theorem 1.7 2) $\Rightarrow$ 3).

The following theorem is a slight modification of Theorem 1.2 and it's proof is therefore omitted. Recall that the set $\mathrm{Loc}(\mathscr{F}) := \{A \subset X: A \cap F \in \mathscr{F}$ for all $F \in \mathscr{F}\}$ is the set of all *local sets* of the system $\mathscr{F}$. Note that $\mathrm{Loc}(\mathscr{F}) = \mathscr{F}$ if $\mathscr{F}$ contains the set X. Moreover $\mathrm{Loc}(\mathscr{F})$ is closed under finite intersections and unions if $\mathscr{F}$ has that property.

THEOREM 1.18. *Let $\tau_0, \mathscr{F}, v$ satisfy condition (*). If $F\backslash U \in \mathscr{F}$ for all $F \in \mathscr{F}$ and $U \in \tau_0$ then the following statements are equivalent:*

1. $m_{\tau_0}^{\mathscr{F}\text{pts}*X}$: $\text{Loc}(\mathscr{F}) \to L(v) \cap \mathscr{F}$ pts $*X$ *is extendible to a σ-homomorphism on* $\sigma[\text{Loc}(\mathscr{F})]$.

2. (II) *is satisfied and for every disjoint sets* $F_1, F_2 \in \text{Loc}(\mathscr{F})$ *and for every* $F \in \mathscr{F}$ *there exists* $U_1, U_2 \in \tau_0$ *with* $F_i \subset U_i$ *for* $i = 1, 2$ *and* $U_1 \cap U_2 \cap F = \emptyset$.

In general, condition (I) does not imply Theorem 1.18 2): take $\mathscr{F}$ as the set of all compact sets of the line $X := \mathsf{R}$. Then $\text{Loc}(\mathscr{F})$ is the set of all closed subsets. Let τ_0 be the set of all open and bounded subsets and choose $F_1 := (-\infty, -1]$ and $F_2 := [1, \infty)$.

2. An application: The inverse standard part map.

We call a topological space *prehausdorff* if $m(x) \cap m(y) \neq \emptyset$ implies $m(x) = m(y)$ for all $x, y \in X$. It is easy to see that every regular space and every Hausdorff space is prehausdorff, cf. [22] for details.

PROPOSITION 2.1. *Let* $v: \mathscr{A} \to *[0, \infty)$ *be an internal finite content and* τ_0 *be a subbase of a topology* τ *with* ${}^{\sigma}\tau_0 \subset \mathscr{A}$. *Assume that* $\text{st}^{-1}(U) \in L(v) \cap ns *X$ *for every* $U \in \tau$. *Then the following statements are equivalent:*

1. $\text{st}^{-1}: \sigma[\tau] \to L(v) \cap \text{ns} *X$ *is a σ-homomorphism.*
2. $\text{st}^{-1}: \sigma[\tau_0] \to L(v) \cap \text{ns} *X$ *is a σ-homomorphism.*
3. τ *is prehausdorff.*

PROOF. The direction 1) $\Rightarrow$ 2) is trivial. For 2) $\Rightarrow$ 3) it suffices to show that $m(x) \cap m(y) \neq \emptyset$ implies $*x \in m(y)$. Suppose there exists $U \in \tau$ with $y \in U$ and $*x \notin *U$. Since τ_0 is a subbase we can assume that $U \in \sigma[\tau_0]$ and by 2) we have $\text{st}^{-1}(U) \cap \text{st}^{-1}(X \setminus U) = \emptyset$. But for $z \in m(y) \cap m(x)$ we have $z \in \text{st}^{-1}(U)$ and $z \in \text{st}^{-1}(X \setminus U)$, a contradiction. For 3) $\Rightarrow$ 1) apply Theorem 1.10 with $\mathscr{B} := \mathscr{P}(X)$, $\tilde{\mathscr{B}} := L(v)$ and $T := \text{st}^{-1}$. Then it suffices to show that σ_T contains every open set $U \in \tau$. By assumption we have $\text{st}^{-1}(U) \in \tilde{\mathscr{B}}$. Suppose that there exists $z \in \text{st}^{-1}(U) \cap \text{st}^{-1}(U^c)$. Then there exists $x \in U$, $y \in U^c$ with $z \approx x$ and $z \approx y$. The prehausdorffness yields $*y \approx x \in U$. Since U is open we obtain $*y \in *U$, a contradiction. The proof is complete.

Now let τ_0 be a subbase of a *regular* topology. We show that the assumption $\text{st}^{-1}(U) \in L(v) \cap \text{ns} *X$ for $U \in \tau$ is satisfied: For $x \in U$ choose $V_{x,1}, \ldots, V_{x,n} \in \tau_0$ such that $x \in V_x := V_{x,1} \cap \ldots \cap V_{x,n} \subset \overline{V_x} \subset U$. Now it is easy to see that

$$\text{st}^{-1}(U) = \bigcup_{x \in U} *V_x \cap \text{ns} *X \tag{9}$$

But $\cup_{x \in U} *V_x$ is Loeb measurable since $*V_x \in \mathscr{A}$.

COROLLARY 2.2. *Let* τ_0 *be a subbase of a regular topology and* $v: \mathscr{A} \to *[0, \infty)$

be an internal content with ${}^{\sigma}\tau_0 \subset \mathscr{A}$. *Then* $\mathrm{st}^{-1}: \sigma[\tau] \to L(\nu) \cap \mathrm{ns}\,{}^*X$ *is a* σ*-homomorphism.*

The following example seems to be the simplest in order to show that st^{-1} is in general not a σ-homomorphism and that $\bar{\nu} \circ \mathrm{st}^{-1}$ and $\underline{\nu} \circ \mathrm{st}^{-1}$ are not measures even if $\mathrm{st}^{-1}(U)$ is Loeb measurable for every open set and ν is a Radon measure.

EXAMPLE 2.3. Let $X := \{0, 1\}$ be endowed with the Sierpinski topology $\{\emptyset, \{0\}, X\}$ and $\delta_i: \mathscr{P}(X) \to \{0, 1\}$ be the Dirac measure at $i = 0, 1$. Observe that both measures are Radon measures. But $\nu := \delta_0 \circ \mathrm{st}^{-1}$ is not additive since $\nu(\{0\}) + \nu(\{1\}) = 1 + 1$.

An example in [12] shows that Corollary 2.2 is even not true for the class of all Hausdorff spaces. On the other side it is well-known that $\overline{{}^*\mu} \circ \mathrm{st}^{-1}$ is a Borel measure provided that μ is a Radon measure on a Hausdorff space, cf. e.g. Theorem 4.4. For further results in this direction we refer to [3].

Finally we mention a slight modification of the inverse standard part map: define $\mathrm{cst}^{-1}: \mathscr{P}(X) \to \mathscr{P}({}^*X)$ by $\mathrm{cst}^{-1}(Q) := \mathrm{cpt}\,{}^*X \cap \mathrm{st}^{-1}(Q)$, where $\mathrm{cpt}\,{}^*X$ is the set of all *compact points*, i.e. the union of all ${}^*K$ with $K \subset X$ compact. It is not very difficult to see that a formula analogous to (9) is valid replacing st^{-1} by cst^{-1} and $\mathrm{ns}\,{}^*X$ by $\mathrm{cpt}\,{}^*X$ using the fact that every compact subset of a prehausdorff space is a regular subspace. Hence we have proved

COROLLARY 2.4. *Let* ${}^{\sigma}\tau_0 \subset \mathscr{A}$ *be a subbase of a prehausdorff space. Then the map* $\mathrm{cst}^{-1}: \sigma[\tau] \to L(\nu) \cap \mathrm{cpt}\,{}^*X$ *is a* σ*-homomorphism.*

Let X be a topological space and let $\mathscr{I}$ be the system of all internal subsets of ${}^*X$ and $\sigma[\mathscr{I}]$ the generated σ-algebra. Assume that $A \subset X$ is a *strong* G_δ*-set*, i.e., that there exists a sequence $(U_n)_{n\in\mathbb{N}}$ of open sets such that $A = \cap_{n=1}^{\infty} \overline{U_n} = \cap_{n=1}^{\infty} U_n$. It is easy to see that $\mathrm{st}^{-1}(A) = \cap_{n=1}^{\infty} {}^*U_n \cap \mathrm{ns}\,{}^*X$. Hence $\mathrm{st}^{-1}(A) \in \sigma[\mathscr{I}] \cap \mathrm{ns}\,{}^*X$ for every strong G_δ-set A in X. Applying this result to the system $\mathscr{Z}$ of all zero-sets we obtain that

$$\mathrm{st}^{-1}: \sigma[\mathscr{Z}] \to \sigma[{}^*Z: Z \in \mathscr{Z}] \cap \mathrm{ns}\,{}^*X \subset \sigma[\mathscr{I}] \cap \mathrm{ns}\,{}^*X. \tag{10}$$

It is a natural question whether $\mathrm{st}^{-1}: \sigma[\mathscr{Z}] \to \sigma[\mathscr{I}] \cap \mathrm{ns}\,{}^*X$ can be extended to a larger σ-algebra $\mathscr{M} \supset \sigma[\mathscr{Z}]$. Indeed, there exists a maximal σ-algebra $\mathscr{M}$ such that $\mathrm{st}^{-1}: \mathscr{M} \to \sigma[\mathscr{I}] \cap \mathrm{ns}\,{}^*X$ is a σ-homomorphism: define

$$\mathscr{M} := \{A \subset X: \mathrm{st}^{-1}(A) \in \sigma[\mathscr{I}] \cap \mathrm{ns}\,{}^*X \quad \text{and} \quad \mathrm{st}^{-1}(A) \cap \mathrm{st}^{-1}(A^c) = \emptyset\} \tag{11}$$

and observe that $\mathscr{M}$ is a σ-algebra, cf. Theorem 1.10. Henson has proved in [8] the remarkable result that $\mathscr{M}$ coincides with the Baire σ-algebra $\sigma[\mathscr{Z}]$ provided that X is a compact Hausdorff space. Furthermore he proved for a completely regular Hausdorff space X that $\mathrm{st}^{-1}(X)$ is in $\sigma[\mathscr{I}]$ iff X is a Baireset in the Stone-Čech compactification. A completely regular Hausdorff space with this property is called *bianalytic*. It follows from Theorem 9.10 in [6] that a bianalytic space is necessarily Lindelöf. But observe that the following Theorem is valid for an *arbitrary* topological space.

THEOREM 2.5. *Let A be a subset of a topological space and assume that* $\mathrm{st}^{-1}(A)$ *is in the σ-algebra $\sigma[\mathscr{I}]$ generated by the set $\mathscr{I}$ of all internal subsets. Then A is Lindelöf.*

PROOF. As pointed out in [8] we can assume that there exists a *countable* subalgebra $\mathscr{I}_0$ of the algebra $\mathscr{I}$ with $\mathrm{st}^{-1}(A)\in\sigma[\mathscr{I}_0]$. Then $[x] := \cap_{I\in\mathscr{I}_0, x\in I} I$ is in $\sigma[\mathscr{I}_0]$ and $y \sim x :\Leftrightarrow y\in[x]$ defines an equivalence relation on $^*X$ which is identical with the relation $x\in B \Rightarrow y\in B$ for all $B\in\sigma[\mathscr{I}_0]$. Now let $(U_j)_{j\in J}$ be an open covering of A and assume that $\mathrm{st}^{-1}(A)\in\sigma[\mathscr{I}_0]$. Let $\mathscr{I}_1$ be the set of all $I\in\mathscr{I}_0$ such that there exists a finite number of sets in $\{U_j: j\in J\}$, say $U_{I_1},\ldots,U_{I_n}$, with $I\subset {^*U_{I_1}}\cup\ldots\cup {^*U_{I_n}}$ and define U_I as the union of $U_{I_1},\ldots,U_{I_n}$. We show that the system of all open sets occurring in the finite union of U_I with $I\in\mathscr{I}_1$ is the desired countable open covering. Let $x\in A$. Then $[^*x] = \cap_{I\in\mathscr{I}_0} I \subset \mathrm{st}^{-1}(A) \subset \cup_{j\in J} {^*U_j}$ and by saturation there exists $I\in\mathscr{I}_0$ with $[^*x]\subset I\subset {^*U_I}$. Since I is in $\mathscr{I}_1$ the proof is complete.

COROLLARY 2.6. *Let X be completely regular and* $\mathrm{ns}\,^*X\in\sigma[\mathscr{I}]$, (e.g. *if X is compact). Then every Baire set is Lindelöf.*

COROLLARY 2.7. *A regular space X with* $\mathrm{ns}\,^*X\in\sigma[\mathscr{I}]$ *is completely regular.*

PROOF. A regular Lindelöf space is completely regular, cf. [30].

COROLLARY 2.8. *Let (X,τ) be a regular Hausdorff space. Then the following statements are equivalent:*

1. $\mathrm{st}^{-1}: \sigma[\tau]\to\sigma[\mathscr{I}]\cap \mathrm{ns}\,^*X$ *is a σ-homomorphism and* $\mathrm{st}^{-1}(X)\in\sigma[\mathscr{I}]$.
2. $\mathrm{st}^{-1}(U)\in\sigma[\mathscr{I}]$ *for every open set U.*
3. *Every subspace of X is Lindelöf and X is bianalytic.*
4. $\sigma[\mathscr{Z}] = \sigma[\tau]$ *and* $\mathrm{ns}\,^*X\in\sigma[\mathscr{I}]$.

PROOF. 1) $\Rightarrow$ 2) is trivial and for 2) $\Rightarrow$ 3) observe that by Theorem 2.5 every open subspace is Lindelöf; but it is well-known that then every subspace is Lindelöf. Henson's Theorem yields the equivalence of $\mathrm{st}^{-1}(X)\in\sigma[\mathscr{I}]$ and the bianalyticity of X. Also 3) $\Rightarrow$ 4) is clear since every open set is the countable union of cozero-sets. For 4) $\Rightarrow$ 1) observe that $\mathrm{st}^{-1}: \sigma[\mathscr{Z}]\to\sigma[\mathscr{I}]\cap\mathrm{ns}\,^*X$ is always a σ-homomorphism.

By formula (10) $\{\mathrm{st}^{-1}(A): A\in\sigma[\mathscr{Z}]\}$ is a σ-algebra contained in $\sigma[^*Z: Z\in\mathscr{Z}]\cap\mathrm{ns}\,^*X$. However, for Hausdorff spaces these σ-algebras are often different: they coincide if and only every $Z\in\mathscr{Z}$ is open; more general, let $F\subset X$ be such that $^*F\cap\mathrm{ns}\,^*X$ is in $\{\mathrm{st}^{-1}(A): A\in\sigma[\mathscr{Z}]\}$. Then there exists $A\subset X$ such that $^*F \cap \text{ns}\, ^*X = \text{st}^{-1}(A)$. But this identity implies $F = A$ by the Hausdorff property and F is open since $^*F$ contains every monad $m(x)$ for $x \in F = A$.

3. Representations of standard measures.

In our main result Theorem 3.4 of this section we make the following assumptions: Let $\mathscr{F}$ be a system of subsets and τ_0 be a system of subsets closed under finite unions such that $\tau_0^c \subset \mathscr{F}$ and let $v: \mathscr{A} \to {}^*[0, \infty)$ be an internal finite content with ${}^\sigma\tau_0 \subset \mathscr{A}$ and assume that we have constructed a σ-homomorphism $\phi: \sigma[\mathscr{F}] \to L(v) \cap Y$ such that

$$\phi(F) = \bigcap_{U \in \tau_0, F \subset U} {}^*U \cap Y \quad \text{for all} \quad F \in \mathscr{F}, \tag{12}$$

where Y is a fixed subset of $^*X$. It follows that $\bar{v} \circ \phi$ and $\underline{v} \circ \phi$ are measures on $\sigma[\mathscr{F}]$. Note also that $\sigma[\tau_0] \subset \sigma[\mathscr{F}]$. Our result answers the following question: if $\mu: \sigma[\tau_0] \to [0, \infty)$ is a (standard) measure with $v(^*X) \approx \mu(X)$, under which conditions holds the equality

$$\mu(A) = \bar{v} \circ \phi(A) \quad \text{for all} \quad A \in \sigma[\tau_0]. \tag{13}$$

We call $\bar{v} \circ \phi$ an *outer Loeb measure representation of* μ (with respect to ϕ) if $v(^*X) \approx \mu(X)$ and the last equality holds, and similar one defines an *inner Loeb measure representation.*

DEFINITION 3.1. Let $\mathscr{F}_0$ and τ_0 be systems of sets and let $\mu: \sigma[\mathscr{F}_0 \cup \tau_0] \to [0, \infty)$ be a function. Then μ is called τ_0-*outer regular at* $F \in \mathscr{F}_0$ if the equation $\mu(F) = \inf\{\mu(U): U \in \tau_0, F \subset U\}$ holds.

PROPOSITION 3.2. *Let $\mathscr{F}_0$ be a system of subsets closed under finite unions and τ_0 be a system of subsets with $\tau_0^c \subset \mathscr{F}_0$ and let $\mu_1, \mu_2: \sigma[\mathscr{F}_0] \to [0, \infty)$ be measures with $\mu_1(X) = \mu_2(X) < \infty$. If $\mu_1(F) \leqq \mu_2(F)$ for all $F \in \mathscr{F}_0$ and μ_1 is τ_0-outer regular at every $F \in \mathscr{F}_0$ then $\mu_1 = \mu_2$.*

The proof of proposition 3.2 is omitted. Moreover we need a technical Lemma.

LEMMA 3.3. *Let $v: \mathscr{A} \to {}^*[0, \infty)$ be an internal finite content and $Y \subset {}^*X$. If $v(^*X) = \bar{v}(Y)$ then $\bar{v}(A \cap Y) = \bar{v}(A)$ for all $A \in \mathscr{A}$. The same holds for $\underline{v}$.*

PROOF. Let $A \in \mathscr{A}$. By formula (4) we know that $\bar{v}(^*X) = \bar{v}(Y) = \bar{v}(Y \cap A) + \bar{v}(Y \cap A^c) \leqq \bar{v}(Y \cap A) + \bar{v}(A^c)$. Hence $\bar{v}(A) = \bar{v}(^*X) - \bar{v}(A^c) \leqq \bar{v}(Y \cap A)$. Now let $\underline{v}(Y) = \underline{v}(^*X)$. Then $\underline{v}(Y) = \bar{v}(Y) = \underline{v}(^*X)$, in particular Y is Loeb measurable. Hence we obtain $\underline{v}(Y) = \underline{v}(Y \cap A) + \underline{v}(Y \cap A^c)$ and we can now repeat the above argument.

THEOREM 3.4. *Under the above assumptions $\bar{v} \circ \phi$ ($\underline{v} \circ \phi$ respectively) is an outer* *(inner respectively) Loeb measure representation of μ on $\sigma[\tau_0]$ iff the following properties are satisfied:*

1. $\bar{\nu}(Y) = \bar{\nu}(^*X)$ ($\underline{\nu}(Y) = \underline{\nu}(^*X)$ *resp.*).
2. *μ is τ_0-outer regular at every $F \in \tau_0^c$.*
3. *$\nu(^*F) \precsim \mu(F)$ for all $F \in \tau_0^c$.*

PROOF. Let $\tilde{\nu}$ be either $\bar{\nu}$ or $\underline{\nu}$ and assume that $\mu(A) = \tilde{\nu} \circ \phi(A)$ holds for all $A \in \sigma[\tau_0]$. Since $\phi(X) = Y$ we obtain $\nu(^*X) \approx \mu(X) = \tilde{\nu}(\phi(X)) = \tilde{\nu}(Y)$. This proves 1). For $F \in \tau_0^c$ we have $^*F \cap Y \subset \phi(F)$, and our previous lemma yields $\tilde{\nu}(^*F) = \tilde{\nu}(^*F \cap Y) \leqq \tilde{\nu}(\phi(F)) = \mu(F)$ and 3) is proved. The equation (3) shows that

$$\mu(F) = \tilde{\nu}(\phi(F)) = \tilde{\nu}\Big(\bigcap_{U \in \tau_0,\, F \subset U} {}^*U \cap Y\Big) = \inf_U \tilde{\nu}(^*U \cap Y). \tag{14}$$

But $\tilde{\nu}(^*U \cap Y) = \tilde{\nu}(^*U)$ by Lemma 3.3 and $\nu(^*U) \succsim \mu(U)$ (take complements in condition 3)), hence we have $\mu(F) \geqq \inf\{\mu(U): U \in \tau_0, F \subset U\}$.

Now assume that 1), 2) and 3) are satisfied. Lemma 3.3 and 1) yields the equation $\tilde{\nu}(^*U \cap Y) = \tilde{\nu}(^*U)$ for all $U \in \tau_0$. Let $F \in \tau_0^c$. By our general assumption F is in $\mathscr{F}$ and with 2) and 3) we obtain

$$\mu(F) = \inf_U \mu(U) \leqq \inf_U \tilde{\nu}(^*U) = \inf_U \tilde{\nu}(^*U \cap Y) = \tilde{\nu}(\phi(F)). \tag{15}$$

For the last equality we have used equation (3). Hence we have proved that $\mu(F) \leqq \tilde{\nu} \circ \phi(F)$ for all $F \in \tau_0^c$. Since $\mu(X) \approx \nu(^*X) \approx \tilde{\nu}(Y) = \tilde{\nu} \circ \phi(X)$ an application of Proposition 3.2 with $\mathscr{F}_0 := \tau_0^c$ completes the proof.

REMARK 3.5. A short review of the proof shows that for the necessity part we did not used the fact that $\tilde{\nu} \circ \phi$ is a measure.

COROLLARY 3.6. *Let (X, τ) be normal and countably compact and ϕ the extension of m_τ as in Corollary 1.3 and $\mu: \sigma[\tau] \to [0, \infty)$ be a Borel measure. Then $\overline{^*\mu} \circ \phi$ is a Borel measure. The equality $\overline{^*\mu} \circ \phi = \mu$ holds iff μ is regular.*

PROOF. Apply Theorem 3.4 with $Y := {}^*X$, $\mathscr{F}$ as the set of all closed subsets and $\tau_0 := \tau$ and use Corollary 1.3.

COROLLARY 3.7. *Let (X, τ) be normal and countably compact. Then for every Baire measure μ there exists a Borel extension.*

PROOF. Apply Theorem 3.4 with $Y, \mathscr{F}$ as before and $\tau_0 := \mathscr{Z}^c$. By Theorem 1.2 we can extend $m_{\mathscr{Z}^c}: \mathscr{F} \to L(^*\mu)$ to a σ-homomorphism ϕ. Then $\overline{^*\mu} \circ \phi$ is a Borel measure and by Theorem 3.4 "$\Leftarrow$" it is an extension.

Representations of standard measures via the standard part map (for Hausdorff spaces) were intensively studied in [4, 12, 13, 16, 17]. It is now not very surprising that most of the results carry over to the class of regular spaces. As pointed out in [5, 12, 17] Loeb representations can be used to give nice characterizations of weakly compact subsets of the space of all τ-smooth measures and of all Radon measures respectively on completely regular Hausdorff spaces. These characterizations are still true for regular spaces and we include here the proof because all these results can be proved in a rather unified way.

We call a *subbase* τ_0 of a topological space *regular* if every closed subset and every point outside can be separated by τ_0-open sets. This is equivalent to say that $\mathrm{st}^{-1}(F) = m_{\tau_0}(F) \cap \mathrm{ns}\, {}^*X$ for every closed set F. Clearly every regular subbase induces a regular topology but the converse fails in general. The following Theorem was proved in [12] for the special case $\nu := {}^*\mu$.

THEOREM 3.8. *Let τ_0 be a regular base of a topology τ closed under finite unions and intersections and let $\mu: \sigma[\tau_0] \to [0, \infty)$ be a τ_0-outer regular measure at every $F \in \tau_0^c$. If $\nu: \mathscr{A} \to {}^*[0, \infty)$ is an internal finite content with ${}^\sigma\tau_0 \subset \mathscr{A}$ and $\nu({}^*X) \approx \mu(X)$ then the following statements are equivalent:*

1. $\bar{\nu} \circ \mathrm{st}^{-1}: \sigma[\tau] \to [0, \infty)$ *is a τ-smooth Borel extension of μ.*
2. $\nu({}^*F) \lesssim \mu(F)$ *for every $F \in \tau_0^c$ and $\mu(X) = \sup_{S \in \mathscr{S}} \mu(S)$ for every upward directed system $\mathscr{S} \subset \tau_0$ with $\mathscr{S} \uparrow X$.*
3. $\nu({}^*F) \lesssim \mu(F)$ *for all $F \in \tau_0^c$ and $\bar{\nu}(\mathrm{ns}\, {}^*X) = \bar{\nu}({}^*X)$.*
4. $\bar{\nu} \circ \mathrm{st}^{-1} = \mu$ *on $\sigma[\tau_0]$.*

PROOF. For 1) $\Rightarrow$ 2) apply Theorem 3.4 "$\Rightarrow$" with $\phi := \mathrm{st}^{-1}$ and $\mathscr{F} := \tau^c$. For 2) $\Rightarrow$ 3) let $A \in \mathscr{A}$ with $\mathrm{ns}\, {}^*X \subset A$; we have to prove that $\nu(A) \approx \nu({}^*X)$. Since $m(x) \subset A$ and A is internal we can find $U_x \in \tau_0$ with ${}^*U_x \subset A$. For every finite subset $E \subset X$ define $U_E := \cup_{x \in E} U_x$. Then $(U_E)_E \uparrow X$ and we obtain

$$\bar{\nu}(A) \geqq \bar{\nu}({}^*U_E) \geqq \mu(U_E) \uparrow \mu(X). \tag{16}$$

Hence $\bar{\nu}(\mathrm{ns}\, {}^*X) = \bar{\nu}({}^*X)$. The implication 3) $\Rightarrow$ 4) follows from Theorem 3.4 "$\Leftarrow$" with $\phi := \mathrm{st}^{-1}$. For 4) $\Rightarrow$ 1) we know that $\bar{\nu} \circ \mathrm{st}^{-1}$ is a Borel measure. Theorem 4 in [12] shows that $\bar{\nu} \circ \mathrm{st}^{-1}$ is τ-smooth.

We give now two applications of Theorem 3.8:

(i) *If τ_0 is the system of all cozero-sets of a completely regular space and μ is a Baire measure with the property that $\mu(X) = \sup_{S \in \mathscr{S}} \mu(S)$ for all $\mathscr{S} \subset \tau_0$ with $\mathscr{S} \uparrow X$ then $\overline{{}^*\mu} \circ \mathrm{st}^{-1}$ is a τ-smooth Borel extension.*

PROOF. Use 2) $\Rightarrow$ 1 and $\nu := {}^*\mu$.

(ii) *If τ is a regular topology and μ is a regular Borel measure then $\overline{{}^*\mu} \circ \mathrm{st}^{-1} = \mu$ holds iff μ is τ-smooth.*

PROOF. Use 2) $\Leftrightarrow$ 4) with $\tau_0 := \tau$.

Let X be a regular space and $M_\tau(X)$ be the set of all non-trivial τ-smooth measures on X. The weak topology is the weakest topology such that every map

$\hat{U}: M_\tau(X) \to [0, \infty)$ defined by $\hat{U}(\mu) := \mu(U)$ with $U \in \tau$ is lower semicontinuous and $\hat{X}: M_\tau(X) \to [0, \infty)$ is continuous. Hence $\nu \in {}^*M_\tau(X)$ is infinitesimal near to $\mu \in M_\tau(X)$ iff $\nu({}^*U) \gtrsim \mu(U)$ for all $U \in \tau$ and $\nu({}^*X) \approx \mu(X)$. Hence we have characterized in Theorem 3.8 the monads of the weak topology. As usual let $\text{fin}\,{}^*\mathrm{R} := \{x \in {}^*\mathrm{R}: |x| \leqq n \text{ for some } n \in \mathrm{N}\}$. We prove now that

$$\text{(17)} \qquad \text{ns}\,{}^*M_\tau(X) = \{\nu \in {}^*M_\tau(X): \bar{\nu}(\text{ns}\,{}^*X) \approx \nu({}^*X) \in \text{fin}\,{}^*\mathrm{R}\}.$$

PROOF. Let $\nu \in \text{ns}\,{}^*M_\tau(X)$. Then there exists $\mu \in M_\tau(X)$ with $\nu({}^*U) \gtrsim \mu(U)$ and $\nu({}^*X) \approx \mu(X)$. Theorem 3.8. 2) $\Rightarrow$ 3) shows that $\bar{\nu}(\text{ns}\,{}^*X) = \bar{\nu}({}^*X)$. Now let ν be in the set on the right hand side of (17). Define $\mu := \bar{\nu} \circ \text{st}^{-1}$ and apply Theorem 3.8 4) $\Rightarrow$ 1 $\Rightarrow$ 3) using the fact that $\bar{\nu}(\text{ns}\,{}^*X) = \bar{\nu} \circ \text{st}^{-1} = \mu$ is a regular measure, cf. Theorem 3.4.

It is well-known that $M_\tau(X)$ is a regular Hausdorff space if X is regular. Hence equation (17) yields the following nonstandard compactness criterion (proved in [12]): a family $\mathscr{P} \subset M_\tau(X)$ is weakly relatively compact iff $\bar{\nu}(\text{ns}\,{}^*X) = \bar{\nu}({}^*X) \in \text{fin}\,{}^*\mathrm{R}$ for every $\nu \in {}^*\mathscr{P}$. Lemma 2.4 in [12] shows that this is equivalent to the statement that $\sup_{S \in \mathscr{S}} \inf_{\mu \in \mathscr{P}} \{\mu(S)/\mu(X)\} = 1$ for every system $\mathscr{S} \subset \tau$ with $\mathscr{S} \uparrow X$ and $|\mu(X)| \leqq c$ for all $\mu \in \mathscr{P}$ and some $c \in \mathrm{R}$, i.e., that $\mathscr{P}$ is *uniformly τ-smooth* and *bounded*.

THEOREM 3.9. *Under the assumptions of Theorem* 3.8 *the following statements are equivalent:*

1. $\underline{\nu} \circ \text{st}^{-1}: \sigma[\tau] \to [0, \infty)$ *is a Radon extension of* μ.
2. $\nu({}^*F) \lesssim \mu(F)$ *for every* $F \in \tau_0^c$ *and for every* $\varepsilon > 0$ *there exists* $K \subset X$ *compact such that for all* $U \in \tau_0$ *with* $K \subset U$ *the relation* $\mu(U) \geqq \mu(X) - \varepsilon$ *holds.*
3. $\nu({}^*F) \lesssim \mu(F)$ *for all* $F \in \tau_0^c$ *and* $\underline{\nu}(\text{ns}\,{}^*X) = \underline{\nu}({}^*X)$.
4. $\underline{\nu} \circ \text{st}^{-1} = \mu$ *on* $\sigma[\tau_0]$.

PROOF. For 1) $\Rightarrow$ 2) apply Theorem 3.4 "$\Rightarrow$" and for 2) $\Rightarrow$ 3) choose a compact set K with the property stated in 2). Then $\underline{\nu}(\text{ns}\,{}^*X) \geqq \underline{\nu}(m_{\tau_0}(K)) = \inf_{U \in \tau_0, K \subset U} \underline{\nu}({}^*U) \geqq \inf_U \mu(U) \geqq \mu(X) - \varepsilon$. For the next implication 3) $\Rightarrow$ 4) apply Theorem 3.4 "$\Leftarrow$". For 4) $\Rightarrow$ 1) observe that $\underline{\nu} \circ \text{st}^{-1}$ is a Borel measure and by Theorem 4 in [12] it is a Radon measure.

Let X be a regular space and $M_{\mathrm{R}}(X)$ be the set of all Radon measures on X. An application of Theorem 3.9 shows that

$$\text{(18)} \qquad \text{ns}\,{}^*M_{\mathrm{R}}(X) = \{\nu \in {}^*M_{\mathrm{R}}(X): \underline{\nu}(\text{ns}\,{}^*X) \approx \nu({}^*X) \in \text{fin}\,{}^*\mathrm{R}\}.$$

Hence a subset $\mathscr{P} \subset M_{\mathrm{R}}(X)$ is weakly relatively compact if and only if $\underline{\nu}(\text{ns}\,{}^*X) = \underline{\nu}({}^*X) \in \text{fin}\,{}^*\mathrm{R}$ for all $\nu \in {}^*\mathscr{P}$. A family $\mathscr{P} \subset M_{\mathrm{R}}(X)$ is called *uniformly tight* if the relation $\sup_{K \text{ compact}} \inf_{\mu \in \mathscr{P}} \mu(K)/\mu(X) = 1$ holds. By Lemma 2.4 in [12] this is equivalent to the statement $\underline{\nu}(\text{cpt}\,{}^*X) = \underline{\nu}({}^*X)$ for all $\nu \in {}^*\mathscr{P}$. Hence every uniformly tight family of probability measures on a regular space is weakly relatively compact. A regular space for which the converse is also true is called a *Prohorov space.*

THEOREM 3.10. *Let X be a normal, countably compact space and $M_{\mathrm{reg}}(X)$ be the set of all regular Borel measures on X. Then $B_r := \{\mu \in M_{\mathrm{reg}}(X): \mu(X) \leqq r\}$ is compact in the weak topology for every $r \in [0, \infty)$.*

PROOF. Let $\nu \in {}^*B_r$ and let $\phi: \sigma[\tau] \to L(\nu)$ be the σ-homomorphism constructed in Corollary 1.3. Define $\mu := \bar{\nu} \circ \phi$. Theorem 3.4 "$\Rightarrow$" shows that μ is regular and that ν is infinitesimal near to μ.

A detailed discussion of the local compactness of all *Baire measures* can be found in [19]. The literature about compactness Theorems is very large and we refer to [11, 27, 28, 29] and the references given there.

The following Theorem generalizes slightly Theorem 3.8 in [2] which was formulated for $\tau_0 = \mathscr{Z}^c$. The proof rests on the results in [12].

THEOREM 3.11. *Let τ_0 be a base of a regular topology. Then the following statements are equivalent:*

1. ns $^*X \in L(\nu)$ *for all internal finite contents* $\nu: \mathscr{A} \to {}^*[0, \infty)$ *with* ${}^\sigma\tau_0 \subset \mathscr{A}$.
2. ns $^*X \in L(^*\mu)$ *for all Borel measures* μ.
3. *X is pre-radon.*

PROOF. The implication 1) $\Rightarrow$ 2) is trivial. Now let μ be a τ-smooth Borel measure. Then $\mu = \overline{^*\mu} \circ \mathrm{st}^{-1}$ and since ns $^*X \in L(^*\mu)$ we have $\overline{^*\mu} \circ \mathrm{st}^{-1} = \underline{^*\mu} \circ \mathrm{st}^{-1}$. Hence μ is Radon. Now let us prove 3) $\Rightarrow$ 1. Let ν be as in 1) and choose $A_2 \in \mathscr{A}$ with ns $^*X \subset A_2$ and $\nu(A_2) \leqq \bar{\nu}(\mathrm{ns}\, ^*X) + \varepsilon/2$. It suffices to construct $A_1 \subset \mathrm{ns}\, ^*X$ with $A_1 \in \mathscr{A}$ and $\nu(A_2) - \nu(A_1) < \varepsilon$. But $\bar{\nu} \circ \mathrm{st}^{-1}$ is τ-smooth and by 3) it is Radon. Choose K compact such that $\bar{\nu} \circ \mathrm{st}^{-1}(K) \geqq \bar{\nu} \circ \mathrm{st}^{-1}(X) - \varepsilon/2 =: \alpha$. We can assume that K is closed. Let τ_1 be the system of all finite unions of sets in τ_0. We know that $\mathrm{st}^{-1}(K) = \bigcap_{U \in \tau_1, K \subset U} {}^*U =: m_{\tau_1}(K)$; hence for every $U \in \tau_1$ with $K \subset U_1$ there exists $Z \in \mathscr{A}$ with $^*K \subset Z \subset {}^*U$ and $\nu(Z) \geqq \alpha$. By saturation we can find $Z \in \mathscr{A}$ such that $\nu(Z) \geqq \alpha$ and $Z \subset m_{\tau_1}(K) \subset \mathrm{ns}\, ^*X$. Define $A_1 = Z$ and observe that $\nu(A_2) - \nu(A_1) \leqq \bar{\nu}(\mathrm{ns}\, ^*X) + \varepsilon/2 - \alpha \leqq \varepsilon$.

4. Representability via partial maps.

THEOREM 4.1. *Let $\nu: \mathscr{A} \to \{0, 1\}$ be an internal two-valued content over $^*X$ and $s: Y \to X$ be a function with $Y \subset {}^*X$. Then either $\bar{\nu} \circ s^{-1}(x) = 1$ for some $x \in X$ or $\bar{\nu} \circ s^{-1} = 0$.*

PROOF. If $\bar{\nu}(s^{-1}(x)) \neq 0$ for some $x \in X$ it follows immediately that $\bar{\nu}(s^{-1}(x)) = 1$ since ν is two-valued. Assume now that $\bar{\nu}(s^{-1}(x)) = 0$ for all $x \in X$. Hence there exists $A_x \in \mathscr{A}$ with $s^{-1}(x) \subset A_x$ and $\nu(A_x) = 0$ (here we use that ν is two-valued). By Lemma 5 in [14] the set $A := \cup_{x \in X} A_x$ is again a null-set. Now $s^{-1}(X) \subset A$ implies that $\bar{\nu}(s^{-1}(X)) = 0$. Since $\bar{\nu} \circ s^{-1}$ is monotone the proof is complete.

A measure $\mu: \mathscr{B} \to [0, \infty)$ possesses a *Loeb representation by a partial map* if there exists a function $s: Y \to X$ (where ${}^{\sigma}X \subset Y \subset {}^*X$) with $s({}^*x) = x$ for all $x \in X$ satisfying the relations $s^{-1}(B) \in L({}^*\mu)$ for all $B \in \mathscr{B}$ and $\overline{{}^*\mu}(s^{-1}(B)) = \mu(B)$ for all $B \in \mathscr{B}$. Since s is measurable the latter condition is equivalent to $\underline{{}^*\mu}(s^{-1}(B)) = \mu(B)$ for all $B \in \mathscr{B}$.

COROLLARY 4.2. *Let $\mu: \mathscr{B} \to \{0, 1\}$ be a non-trivial two-valued measure with $\{x\} \in \mathscr{B}$ and $\mu(\{x\}) = 0$. Then μ does not possess a Loeb representation by a partial map.*

PROOF. Assume that μ has a representation. Then $\overline{{}^*\mu} \circ s^{-1}(x) = \mu(\{x\}) = 0$. Theorem 4.1 shows that μ is trivial, a contradiction.

As a concrete example one can take the measure μ defined by $\mu(A) = 0$ or 1 according as A is a countable or co-countable subset of an uncountable set X. On the other side, Lindstrøm has shown in [15] that every measure $\mu: \mathscr{B} \to [0, \infty)$ has a *weak Loeb measure representation*, i.e., that there exists a σ-algebra $\tilde{\mathscr{B}} \subset L({}^*\mu)$ and a *surjective* σ-homomorphism $\theta: \tilde{\mathscr{B}} \to B$ such that $\overline{{}^*\mu}(\tilde{B}) = \mu(\theta(\tilde{B}))$ for all $\tilde{B} \in \tilde{\mathscr{B}}$. His proof uses the above-mentioned extension theorem of R. Sikovski. This homomorphism can be explicitly described: define $\tilde{\mathscr{B}} := \sigma[{}^*B: B \in \mathscr{B}]$ and $\theta(\tilde{B}) := \tilde{B} \cap X$. Obviously $\theta: \tilde{\mathscr{B}} \to \mathscr{B}$ is a surjective σ-homomorphism and by Theorem II.2 in [10] we obtain the desired equation $\overline{{}^*\mu}(\tilde{B}) = \mu(\theta(\tilde{B}))$.

It is an interesting question which measures can be represented by a partial map. As in [24] a measure $\mu: \mathscr{B} \to [0, \infty)$ is called *compact* if it is regular with respect to a *compact family* $k \subset \mathscr{B}$. A family k of subsets is called *compact* if the intersection of every subfamily of k with the finite intersection property is non-empty. D. Ross has shown in [24] that every compact measure has a representation by a partial map; whether the converse is true is unclear and our result Theorem 4.6 concerns only the simple case of two-valued measures. A key observation is the following Proposition which implication part was proved in [24] and whose proof is included here only for completeness.

PROPOSITION 4.3. *Let $\mathscr{F}$ be a system of subsets of X. Then $\mathscr{F}$ is compact iff there exists a partial map $s: {}^*X \to X$ with ${}^*F \subset s^{-1}(F)$ for all $F \in \mathscr{F}$.*

PROOF. For ${}^*x \in {}^{\sigma}X$ define $s({}^*x) = x$. Let $y \in \mathscr{F}$ pts ${}^*X \setminus {}^{\sigma}X$. Then $\{F \in \mathscr{F}: y \in {}^*F\}$ has the finite intersection property. Since $\mathscr{F}$ is compact we can choose $x \in \cap_{F \in \mathscr{F}, y \in {}^*F} F$ and we put $s(y) := x$. For $y \in {}^*X \setminus \mathscr{F}$ pts ${}^*X$ we define $s(y) \in X$ arbitrary. Now it is easy to check that ${}^*F \subset s^{-1}(F)$ for all $F \in \mathscr{F}$. For the converse let $\mathscr{F}_0$ be a subfamily of $\mathscr{F}$ with the finite intersection property. By saturation there exists $y \in \cap_{F \in \mathscr{F}_0} {}^*F$. Define $x := s(y)$. Since $y \in {}^*F$ for all $F \in \mathscr{F}_0$ we obtain $x = s(y) \in s({}^*F) \subset s(s^{-1}(F)) = F$. Hence $x \in \cap_{F \in \mathscr{F}_0} F$.

The following Theorem is an abstract formulation of the fact that every Radon measure on a Hausdorff space is Loeb representable by the standard part map.

THEOREM 4.4 *Let $\mu: \mathscr{B} \to [0, \infty)$ be a finite measure and regular with respect to a system $\mathscr{F} \subset \mathscr{B}$ and let $v: \mathscr{A} \to {}^*[0, \infty)$ be an internal content with $v({}^*F) \approx \mu(F)$ for all $F \in \mathscr{F}$. If Y is a subset of ${}^*X$ with $v(Y) \approx v({}^*X)$ and $s: Y \to X$ satisfies the relation ${}^*F \cap Y \subset s^{-1}(F)$ for all $F \in \mathscr{F}$ then $s^{-1}(B) \in L(v)$ for all $B \in \mathscr{B}$ and $\underline{v} \circ s^{-1}(B) = \mu(B)$ for all $B \in \mathscr{B}$.*

PROOF. Let $B \in \mathscr{B}$. For every $\varepsilon > 0$ we can find $F_1, F_2 \in \mathscr{F}$ with $F_1 \subset B \subset F_2^c$ and $\mu(F_2^c) - \mu(F_1) < \varepsilon$ by regularity. On the other side we have

$$(19) \qquad {}^*F_1 \cap Y \subset s^{-1}(F_1) \subset s^{-1}(B) \subset s^{-1}(F_2^c) \subset {}^*F_2^c$$

Lemma 3.3 yields $\underline{v}({}^*F_1 \cap Y) = \underline{v}({}^*F_1)$ and therefore we have $\underline{v}({}^*F_2^c) - \underline{v}({}^*F_1 \cap Y) \leqq v(F_2^c) - \underline{v}({}^*F_1) \approx \mu(F_2^c) - \mu(F_1) < \varepsilon$. Since Y and therefore ${}^*F_1 \cap Y$ are Loeb measurable we infer the Loeb measurability of $s^{-1}(B)$. The above inequality and (19) yield also the equation $\underline{v}(s^{-1}(B)) = \mu(B)$.

Proposition 4.3 and Theorem 4.4 (with $v := {}^*\mu$, $Y := {}^*X$) yield now the following result due to D. Ross [24]:

COROLLARY 4.5. *Every compact measure possesses a Loeb representation by a partial map.*

Finally we show that for two-valued measures the converse is also true:

THEOREM 4.6. *A two-valued measure $\mu: \mathscr{B} \to \{0, 1\}$ possesses a Loeb representation by a partial map iff it is a Dirac measure iff it is compact.*

PROOF. Assume that μ possesses a Loeb representation. By Theorem 4.1 we have $\overline{{}^*\mu} \circ s^{-1}(\{x\}) = 1$ for some $x \in X$. Let $B \in \mathscr{B}$. If $x \in B$ then $\mu(B) = \overline{{}^*\mu}(s^{-1}(B)) \geqq \overline{{}^*\mu}(s^{-1}(\{x\}) = 1$. If $x \notin B$ then $x \in B^c$ and hence $\mu(B^c) = 1$, i.e. $\mu(B) = 0$. It follows that μ is the dirac measure δ_x. For the next implication observe that every Dirac measure δ_x is compact: choose $k := \{B \in \mathscr{B}: x \in B\}$. Corollary 4.5 yields the last implication.

J. Aldaz constructed in [2] a (two-valued) measure μ over N such that the Loeb measure $\overline{{}^*\mu}$ is not a compact measure therefore answering a question of D. Ross negatively. The following result is more general but it uses essentially the same proof.

THEOREM 4.7. *Let $v: \mathscr{A} \to \{0, 1\}$ be an internal two-valued non-trivial content* *with* $\{x\} \in \mathscr{A}$ *and* $v(\{x\}) = 0$ *for all* $x \in {}^*X$. *Then the Loeb measure* $\bar{v}: L(v) \to \{0, 1\}$ *is not compact.*

PROOF. Suppose there exists a compact family $k \subset L(v)$ such that $\bar{v}$ is regular with respect to k. Since $\bar{v}(\{x\}) = 0$ there exists $K_x \in k$ with $K_x \subset {}^*X \backslash \{x\}$ such that $1 = \bar{v}({}^*X \backslash \{x\}) = \bar{v}(K_x)$. It is easy to see that $\{K_x : x \in {}^*X\}$ has the finite intersection property. But $\cap_{x \in {}^*X} K_x \subset \cap_{x \in {}^*X} {}^*X \backslash \{x\} = \emptyset$, a contradiction.

5. Generalized set functions.

The definition of a content requires that the domain of the set function is an algebra. This section deals with set functions which are only defined on a *lattice*, i.e. a system of subsets containing $\emptyset$ and X which is closed under finite unions and finite intersections. Moreover we drop the finiteness condition of the set function. We call v a *set function* if it is a function on a lattice $\mathscr{L}$ into $[0, \infty]$ such that $v(\emptyset) = 0$ and v is *monotone*, i.e., that $A \subset B$ implies $v(A) \leqq v(B)$ for all $A, B \in \mathscr{L}$. The meaning of an *internal set function* should now be clear. We mention the following example which illustrates the advantages of our weakened assumptions: let $C(X)$ be the set of all continuous real-valued functions on a compact Hausdorff space X and $L: C(X) \to \mathsf{R}$ be a positive linear functional. Then

$$(20) \qquad v: {}^*\mathscr{Z} \to {}^*[0, \infty) \quad v(Z) := \text{internal} \inf\{{}^*L(f): f \in {}^*C(X), \chi_z \leqq f\}$$

defines an internal set function where χ_z denotes the characteristic function of $Z \in {}^*\mathscr{Z}$. It can be shown that $\bar{v} \circ \mathrm{st}^{-1}$ is a Radon measure with $L(f) = \int f d(\bar{v} \circ \mathrm{st}^{-1})$ for every $f \in C(X)$. This is of course the Riesz-Alexandroff representation Theorem, cf. [2, 16, 23, 31].

Now let $v: \mathscr{L} \to {}^*[0, \infty]$ be an internal set function over an internal set Z. As before we can define the *outer Loeb function* $\bar{v}: \mathscr{P}(Z) \to {}^*[0, \infty]$ and the *inner Loeb function* $\underline{v}: \mathscr{P}(Z) \to {}^*[0, \infty]$, if we use the extended real standard part map st: ${}^*[0, \infty] \to [0, \infty]$. A subset $Q \subset Y$ is called *Loeb integrable* if for every $\varepsilon > 0$ there exists $A_1, A_2 \in \mathscr{L}$ with $A_1 \subset Q \subset A_2$ and $v(A_2) \in \text{fin}\, {}^*\mathsf{R}$ and $v(A_2) - v(A_1) < \varepsilon$. We denote the set of all Loeb integrable subsets by $L(v)$; obviously $L(v)$ contains all $A \in \mathscr{L}$ with $v(A) \in \text{fin}\, {}^*\mathsf{R}$ but $L(v)$ is in general *not* an algebra even if v is a finite set function as the following example shows:

EXAMPLE 5.1. Let X be an uncountable set and $\mathscr{F}$ be the system of all finite subsets of X. Define a set function $\mu: \mathscr{F} \cup \{X\} \to [0, \infty)$ by $\mu(F) = 0$ for all $F \in \mathscr{F}$ and $\mu(X) = 1$. Then μ even satisfies the condition

$$(21) \qquad \mu(A \cup B) + \mu(A \cap B) = \mu(A) + \mu(B) \quad \text{for all} \quad A, B \in \mathscr{F} \cup \{X\}$$

It is easy to check that $L(^*\mu) = \{A \subset {}^*X : \exists E \text{ hyperfinite with } A \subset E\} \cup \{X\}$. Hence $L(^*\mu)$ is not an algebra.

An internal set function $v: \mathscr{L} \to {}^*[0, \infty]$ is called *subadditive* if $v(A \cup B) \leqq v(A) + v(B)$ for all $A, B \in \mathscr{L}$ and *additive* if $A \cap B = \emptyset$ implies $v(A \cup B) = v(\mathrm{A}) + v(\mathrm{B})$ for all $A, B \in \mathscr{L}$. A set function is called *modular* if the equation (21) holds. Trivially every modular set function is additive and subadditive.

It turns out that the set $L(v)$ is not so important as in the case of contents. Despite of this fact we prove the following theorem about $L(v)$.

THEOREM 5.2. *Let $v: \mathscr{L} \to {}^*[0, \infty)$ be an internal finite set function. If $\mathscr{S} \subset \mathscr{L}$ is a family of admissible cardinality then $\cup \mathscr{S}$ and $\cap \mathscr{S}$ are in $L(v)$. If v is modular then $L(v)$ is closed under countable unions and intersections.*

PROOF. The first statement follows by a modification of the proof of Theorem 1 in [12], cf. Theorem 3.5 in [2]. For the second statement let $B_i, C_i \in \mathscr{L}$ be given with $B_i \subset C_i$ for $i = 1, \ldots, n$ and let $\varepsilon_1, \ldots, \varepsilon_n \in [0, \infty)$. The modularity and an easy induction argument shows that $v(C_i) - v(B_i) < \varepsilon_i$ for $i = 1, \ldots, n$ implies that $v(C_1 \cup \ldots \cup C_n) - v(B_1 \cup \ldots \cup B_n) \leqq \varepsilon_1 + \ldots + \varepsilon_n$. It follows that $L(v)$ is closed under finite unions. Now let $(A_n)_{n \in \mathrm{N}}$ be a sequence in $L(v)$. We can assume that $A_n \subset A_{n+1}$ and let $\varepsilon > 0$. Choose $B_n, C_n \in \mathscr{L}$ with $B_n \subset A_n \subset C_n$ and $v(C_n) - v(B_n) \leqq \varepsilon 2^{-n}$. Let $\tilde{C}_n := C_1 \cup \ldots \cup C_n$ and $\tilde{B}_n := B_1 \cup \ldots \cup B_n$. Then $v(\tilde{C}_n) - v(\tilde{B}_n) \leqq \varepsilon$. Let $\alpha := \lim_{n \in \mathrm{N}} \bar{v}(\tilde{B}_n)$ and choose $n_0 \in \mathrm{N}$ with $v(\tilde{B}_n) - v(\tilde{B}_{n_0}) < \varepsilon$. For every $n \in \mathrm{N}$ with $n \geqq n_0$ we have $v(\tilde{C}_n) \leqq v(\tilde{B}_n) + \varepsilon \leqq v(\tilde{B}_{n_0}) + 2\varepsilon$. By the overflow principle there exists $N \in {}^*\mathrm{N} - \mathrm{N}$ with $v(\tilde{C}_N) - v(\tilde{B}_{n_0}) \leqq 2\varepsilon$. Since $\tilde{B}_{n_0} \subset \cup_{n=1}^{\infty} A_n \subset \tilde{C}_N$ the proof is complete. The proof for intersections is quite similar.

J. Aldaz proved in [2] that the function $\bar{v} \circ \mathrm{st}^{-1}$ restricted to $\sigma[\tau]$ is a Borel measure for an internal subadditive additive finite set function v *provided that* ns $^*X$ is Loeb integrable. We show in the sequel that this assumption is redundant. The following result is an easy consequence of the saturation principle.

PROPOSITION 5.3. *Let $v: \mathscr{L} \to {}^*[0, \infty]$ be an internal set function. For every upward directed system $\mathscr{S} \subset \mathscr{L}$ of admissible cardinality we have $\bar{v}(\cup_{S \in \mathscr{S}} S) = \sup_{S \in \mathscr{S}} \bar{v}(S)$. If $\mathscr{S} \subset \mathscr{L}$ is a downward directed system of admissible cardinality and $\bar{v}(S) < \infty$ for all $S \in \mathscr{S}$ then $\bar{v}(\cap_{S \in \mathscr{S}} S) = \inf_{S \in \mathscr{S}} \bar{v}(S)$. The same results are true for v.*

The following Theorem is due to P. Loeb, cf. [16].

THEOREM 5.4. *Let $v: \mathscr{L} \to {}^*[0, \infty]$ be an internal subadditive set function. Then $\bar{v}: \mathscr{P}(Z) \to [0, \infty]$ is an outer measure and the following equality holds:*

(22) $$\bar{\nu}(Q) = \inf\left\{\sum_{n=1}^{\infty} \operatorname{st} \nu(A_n): n \in \mathbb{N},\ A_n \in \mathscr{L}, Q \subset \cup_{n=1}^{\infty} A_n\right\}.$$

More can be proved if we assume that $\nu: \mathscr{L} \to {}^*[0, \infty]$ is an internal modular set function: then $\bar{\nu}$ is monotone and satisfies the following two conditions:

(i) $\bar{\nu}(\cup_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \bar{\nu}(A_n)$ for every increasing sequence $(A_n)_{n\in\mathbb{N}}$.

(ii) $\bar{\nu}(\cap_{n=1}^{\infty} B_n) = \lim_{n\to\infty} \bar{\nu}(B_n)$ for every decreasing sequence $(B_n)_{n\in\mathbb{N}}$ in $\mathscr{L}$ with $\bar{\nu}(B_n) < \infty$.

In other words: $\bar{\nu}$ is a non-negative $\mathscr{L}$-capacity in the sense of [7, p. 52] if ν is an internal finite modular set function. Here Property (ii) follows from Proposition 5.3 and the proof of (i) is omitted.

Let $\nu: \mathscr{L} \to {}^*[0, \infty]$ be an internal subadditive set function. We have seen that $L(\nu)$ is in general not an algebra and the equation (4) is not longer valid. We define $M(\bar{\nu})$ as the set of all $Q \subset Z$ such that the *outer measurability condition* $\bar{\nu}(A \cap Q) + \bar{\nu}(A \cap Q^c) = \bar{\nu}(A)$ is valid for all $A \subset Z$. It is well-known that $M(\bar{\nu})$ is an algebra and that $\bar{\nu}$ restricted to $M(\bar{\nu})$ is a content and $M(\bar{\nu})$ is the largest algebra with this property. If $\bar{\nu}$ is an outer measure then $M(\bar{\nu})$ is an σ-algebra and $\bar{\nu}$ restricted to $M(\bar{\nu}) \cap Y$ is a measure for all $Y \subset Z$. Moreover a subset $Q \subset Z$ is in $M(\bar{\nu})$ iff

(23) $$\nu(F) \geqq \bar{\nu}(F \cap Q) + \bar{\nu}(F \cap Q^c) \quad \text{for all} \quad F \in \mathscr{L}.$$

We emphasize that in general the sets $L \in \mathscr{L}$ are *not* contained in $M(\bar{\nu})$, cf. Proposition 5.9. Hence the measurability of the inverse standard part map must be proved in a different way as before. The following result is a sufficient criterion for outer measurability.

THEOREM 5.5. *Let $\nu: \mathscr{L} \to {}^*[0, \infty]$ be an internal subadditive and additive set function. Assume that Q is the intersection of a family $\mathscr{S} \subset \mathscr{L}^c$ of admissible cardinality and that for every $S \in \mathscr{S}$ there exists $Z \in \mathscr{L}$ with $Q \subset Z \subset S$. Then Q is in $M(\bar{\nu})$.*

PROOF. By assumption we have $Q = \cap_{S\in\mathscr{S}} S$ and we can assume that $\mathscr{S}$ is closed under finite intersections. We have to prove the inequality (23) for every $F \in \mathscr{L}$. For $S \in \mathscr{S}$ and $F \in \mathscr{L}$ choose $Z \in \mathscr{L}$ with $Q \subset Z \subset S$. Since $F \cap S^c$ and $F \cap Z$ are disjoint sets in $\mathscr{L}$ we obtain $\nu(F \cap S^c) + \nu(F \cap Z) \leqq \nu(F)$. Hence $\bar{\nu}(F \cap S^c) + \bar{\nu}(F \cap Q) \leqq \nu(F)$ since $Q \subset Z$. But $Q^c = \cup_{S\in\mathscr{S}} S^c$ and now Proposition 5.3 yields $\bar{\nu}(F \cap Q^c) + \bar{\nu}(F \cap Q) \leqq \bar{\nu}(F)$.

THEOREM 5.6. *Let τ_0 be a regular base of a topology τ and let $\nu: \mathscr{L} \to {}^*[0, \infty]$ be an internal subadditive and additive set function with ${}^{\sigma}\tau_0 \subset \mathscr{L}^c$. Then the inverse standard part function $\operatorname{st}^{-1}: \sigma[\tau] \to M(\bar{\nu}) \cap \operatorname{ns} {}^*X$ is a σ-homomorphism and $\bar{\nu} \circ \operatorname{st}^{-1}$*

is a measure on the Borel σ-algebra. If v is finite, or if v is modular and there exists a sequence of sets $A_n \subset {}^*X$ *with* $\mathrm{ns}\,{}^*X \subset \cup_{n=1}^{\infty} A_n$ *and* $\bar{v}(A_n) < \infty$, *then* $\bar{v} \circ \mathrm{st}^{-1}$ *is τ-smooth.*

PROOF. For the first statement it suffices to show that $\mathrm{st}^{-1}(F) \in M(\bar{v}) \cap \mathrm{ns}\,{}^*X$ for all closed subsets $F \subset X$. Let Q be the intersection of all ${}^*U$ with $U \in \tau_0$ such that there exists $Z \in \tau_0^c$ with $F \subset Z \subset U$. By Theorem 5.5 Q is in $M(\bar{v})$ and it suffices to show that $Q \cap \mathrm{ns}\,{}^*X = \mathrm{st}^{-1}(F)$. Let $y \in Q \cap \mathrm{ns}\,{}^*X$. Then there exists $x \in X$ with $y \approx x$. Suppose that $x \notin F$. Then there exists $U \in \tau_0$ with $x \in U$ which is disjoint to F. Choose a closed neighborhood $Z \in \tau_0^c$ with $x \in Z \subset U$. Since $y \approx x$ we have $y \in {}^*Z$, but $y \in Q$ and $F \subset U^c \subset Z^c$ implies $y \in {}^*Z^c$, a contradiction.

Now let $\mathscr{S} \subset \tau$ be upward directed. It suffices to show that $\bar{v} \circ \mathrm{st}^{-1}(\cup_{S \in \mathscr{S}} S) \leqq \sup_{S \in \mathscr{S}} \bar{v} \circ \mathrm{st}^{-1}(S)$ since $\bar{v}$ is monotone. For every $S \in \mathscr{S}$ and $x \in S$ we can find $U_x, V_x \in \tau_0$ and $Z_x \in \tau_0^c$ such that $x \in U_x \subset Z_x \subset V_x \subset {}^*\overline{V_x} \subset {}^*S$. It is easy to see that $\mathrm{st}^{-1}(\cup_{S \in \mathscr{S}} S) = \cup_{x \in X} {}^*U_x \cap \mathrm{ns}\,{}^*X$. We first prove that $\bar{v}(\mathrm{st}^{-1}(\cup \mathscr{S}) \cap A) \leqq \sup_{S \in \mathscr{S}} \bar{v}(\mathrm{st}^{-1}(S) \cap A)$ for every $A \subset \mathrm{ns}\,{}^*X$ with $\bar{v}(A) < \infty$. Choose $E \in \mathscr{L}$ with $A \subset E$ and $\bar{v}(E) \leqq \bar{v}(A) + \varepsilon$. By Proposition 5.3 $\bar{v}(\mathrm{st}^{-1}(\cup \mathscr{S} \cap A)) \leqq \bar{v}(\cup_{x \in X} {}^*U_x \cap E) = \sup_{x_1, \ldots, x_n \in X} \bar{v}(({}^*U_{x_1} \cup \ldots \cup {}^*U_{x_n}) \cap E)$. Let Q_{x_i} be the intersection of the family $\{{}^*V: V \in \tau_0, \exists Z \in \tau_0^c, U_{x_i} \subset Z \subset V\}$. Then ${}^*U_{x_i} \subset Q_{x_i} \subset {}^*V_{x_i} \subset {}^*S$ and $Q_{x_i} \in M(\bar{v})$. Now $\bar{v}(({}^*U_{x_1} \cup \ldots \cup {}^*U_{x_n}) \cap E) \leqq \bar{v}((Q_{x_1} \cup \ldots \cup Q_{x_n}) \cap E)$ and by the following Lemma we obtain as an upper bound $\bar{v}((Q_{x_1} \cup \ldots \cup Q_{x_n}) \cap A) + \varepsilon \leqq \bar{v}(({}^*V_{x_1} \cup \ldots \cup {}^*V_{x_n}) \cap A) + \varepsilon \leqq \bar{v}(\mathrm{st}^{-1}(S)) + \varepsilon$ for some $S \in \mathscr{S}$ since $\mathscr{S}$ is upward directed and $A \subset \mathrm{ns}^*X$ and ${}^*\overline{V_x} \subset {}^*S$. Now let $A_n \subset {}^*X$ with $\mathrm{ns}\,{}^*X \subset \cup_{n=1}^{\infty} A_n$ and $\bar{v}(A_n) < \infty$. Obviously we can assume that $A_n \subset \mathrm{ns}\,{}^*X$ and $A_n \subset A_{n+1}$. By property (i) we have $\bar{v}(\mathrm{st}^{-1}(\cup \mathscr{S})) = \lim_{n \to \infty} \bar{v}(\mathrm{st}^{-1}(\cup \mathscr{S}) \cap A_n)$ and we can apply the previous case. The result follows now immediately.

LEMMA 5.7. *Let* $v: \mathscr{L} \to {}^*[0, \infty]$ *be an internal subadditive set function and let* $E \in \mathscr{L}$ *and* $A \subset E$ *such that* $v(E) \leqq \bar{v}(A) + \varepsilon < \infty$. *Then* $\bar{v}(E \cap Q) \leqq \bar{v}(A \cap Q) + \varepsilon$ *for every* $Q \in M(\bar{v})$.

PROOF. We have $\bar{v}(E \cap Q) + \bar{v}(E \cap Q^c) = \bar{v}(E) \leqq \bar{v}(A) + \varepsilon \leqq \bar{v}(A \cap Q) + \bar{v}(A \cap Q^c) + \varepsilon \leqq \bar{v}(A \cap Q) + \bar{v}(E \cap Q^c) + \varepsilon$.

COROLLARY 5.8. *Let* τ_0 *be a regular base of a topology* τ *on X. Then X is pre-radon iff* $\mathrm{ns}\,{}^*X \in L(v)$ *for every internal subadditive additive finite setfunction* $v: \mathscr{L} \to {}^*[0, \infty)$ *with* ${}^\sigma\tau_0 \subset \mathscr{L}^c$.

PROOF. Use Theorem 5.6 and repeat the proof of Theorem 3.11 3) ⇒ 1).

PROPOSITION 5.9. *Let* $\mu: \mathscr{L} \to [0, \infty)$ *be a finite set function. Then* ${}^\sigma\mathscr{L}$ *is contained in* $M(\overline{{}^*\mu})$ *if and only if for every* $F_1, F_2 \in \mathscr{L}$ *with* $F_1 \subset F_2$ *the equality* $\mu(F_2) = \mu(F_1) + \inf_{L \in \mathscr{L}, F_2 \setminus F_1 \subset L} \mu(L)$ *holds.*

PROOF. For the necessity let $F_1 \subset F_2$ in $\mathscr{L}$. Since $^*F_1 \in M(\overline{^*\mu})$ the outer measurability condition (with $A := {}^*F_2$) yields $\mathrm{st}({}^*\mu({}^*F_2 \cap {}^*F_1)) + \overline{^*\mu}({}^*F_2 \setminus {}^*F_1) = \mathrm{st}({}^*\mu(F_1))$. Hence for every $\varepsilon > 0$ there exists $L \in {}^*\mathscr{L}$ with ${}^*F_2 \setminus {}^*F_1 \subset L$ and ${}^*\mu(L) \leqq {}^*\mu({}^*F_2) - {}^*\mu({}^*F_1) + \varepsilon$. Apply now the Transfer principle and let ε be arbitrary. For the converse let $Q \in {}^*\mathscr{L}$. By (23) it suffices to show that $\mathrm{st}\,{}^*\mu(F) = \mathrm{st}\,{}^*\mu(F \cap Q) + \overline{^*\mu}(F \cap Q^c)$ for all $F \in {}^*\mathscr{L}$. Define $F_2 := F$ and $F_1 := F \cap Q$. Then $F_1, F_2 \in {}^*\mathscr{L}$ and the Transfer applied to our assumption yields ${}^*\mu(F) = {}^*\mu(F \cap Q) + \text{internal } \inf_{L \in {}^*\mathscr{L}, F \cap Q^c \subset L} {}^*\mu(L)$. Taking standard parts the above equation is proved.

A set function $\mu: \mathscr{L} \to [0, \infty]$ satisfying the condition in Proposition 5.9 is usually called *complementary tight*.

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