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COMPUTING AND ESTIMATING THE GLOBAL DIMENSION IN CERTAIN CLASSES OF BANACH ALGEBRAS

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Let A be a Banach algebra (always nonzero). In this work, we shall study the homological dimension of a certain Banach A-module $M_+(A)$ of right multipliers on A. It turns out that the inequality

$$\mathrm{dh}_A M_+(A) \leqq \mathrm{dh}_A A + 2$$

holds and that this inequality often becomes equality. In that case A is not projective, we have $\mathrm{dh}_A M_+(A) \geqq 3$ and, as a consequence,

$$\mathrm{dg}\, A \geqq 3.$$

The latter estimate holds, for example, for all topologically nilpotent commutative Banach algebras and for a wide class of algebras $l^1(\omega)$, where ω is a radical weight and multiplication is by convolution.

The key result of the paper is Theorem 1. In this theorem, the homological dimension of an A-module X is calculated, given that the reduced module X_Π has certain properties. In Corollary 2, Theorem 1 is used to prove the homological infinite-dimensionality of A provided that $\mathrm{dh}_A A_\Pi \leqq \mathrm{dh}_A A$ and the operator

$$\sigma : A \mathop{\hat{\otimes}}_A A \to A^2 : a \mathop{\otimes}_A b \mapsto ab$$

is not an isomorphism; in particular, we have $\mathrm{dh}_A A = \infty$ for algebras such as the sequence algebra l_2 with coordinatewise multiplication and the algebra $\mathscr{H}\mathscr{S}(H)$ of Hilbert-Schmidt operators on a Hilbert space H. In Theorem 2, it is shown that $\mathrm{dh}_A A = \infty$ for all nilpotent Banach algebras. The homological dimension of the A-module $M_+(A)$ is calculated in Theorem 3. Finally, Theorems 4–6 are devoted to estimating the global dimension of algebras $l^1(\omega)$ and topologically nilpotent Banach algebras.

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§1. Preliminaries.

Let A be a Banach algebra, not necessarily with an identity, and let A_+ be the Banach algebra obtained by adjoining an identity to A. By an A-module we mean a left Banach module over A. The categories of A-modules and Banach spaces will be denoted by A-mod and Ban; the corresponding sets of morphisms from X to Y will be denoted by ${}_Ah(X, Y)$ and $\mathscr{B}(X, Y)$. The fundamental homological concepts for the categories of Banach modules (the homological dimension, $\mathrm{dh}_A X$, of $X \in A$-mod, projectivity, the global dimension, $\mathrm{dg}\, A$, of A, the cohomology groups of A and others) are assumed to be known; they are set out in detail in [1]. We review some considerations from this book.

The canonical morphism for an A-module X means the morphism $\pi \in {}_Ah(A \hat{\otimes} X, X)$ defined by $\pi(a \otimes x) = a \cdot x$ $(a \in A, x \in X)$. Here $A \hat{\otimes} X$ is the A-module with the left outer multiplication given by $a \cdot (b \otimes x) = ab \otimes x$ $(a, b \in A, x \in X)$, where $\hat{\otimes}$ denotes the projective tensor product of Banach spaces (see [2]).

The closure of the image of the morphism π is called the essential part of the A-module X and is denoted by $A \cdot X$. An A-module X is said to be essential if $A \cdot X = X$, and annihilator if $A \cdot X = 0$. We note that an essential A-module X is projective if and only if the morphism $\pi: A \hat{\otimes} X \to X$ is a retraction in A-mod.

We denote by A^2 the essential part, $A \cdot A$, of the A-module A. For each $n > 2$, A^n denotes $A \cdot A^{n-1}$. A Banach algebra A is said to be idempotent if $A^2 = A$, and nilpotent if $A^n = 0$ for some n.

Let E be a Banach space. $\mathscr{B}(E)$ and $\mathscr{K}(E)$ will denote the Banach algebras of all continuous and all compact operators on E respectively, and $\mathscr{N}(E)$ will denote the Banach algebra of all nuclear operators on E. We recall that a Banach space E is said to have the approximation property if every compact operator from an arbitrary Banach space into E can be approximated in norm by finite rank operators. The property is discussed in [1], [2], [3] and [4].

We denote by c_0 the Banach algebra of all sequences tending to zero, with coordinatewise multiplication. Finally, the sequence algebra $l_p (1 \leqq p < +\infty)$ consists of those $\xi = \{\xi_n\}$ for which $\|\xi\| = (\sum_{n=1}^{\infty} |\xi_n|^p)^{1/p}$ is finite.

§2. The reduced module and the homological dimension.

We recall (see [1, II, §5.3]) that there is the so-called reduced module $X_\Pi = A \hat{\otimes}_A X$ associated with any left A-module X. Let $\mathscr{X}$: $X_\Pi \to X$ be the morphism of A-modules defined by $\mathscr{X}(a \otimes_A x) = a \cdot x (a \in A, x \in X)$.

We shall prove the following theorem.

THEOREM 1. *Let A be a Banach algebra such that* $\mathrm{dh}_A A = n < \infty$, *and let* $X \in A$-mod. *Then, if* $\mathrm{dh}_A X_\Pi > n + 1$, *we have* $\mathrm{dh}_A X = \mathrm{dh}_A X_\Pi$, *and if* $\mathrm{dh}_A X_\Pi \leqq n$, *then:*

(i) $\mathrm{dh}_A X \leqq n + 2$;

(ii) *if* $\mathscr{X}: X_\Pi \to X$ *is a coretraction in* Ban, *then* $\mathrm{dh}_A X < n + 2$;

(iii) *if* $\mathscr{X}: X_\Pi \to X$ *is not a topologically injective operator, and A does not have a right identity, then* $\mathrm{dh}_A X = n + 2$.

We preface to the proof of Theorem 1 a lemma, which is related to [5, Theorem 1].

LEMMA 1. *Let A be a Banach algebra, and let* $\tau: X_0 \to X$ ($X_0, X \in A$-mod) *be a morphism of A-modules. Further, let E_0 and E be Banach spaces, and* $v: E_0 \to E$ *an operator which is not topologically injective. Consider the morphism of A-modules*

$$\Delta: X_0 \hat{\otimes} E_0 \to (X \hat{\otimes} E_0) \oplus (X_0 \hat{\otimes} E)$$

$$\Delta(x \otimes y) = (\tau(x) \otimes y, x \otimes v(y)) \; (x \in X_0,\, y \in E_0).$$

Then the following are equivalent:

(i) *the morphism Δ is a coretraction;*

(ii) *the morphism τ is a coretraction.*

PROOF. Trivially, (ii) implies (i). To show that (i) implies (ii), suppose (i) holds. This means that there exists a morphism of A-modules

$$\nabla: (X \hat{\otimes} E_0) \oplus (X_0 \hat{\otimes} E) \to X_0 \hat{\otimes} E_0$$

that is a left inverse to Δ. Let $\varphi: X \hat{\otimes} E_0 \to X_0 \hat{\otimes} E_0$ (respectively, $\psi: X_0 \hat{\otimes} E \to X_0 \hat{\otimes} E_0$) be the restriction of ∇ to the first (respectively, second) direct summand. Then φ and ψ are morphisms of A-modules such that

$$\varphi(\tau(x) \otimes y) + \psi(x \otimes v(y)) = x \otimes y \quad (x \in X_0, y \in E_0). \tag{1}$$

Since the operator v is not topologically injective, there exists a sequence $\{y_n\}_{n=1}^{\infty}$, $y_n \in E_0$, such that for all n $\|y_n\| = 1$, and $\|v(y_n)\| = \alpha_n$, where $\lim_{n\to\infty} \alpha_n = 0$. Let $f_n \in (E_0)^*$, $1 \leqq n < \infty$, be such that $f_n(y_n) = \|f_n\| = 1$. For $n = 1, 2, \ldots, z \in X$, set

$$\varphi_n(z) = (1_{X_0} \hat{\otimes} f_n)\varphi(z \otimes y_n).$$

Then clearly, for each n, $\varphi_n: X \to X_0$ is a morphism of A-modules. From (1), we see that

$$\varphi_n(\tau(x)) + (1_{X_0} \hat{\otimes} f_n)\psi(x \otimes v(y_n)) = x$$

for all n and for all $x \in X_0$. It is clear that

$$\|(1_{X_0} \hat{\otimes} f_n)\psi(x \otimes v(y_n))\| \leqq \|\psi\| \, \|x\| \, \alpha_n$$

for $n = 1, 2, \ldots, x \in X_0$. It follows that for all n

$$\|\varphi_n \circ \tau - 1_{X_0}\|_{\mathscr{B}(X_0)} \leqq \|\psi\| \, \alpha_n.$$

Since $\lim_{n\to\infty} \alpha_n = 0$, there is a number m such that

$$\|\varphi_m \circ \tau - 1_{X_0}\| < 1. \tag{2}$$

Now consider the Banach algebra $B = {}_A h(X_0, X_0)$, which is a closed subalgebra in $\mathscr{B}(X_0)$. It is clear that the element $e = 1_{X_0}$ is an identity of B. From (2), for $b = \varphi_m \circ \tau$ we have $\|b - e\|_B < 1$. It follows that b is an invertible element of the algebra B. Set $\zeta = b^{-1} \in B$. Then clearly $\zeta \circ \varphi_m \circ \tau = 1_{X_0}$. Consequently, the morphism of A-modules $\zeta \circ \varphi_m$ is a left inverse to τ, i.e., (ii) holds.

PROOF OF THEOREM 1. By Theorem V.2.1 of [1], for some A-module W there exist short admissible complexes of A-modules

$$0 \leftarrow W \leftarrow V \overset{\Delta_1}{\leftarrow} A \hat{\otimes} X_\Pi \leftarrow 0 \tag{3}$$

and

$$0 \leftarrow X \leftarrow U \leftarrow W \leftarrow 0, \tag{4}$$

where $V = (A_+ \hat{\otimes} X_\Pi) \oplus (A \hat{\otimes} X)$, $U = (A_+ \hat{\otimes} X) \oplus X_\Pi$,

$$\Delta_1(a \otimes x) = (a \otimes x, a \otimes \mathscr{X}(x)) \quad (a \in A \subset A_+, x \in X_\Pi).$$

Since $\mathrm{dh}_A A = n$, it is clear that $\mathrm{dh}_A A \otimes X_\Pi \leqq n$ and $\mathrm{dh}_A V \leqq n$. Using (3) and Proposition III.5.5 of [1], we have

$$\mathrm{dh}_A W \leqq \max\{\mathrm{dh}_A V, \mathrm{dh}_A A \hat{\otimes} X_\Pi + 1\} \leqq n + 1. \tag{5}$$

Set $m = \mathrm{dh}_A X_\Pi$, and suppose that $n + 1 < m < \infty$. Using Proposition III.5.5 of [1], (4) and (5), we have $\mathrm{dh}_A U = m$ and

$$\mathrm{dh}_A X \leqq \max\{\mathrm{dh}_A U, \mathrm{dh}_A W + 1\} \leqq m. \tag{6}$$

The short admissible complex (4) defines, for any A-module Y, the exact sequence of groups

$$\ldots \to \mathrm{Ext}_A^m(X, Y) \to \mathrm{Ext}_A^m(U, Y) \to \mathrm{Ext}_A^m(W, Y) \to \ldots \tag{7}$$

(see [1, Theorem III.4.4]). Since $\mathrm{dh}_A U = m$, there exists an A-module Y such that $\mathrm{Ext}_A^m(U, Y) \neq 0$. It follows from (5) that $\mathrm{Ext}_A^m(W, Y) = 0$. Since the sequence (7) is exact, we have $\mathrm{Ext}_A^m(X, Y) \neq 0$. In view of (6), $\mathrm{dh}_A X = m = \mathrm{dh}_A X_\Pi$.

We shall prove now that, if $m = \infty$, then $\mathrm{dh}_A X = \infty$.

Indeed, if $\mathrm{dh}_A X < \infty$, then, using (4) and Proposition III.5.5 of [1], we have, in view of (5),

$$m = \mathrm{dh}_A U \leqq \max\{\mathrm{dh}_A X, \mathrm{dh}_A W\} < \infty.$$

Now suppose that $m \leqq n$. Using (4) and (5), we have

$$\mathrm{dh}_A X \leqq \max\{\mathrm{dh}_A U, \mathrm{dh}_A W + 1\} \leqq n + 2,$$

i.e., (*i*) holds. If, in addition, $\mathscr{X} : X_\Pi \to X$ is a coretraction in Ban, then the short exact sequence

$$0 \leftarrow X/A \cdot X \leftarrow X \xleftarrow{\mathscr{X}} X_\Pi \leftarrow 0 \tag{8}$$

is admissible. Using the obvious isomorphism of A-modules between $X/A \cdot X$ and $\mathbf{C} \hat{\otimes} X/A \cdot X$, where $\mathbf{C} = A_+/A$ is the one-dimensional annihilator A-module, we see that $\mathrm{dh}_A X/A \cdot X \leqq n + 1$. Using (8), we have

$$\mathrm{dh}_A X \leqq \max\{\mathrm{dh}_A X/A \cdot X, \mathrm{dh}_A X_\Pi\} \leqq n + 1,$$

i.e., (ii) holds.

We now assume that $m \leqq n$, that $\mathscr{X} : X_\Pi \to X$ is not a topologically injective operator and that A does not have a right identity. To obtain a contradiction, suppose that $\mathrm{dh}_A X < n + 2$. Then, using (4) and Proposition III.5.5 of [1], we have

$$\mathrm{dh}_A W \leqq \max\{\mathrm{dh}_A U, \mathrm{dh}_A X - 1\} < n + 1. \tag{9}$$

Let us consider the case where $n > 0$. The short admissible complex (3) defines, for any A-module Y, the exact sequence of groups

$$\ldots \to \mathrm{Ext}_A^n(V, Y) \to \mathrm{Ext}_A^n(A \hat{\otimes} X_\Pi, Y) \to \mathrm{Ext}_A^{n+1}(W, Y) \to \ldots, \tag{10}$$

where $\mathrm{Ext}_A^{n+1}(W, Y) = 0$, in view of (9). Since the A-module $A_+ \hat{\otimes} X_\Pi$ is projective, it follows that

$$\mathrm{Ext}_A^n(V, Y) = \mathrm{Ext}_A^n(A \hat{\otimes} X, Y),$$

recalling that $n > 0$. Therefore, the segment (10) of the long exact sequence for the group Ext takes the form

$$\mathrm{Ext}_A^n(A \hat{\otimes} X, Y) \xrightarrow{\delta} \mathrm{Ext}_A^n(A \hat{\otimes} X_\Pi, Y) \to 0.$$

Consequently, the morphism of groups $\delta = \mathrm{Ext}_A^n(1_A \hat{\otimes} \mathscr{X}, Y)$ is an epimorphism for any A-module Y.

Since $\mathrm{dh}_A A = n$, there is a projective resolution

$$0 \leftarrow A \leftarrow P_0 \xleftarrow{d_0} P_1 \xleftarrow{d_1} \ldots \tag{11}$$

of the A-module A with $P_k = 0$ for $k > n$. One can compute the morphism δ considered above by using the following commutative diagram

$$\begin{array}{ccccccc} 0 \leftarrow A \hat{\otimes} X_\Pi \leftarrow & P_0 \hat{\otimes} X_\Pi & \xleftarrow{d_0 \hat{\otimes} 1} & P_1 \hat{\otimes} X_\Pi & \xleftarrow{d_1 \hat{\otimes} 1} & \dots \\ \downarrow 1 \hat{\otimes} \mathscr{X} & \downarrow 1 \hat{\otimes} \mathscr{X} & & \downarrow 1 \hat{\otimes} \mathscr{X} & & \\ 0 \leftarrow A \hat{\otimes} X \leftarrow & P_0 \hat{\otimes} X & \xleftarrow{d_0 \hat{\otimes} 1} & P_1 \hat{\otimes} X & \xleftarrow{d_1 \hat{\otimes} 1} & \dots \end{array}$$

obtained from (11). It is easy to check that the morphism of groups δ is induced by the operator

$$\lambda : {}_A h(P_n \hat{\otimes} X, Y) \to {}_A h(P_n \hat{\otimes} X_\Pi, Y),$$

where $\lambda = {}_A h(1_{P_n} \hat{\otimes} \mathscr{X}, Y)$.

Now set $Y = P_n \hat{\otimes} X_\Pi$, and consider the element defined by $1_y \in {}_A h(Y, Y)$, of the group

$$\operatorname{Ext}^n_A (A \hat{\otimes} X_\Pi, Y) = {}_A h(Y, Y)/\operatorname{Im} \psi_\Pi,$$

where $\psi_\Pi = {}_A h(d_{n-1} \hat{\otimes} 1_{X_\Pi}, Y)$. This element belongs to $\operatorname{Im} \delta$, since δ is an epimorphism. It follows that there exist morphisms of A-modules ξ: $P_{n-1} \hat{\otimes} X_\Pi \to Y$ and $\eta : P_n \hat{\otimes} X \to Y$ such that

$$1_y = \psi_\Pi(\xi) + \lambda(\eta).$$

But $\psi_\Pi(\xi) = \xi \circ (d_{n-1} \hat{\otimes} 1_{X_\Pi})$ and $\lambda(\eta) = \eta \circ (1_{P_n} \hat{\otimes} \mathscr{X})$, and hence $x \otimes y = \xi(d_{n-1}(x) \otimes y) + \eta(x \otimes \mathscr{X}(y))$ $(x \in P_n, y \in X_\Pi)$.

Consequently, the morphism

$$\Delta : P_n \hat{\otimes} X_\Pi \to (P_{n-1} \hat{\otimes} X_\Pi) \oplus (P_n \hat{\otimes} X),$$

defined by

$$\Delta(x \otimes y) = (d_{n-1}(x) \otimes y, x \otimes \mathscr{X}(y)) \quad (x \in P_n, y \in X_\Pi),$$

is a coretraction. From Lemma 1 we see that the morphism $d_{n-1} : P_n \to P_{n-1}$ is a coretraction. But then obviously $\operatorname{dh}_A A < n$. Since $n = \operatorname{dh}_A A$, we obtain a contradiction. Thus, if $n > 0$, then $\operatorname{dh}_A X = n + 2$.

Now let $n = 0$. From (9), the A-module W is projective. Therefore, the short admissible complex (3) splits. It follows that the morphism of A-modules $\Delta_1 : A \hat{\otimes} X_\Pi \to V$ is a coretraction. From Lemma 1 we see that the morphism of A-modules $i : A \to A_+$ (the natural embedding of A in A_+) is a coretraction. Then A has a right identity. But we have assumed that A does not have right identity. This finishes the case where $n = 0$; and the theorem is completely proved.

COROLLARY 1. *Let X be a Banach module over a biprojective Banach algebra*

A which does not have a right identity. Then, if $\mathscr{X}: X_{\Pi} \to X$ *is not a topologically injective operator, then* $\mathrm{dh}_A X = 2$.

The above corollary shows, for example, that $\mathrm{dh}_{l_1} c_0 = 2$ (see [6]) and that $\mathrm{dh}_{l_1} l_p = 2 (1 < p < +\infty)$.

The second example is the algebra $A = E \hat{\otimes} E^*$ (E is any infinite-dimensional Banach space) with multiplication given by

$$(x_1 \otimes f_1)(x_2 \otimes f_2) = \langle x_2, f_1 \rangle x_1 \otimes f_2.$$

From Corollary 1 we obtain that $\mathrm{dh}_A \mathscr{K}(E) = \mathrm{dh}_A \mathscr{B}(E) = 2$ (see [4]).

Now set $X = A$, and consider the operator $\sigma: A \hat{\otimes}_A A \to A^2$ defined by $\sigma(a \otimes_A b) = ab$ $(a, b \in A)$. From Theorem 1 we obtain the following corollary.

COROLLARY 2. *Let A be a Banach algebra such that the operator* $\sigma: A \hat{\otimes}_A A \to A^2$ *is not an isomorphism. Then, if* $\mathrm{dh}_A A_{\Pi} \leqq \mathrm{dh}_A A$, *then* $\mathrm{dh}_A A = \infty$.

For example, if $A = l_2$ with coordinatewise multiplication, then $A_{\Pi} = l_1$, and $\mathrm{dh}_A l_1 = 0$. Therefore, $\mathrm{dh}_{l_2} l_2 = \infty$ and hence $\mathrm{dg}\, l_2 = \infty$.

The second example is the algebra $A = \mathscr{HS}(H)$ of Hilbert-Schmidt operators on a Hilbert space H. It is easy to see that $A_{\Pi} = \mathscr{N}(H)$ and hence $\mathrm{dh}_A A_{\Pi} = 0$. By Corollary 2, $\mathrm{dh}_A A = \infty$ and $\mathrm{dg}\, A = \infty$.

The third example is the algebra $A = \mathscr{N}(E)$, where E is a Banach space without the approximation property. One can show that $A_{\Pi} = E \hat{\otimes} E^*$, and $\sigma: A \hat{\otimes}_A A \to A^2$ is the so-called trace homomorphism Tr: $E \hat{\otimes} E^* \to \mathscr{N}(E)$ defined by $\mathrm{Tr}(x \otimes f)(y) = \langle y, f \rangle x$ $(x, y \in E,\ f \in E^*)$. Since E does not have the approximation property, $\mathrm{Ker}\,\mathrm{Tr} \neq 0$ (see [2]). It is easy to see that $\mathrm{dh}_A A_{\Pi} = 0$. From Corollary 2, we have $\mathrm{dh}_{\mathscr{N}(E)} \mathscr{N}(E) = \infty$, and $\mathrm{dg}\, \mathscr{N}(E) = \infty$ (see [4]).

THEOREM 2. *Let A be a nilpotent Banach algebra. Then* $\mathrm{dh}_A A = \infty$ *and, as a consequence,* $\mathrm{dg}\, A = \infty$.

The main part of the proof of Theorem 2 is the following lemma.

LEMMA 2. *Let A be a Banach algebra without a right identity, and let* $\mathrm{dh}_A A = n < \infty$. *Then, if for some* $k \geqq 2$ $\mathrm{dh}_A A/A^k = n + 1$, *then* $\mathrm{dh}_A A/A^{k+1} = n + 1$.

PROOF OF LEMMA. Consider the short admissible complex of A-modules

$$0 \leftarrow A_+/A \leftarrow A_+/A^k \leftarrow A/A^k \leftarrow 0.$$

Using Proposition III.5.5 of [1] and the equality

$\mathrm{dh}_A A/A^k = n + 1$, we have

(12) $$\mathrm{dh}_A A_+/A^k \leqq \max\{\mathrm{dh}_A A_+/A, \mathrm{dh}_A A/A^k\} \leqq n+1.$$

We can assume that $A^k \neq A^{k+1}$. Set $X = A_+/A^k$, then $X_\Pi = A/A^{k+1}$ (see [1, Theorem II.3.17]), and $\operatorname{Ker}\mathscr{X} = A^k/A^{k+1} \neq 0$. Applying Theorem 1 to X, from (12) we find that $\mathrm{dh}_A X_\Pi = n+1$. Hence $\mathrm{dh}_A A/A^{k+1} = n+1$.

Proof of Theorem 2. Let $m \geqq 2$ be such that $A^m = 0$ and $A^{m-1} \neq 0$. To obtain a contradiction, suppose that $\mathrm{dh}_A A = n < \infty$. It is clear that $A \neq A^2$ and that A-module A/A^2 is an annihilator A-module. This implies that

$$\mathrm{dh}_A A/A^2 = \mathrm{dh}_A \mathbf{C} \mathbin{\hat{\otimes}} A/A^2 = n+1.$$

Using Lemma 2, we have $\mathrm{dh}_A A/A^k = n+1$ for each $k \geqq 2$. In particular, if $k = m$, then

$$\mathrm{dh}_A A = \mathrm{dh}_A A/A^m = n+1.$$

But we have assumed that $\mathrm{dh}_A A = n$. Therefore we obtain a contradiction. Consequently, $\mathrm{dh}_A A = \infty$, and the theorem is proved.

We recall that an A-bimodule X is said to be right-annihilator if $x \cdot a = 0$ for all $x \in X, a \in A$. Each right-annihilator Banach A-bimodule X can be regarded as the A-bimodule $\mathscr{B}(\mathbf{C}, X)$. Theorem 2 and the formula

$$\mathscr{H}^n(A, \mathscr{B}(\mathbf{C}, X)) = \operatorname{Ext}^n_A(\mathbf{C}, X)$$

(see [1, Theorem III.4.12]) yield the following corollary.

Corollary 3. *Let A be a nilpotent Banach algebra. Then for any n there exists a right-annihilator Banach A-bimodule X such that $\mathscr{H}^n(A, X) \neq 0$.*

§3. Modules of right multipliers and estimating the global dimension.

Let A be a Banach algebra. If we set

$M_r(A) = {}_Ah(A, A) = \{T \in \mathscr{B}(A) : T(ab) = aT(b),\ a, b \in A\}$, we get a left Banach A-module provided that the outer multiplication is defined by

$$(a \cdot T)(b) = T(ba) \quad (a, b \in A).$$

It is clear that $M_r(A)$ contains the identity operator 1_A. We consider the morphism of A-modules $R\colon A_+ \to M_r(A)$ given by $R(a) = a \cdot 1_A = R_a$, where $R_a(b) = ba$ $(b \in A)$. The closure of the image of this morphism is denoted by $M_+(A)$.

Lemma 3. *Let A be a Banach algebra, and set $X = M_+(A)$. Then, up to an isometric isomorphism of A-modules, the reduced module $X_\Pi = A \mathbin{\hat{\otimes}}_A X$ coincides with A, and the morphism $\mathscr{X} : X_\Pi \to X$ coincides with the restiction of R to A.*

PROOF. For $a \in A$, let $\lambda(a) = a \underset{A}{\otimes} 1_A$. It is clear that λ is a morphism of A-modules from A into X_Π, and that $\|\lambda\| \leqq 1$.

On the other hand, let $S: A \times X \to A$ be the bilinear operator given by $S(a, T) = T(a)$, where $a \in A$, $T \in X \subset \mathscr{B}(A)$. It is easily verified that S is balanced (i.e., $S(ab, T) = S(a, b \cdot T)$ for any $a, b \in A$, $T \in X$). The operator from $A \underset{A}{\hat{\otimes}} X$ into A associated with S is denoted by μ. It is obvious that μ is a morphism of A-modules, that $\|\mu\| \leqq 1$ and that $\mu \circ \lambda = 1_A$. We shall prove now that $\lambda \circ \mu$ is the identity operator on $A \underset{A}{\hat{\otimes}} X$, in which case $\lambda = \mu^{-1}$ and $\mu: A \underset{A}{\hat{\otimes}} X \to A$ is an isometric isomorphism of A-modules.

Indeed, for any $a \in A$, $b \in A_+$ and for $T = R(b) \in X$ we have $(\lambda \circ \mu)(a \underset{A}{\otimes} T) = \lambda(T(a)) = T(a) \underset{A}{\otimes} 1_A = ab \underset{A}{\otimes} 1_A = a \underset{A}{\otimes} T$.

It remains only to note that $\mathscr{X} = R \circ \mu$, and the assertion is proved.

We define the multiplier seminorm $\|\cdot\|_M$ on a Banach algebra A by

$$\|a\|_M = \sup\{\|ba\| : b \in A, \|b\| \leqq 1\}.$$

Clearly $\|a\|_M \leqq \|a\|$ $(a \in A)$. It is easy to see that, if A has a bounded left approximate identity, then $\|\cdot\|$ and $\|\cdot\|_M$ are equivalent. (The converse is false: Willis [7, Example 5] shows that there exists a commutative, separable Banach algebra in which the multiplier seminorm is equivalent to the original norm, but which does not have a bounded approximate identity.)

By combining Theorem 1 with Lemma 3 we get the following theorem.

THEOREM 3. *Let A be a Banach algebra such that $\mathrm{dh}_A A = n < \infty$. Then $\mathrm{dh}_A M_+(A) \leqq n + 2$. If, in addition, $\|\cdot\|$ and $\|\cdot\|_M$ are not equivalent, and A does not have a right identity, then $\mathrm{dh}_A M_+(A) = n + 2$.*

From Theorem 3 we obtain the following corollary.

COROLLARY 4. *Let A be a Banach algebra which does not have a right identity and in which the multiplier seminorm is not equivalent to the original norm. Then $\mathrm{dg}\, A \geqq 2$.*

We recall that the above estimate of the global dimension was known earlier for all commutative Banach algebras with infinite spectrum (see [8]) and also for some other classes of Banach algebras (see [9, Theorem 5] and [10]).

We pick out another corollary of Theorem 3.

COROLLARY 5. *Let A be a non-projective Banach algebra in which the multiplier seminorm is not equivalent to the original norm. Then $\mathrm{dg}\, A \geqq 3$.*

The following corollary is a consequence of Corollary 5, Theorem IV.3.16 of [1] and Lemma of [11].

COROLLARY 6. *Let A be a non-idempotent commutative Banach algebra in which the multiplier seminorm is not equivalent to the original norm. Let A satisfy at least one of the following conditions:*

(i) ∞ belongs to the Shilov boundary of the spectrum of the algebra A_+;
(ii) A is radical.
Then $\operatorname{dg} A \geqq 3$.

For example, let A be the maximal ideal in the (local) Banach algebra $l^1(\omega)$, where ω is a radical weight (see [12]). We recall that the algebra $l^1(\omega)$ consists of those formal power series $a = \sum_{n=0}^{\infty} a_n X^n$ for which

$$\|a\| = \sum_{n=0}^{\infty} |a_n| \omega_n < \infty.$$

Here $\omega = \{\omega_n\}$ is a real-valued function on $\mathbf{Z}^+ = \{0, 1, 2, \ldots\}$ satisfying (i) $\omega_n > 0$ ($n \in \mathbf{Z}^+$), (ii) $\omega_{m+n} \leqq \omega_m \omega_n$ ($m, n \in \mathbf{Z}^+$) and (iii) $\inf \omega_n^{1/n} = 0$. Multiplication in $l^1(\omega)$ is convolution and hence is given by the formula

$$(a * b)_n = \sum_{k=0}^{n} a_k \cdot b_{n-k} \quad (n \in \mathbf{Z}^+).$$

Then $l^1(\omega)$ is a local algebra, and its unique maximal ideal, $A = \{a = \sum a_n X^n \in l^1(\omega) : a_0 = 0\}$ is a radical algebra. It is obvious that the commutative Banach algebra A is always non-idempotent, and therefore (see [11]) the A-module A is not projective.

THEOREM 4. *Let ω be a radical weight for which there exists a constant C such that*

$$\omega_{m+n+1} \leqq C \omega_{m+1} \omega_{n+1} \quad (m, n \in \mathbf{Z}^+), \tag{13}$$

and let A be the maximal ideal in $l^1(\omega)$. Then $\operatorname{dg} A \geqq 3$ and, as a consequence, there exists an A-bimodule X such that $\mathscr{H}^3(A, X) \neq 0$.

PROOF. This follows from Corollary 6, since for such ω the multiplier seminorm on A is not equivalent to the original norm (see [12, Corollary 1.3 and Theorem 1.4]).

It was noted in [12] that a sufficient condition for (13) to hold is that the sequence $\{\omega_{n+1}/\omega_n\}$ be eventually decreasing. For example, set $\omega_n = e^{-\eta_n}$, where $\eta_n = n^\gamma$ ($\gamma > 1$), or set $\omega_n = 1/n^n$ (or $1/n!$); we obtain radical weight sequences on $\mathbf{Z}^+$ such that ω_{n+1}/ω_n is decreasing, and hence we have examples of algebras $l^1(\omega)$ with $\operatorname{dg} l^1(\omega) \geqq 3$.

Thus, for a radical weight function, the "normal" situation is that $\operatorname{dg} l^1(\omega) \geqq 3$. It is not clear to the author whether the bound $\operatorname{dg} l^1(\omega) < 3$ holds for some ω. Gumerov [11] has shown that $\operatorname{dg} l^1(\omega) = \infty$ for $\omega_n = e^{-\eta_n}$, where $\eta_n = n^\gamma$ $(\gamma > 1)$.

Before giving the next result, we introduce some further notation.

For a Banach algebra A, we set

$$N_A(n) = \sup\{\|a_1 a_2 \dots a_n\|^{1/n} : a_i \in A, \|a_i\| \leqq 1 \,(1 \leqq i \leqq n)\}.$$

It is clear that, for all $a_1, a_2, \dots, a_n \in A$,

$$\|a_1 a_2 \dots a_n\| \leqq N_A(n)^n \quad \|a_1\| \, \|a_2\| \dots \|a_n\|.$$

Following [13], we say that a Banach algebra A is topologically nilpotent if $\lim_{n\to\infty} N_A(n) = 0$. For example, the algebra $(C[0,1], *)$ of all continuous complex-valued functions on $[0,1]$, with supremum norm $\|\cdot\|_\infty$ and convolution multiplication

$$(f * g)(t) = \textstyle\int_0^t f(s)\, g(t-s)\, ds,$$

is topologically nilpotent (see [14, Example 2.2]).

LEMMA 4. *Let A be a Banach algebra in which the multiplier seminorm is equivalent to the original norm. Then there is a constant $\alpha > 0$ such that, for all n, $N_A(n) \geqq \alpha$.*

PROOF. Since $\|\cdot\|$ and $\|\cdot\|_M$ are equivalent, there is $C > 0$ with $\|a\| \leqq C\|a\|_M$ $(a \in A)$. Choose $a \in A$ such that $a \neq 0$. Since

$$\|a\|_M = \sup\{\|ba\| : b \in A, \|b\| \leqq 1\},$$

for every $\varepsilon_1 > 0$ there is an element $b_1 \in A$ with $\|b_1\| \leqq 1$, such that $\|a\|_M \leqq \|b_1 a\| + \varepsilon_1$. Hence

$$\|a\| \leqq C\|b_1 a\| + C\varepsilon_1. \tag{14}$$

We then obtain an inequality of type (14) for the element $b_1 a \in A$ to get, for every $\varepsilon_2 > 0$,

$$\|a\| \leqq C^2 \|b_2 b_1 a\| + C^2 \varepsilon_2 + C\varepsilon_1,$$

where $b_2 \in A$ with $\|b_2\| \leqq 1$. Proceeding in this way we obtain that for every n and for every $\varepsilon > 0$ there are some $b_1, \dots, b_n \in A$ with $\|b_i\| \leqq 1$ $(1 \leqq i \leqq n)$, such that

$$\|a\| \leqq C^n \|b_n b_{n-1} \dots b_1 a\| + \varepsilon.$$

Since

$$\|b_n b_{n-1} \dots b_1\| \leqq N_A(n)^n \|b_n\| \, \|b_{n-1}\| \dots \|b_1\| \leqq N_A(n)^n,$$

we deduce that

$$\|a\| \leqq C^n N_A(n)^n \|a\|.$$

It follows that $N_A(n) \geqq \alpha$, where $\alpha = 1/C$.

Theorem 3 and Lemma 4 yield the following corollary.

COROLLARY 7. *Let A be a topologically nilpotent Banach algebra. Then, if* $\mathrm{dh}_A A = n < \infty$, *then* $\mathrm{dh}_A M_+(A) = n + 2$.

The following lemma is proved by Dixon.

LEMMA 5 (see [13, Lemma 4.2]). *Let A be a Banach algebra and X a left Banach A-module such that the multiplication between algebra and module elements induces a surjective mapping $A \hat{\otimes} X \to X$. Then there is a constant $K > 0$ such that, for all n, every $x \in X$ is expressible in the form*

$$x = \sum_{i=1}^{\infty} a_{i1} a_{i2} \dots a_{in} \cdot x_i$$

for some $a_{i1}, a_{i2}, \dots, a_{in} \in A$, $x_i \in X$ $(1 \leqq i < \infty)$ with

$$\sum_{i=1}^{\infty} \|a_{i1}\| \|a_{i2}\| \dots \|a_{in}\| \|x_i\| \leqq K^n \|x\|.$$

THEOREM 5. *Let A be a projective idempotent Banach algebra. Then there is a constant $\alpha > 0$ such that, for all n, $N_A(n) \geqq \alpha$.*

PROOF. Since the left A-module A is essential and projective, the canonical morphism $\pi : A \hat{\otimes} A \to A$ is a retraction in A-mod. It follows that π is surjective. Applying Lemma 5 for the case where $X = A$, we obtain, for any $x \in A$,

$$\begin{aligned} \|x\| &= \Big\| \sum_{i=1}^{\infty} a_{i1} a_{i2} \dots a_{in} \cdot x_i \Big\| \\ &\leqq \sum_{i=1}^{\infty} \|a_{i1} a_{i2} \dots a_{in}\| \|x_i\| \\ &\leqq \sum_{i=1}^{\infty} N_A(n)^n \|a_{i1}\| \|a_{i2}\| \dots \|a_{in}\| \|x_i\| \\ &\leqq N_A(n)^n K^n \|x\|. \end{aligned}$$

If $x \neq 0$, we deduce that $N_A(n) \geqq \alpha$, where $\alpha = 1/K$.

By combining Lemma 4 and Theorem 5 with Corollary 5 we get the following theorem.

THEOREM 6. *Let A be an idempotent, topologically nilpotent Banach algebra. Then A is not projective and* $\operatorname{dg} A \geqq 3$.

For example, let $A = \{f \in C[0,1] : f(0) = 0\}$ with convolution multiplication. It is noted in [14, Example 5.3] that A is idempotent and topologically nilpotent. By Theorem 6, we have $\operatorname{dg} A \geqq 3$.

The following corollary is a consequence of Corollary 6, Lemma 4 and Theorem 6.

COROLLARY 8. *Let A be a topologically nilpotent commutative Banach algebra. Then* $\operatorname{dg} A \geqq 3$ *and, as a consequence, there exists an A-bimodule X such that* $\mathscr{H}^3(\mathscr{A}, X) \neq 0$.

For example, if $A = (C[0,1], *)$, then $\operatorname{dg} A \geqq 3$.

REFERENCES

1. A. Ya. Helemskii, *The homology of Banach and topological algebras*, Kluwer, Dordrecht, 1989. (Russian original 1986.)
2. A. Grothendieck, *Produits tensoriels topologiques et espaces nucléaires*, Mem. Amer. Math. Soc. 16, 1955.
3. P. Enflo, *A counterexample to the approximation problem*, Acta Math. 130 (1973), 309–317.
4. Yu. V. Selivanov, *Homological characterizations of the approximation property for Banach spaces*, Glasgow Math. J. 34 (1992), 229–240.
5. Yu. V. Selivanov, *Homological "additivity formulae" for Banach algebras*, in "*Int. Conf. on Algebra* (Barnaul 20–25 Aug., 1991)", Novosibirsk, 1991.
6. Yu. V. Selivanov, *The values assumed by the global dimension in certain classes of Banach algebras*, Vest. Mosk. Univ. ser. mat. mekh., 30 (1975), 37–42, Moscow Univ. Math. Bull. 30 (1975), 30–34.
7. G. Willis, *Examples of factorization without bounded approximate units*, Proc. London Math. Soc., to appear.
8. A. Ya. Helemskii, *The lowest values taken by the global homological dimensions of functional Banach algebras*, Trudy sem. Petrovsk. 3 (1978), 223–242, Amer. Math. Soc. Trans. (1984).
9. Yu. V. Selivanov, *Biprojective Banach algebras, their structure, cohomology and relation with nuclear operators*, Funkc. anal. i pril. 10 (1976), 89–90; Functional Anal. Appl. 10 (1976), 78–79.
10. Z. A. Lykova, *A lower bound for the global homological dimension of infinite-dimensional* CCR-*algebras*, Uspekhi Matem. Nauk 41 (1986), 197–198.
11. R. V. Gumerov, *Homological dimension of radical algebras of Beurling type with rapidly decreasing weight*, Vest. Mosk. Univ. ser. mat. mekh. 5 (1988), 18–22.
12. W. G. Bade, H. G. Dales, K. B. Laursen, *Multipliers of radical Banach algebras of power series*, Mem. Amer. Math. Soc. 303, 1984.

13. P. G. Dixon, *Topologically nilpotent Banach algebras and factorization*, Proc. Roy. Soc. Edinburgh A, 119A (1991), 329–341.
14. P. G. Dixon, G. A. Willis, *Approximate identities in extensions of topologically nilpotent Banach algebras*, Proc. Roy. Soc. Edinburgh A, to appear.

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