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WEIGHTED PLANCHEREL FORMULA. IRREDUCIBLE UNITARY REPRESENTATIONS AND EIGENSPACE REPRESENTATIONS*

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§1. Introduction.

Let D be the open unit disk in the complex plane $\mathbf{C}$. The Möbius group $G = \mathrm{SU}(1,1)$ consists of all 2×2 complex matrices

$$g = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}$$

such that $|\alpha|^2 - |\beta|^2 = 1$. It acts on D by means of the maps

$$z \mapsto gz = g(z) = \frac{\alpha z + \beta}{\bar{\beta} z + \bar{\alpha}}, \qquad (z \in D).$$

All holomorphic automorphisms are so obtained. Set

$$Z = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.$$

Then $\{Z, A, B\}$ is a basis of the Lie algebra $\mathrm{su}(1,1)$ of $\mathrm{SU}(1,1)$. The Casimir element is

$$\square = Z^2 - A^2 - B^2.$$

We consider the function space $L^2(D, d\mu_\nu)$, where $d\mu_\nu(z) = (1 - |z|^2)^{\nu - 2} dm(z)$, $dm(z) = dxdy (z = x + iy)$ is the Lebesgue measure on D. For every $f \in L^2(D, d\mu_\nu)$, we define

$$\|f\|_\nu = \left\{ \int_D |f(z)|^2 d\mu_\nu(z) \right\}^{\frac{1}{2}};$$

then $L^2(D, d\mu_\nu)$ becomes a Hilbert space. For $g \in \mathrm{SU}(1,1)$, we define

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(1) $$T^{\nu}(g)\colon f(z) \mapsto f(gz)\{g'(z)\}^{\frac{\nu}{2}} = f\left(\frac{\alpha z + \beta}{\bar{\beta} z + \bar{\alpha}}\right)(\bar{\beta} z + \bar{\alpha})^{-\nu}.$$

Then T^{ν} gives a projective representation of the group SU(1, 1) for ν non integral and a genuine representation of the universal covering group of SU(1, 1). If $\nu \in \mathbf{Z}$, then T^{ν} gives a continuous unitary representation of the group SU(1, 1). T^{ν} induces a representation of the Lie algebra su(1, 1) and its universal enveloping algebra on the space of C^{∞}-vectors for T^{ν}, which will be denoted by T^{ν} also. It is easy to get

$$T^{\nu}(Z) = 2iz\frac{\partial}{\partial z} - 2i\bar{z}\frac{\partial}{\partial \bar{z}} + i\nu,$$

$$T^{\nu}(A) = (1 - z^2)\frac{\partial}{\partial z} + (1 - \bar{z}^2)\frac{\partial}{\partial \bar{z}} - \nu z,$$

$$T^{\nu}(B) = i(1 + z^2)\frac{\partial}{\partial z} - i(1 + \bar{z}^2)\frac{\partial}{\partial \bar{z}} + i\nu\bar{z}.$$

Therefore,

(2) $$\Box_{\nu} = T^{\nu}(\Box) = -4(1 - |z|^2)^2\frac{\partial^2}{\partial z \partial \bar{z}} + 4\nu(1 - |z|^2)\bar{z}\frac{\partial}{\partial \bar{z}} - \nu^2 + 2\nu.$$

(also see [5]). We shall call $\Box_{\nu}$ the Casimir operator or invariant Laplacian. If $\nu \neq 0$, it is equivalent to the Schrödinger operator with the Morse potential. If $\nu = 0$, it is the case studied by Helgason [4]. For that case [4] establishes Plancherel formula, which is equivalent to the irreducible decomposition of the unitary representation T^{ν}, and gives the results of eigenspace representations as well. Peetre, Peng and Zhang [5] studies the case of $\nu > 0$ and ν non odd integer, and establishes a corresponding weighted Plancherel formula. It is different from the case of $\nu = 0$, in that, for the case of $\nu > 1$ the Casimir operator $\Box_{\nu}$ has not only continuous spectrum, but also finite discrete spectra.

The aim of this paper is to study all of the other cases for $\nu \in \mathbf{R}$. We will establish Plancherel formulas and give the results of eigenspace representations, then we will give the irreducible decomposition of the unitary representation T^{ν}. The results will show that although the Plancherel formulas have unified form, from the view point of representation theory there are important differences for the different ν.

As is well known D is holomorphically equivalent to the upper half plane $U = \{z; \operatorname{Im} z > 0\}$, and SU(1, 1) is isomorphic to the holomorphic automorphic group SL(2, $\mathbf{R}$) of U. The classification of the irreducible unitary representations of SU(1, 1) is due to Bargmann [1]. With the notation of Taylor [6], the non-trivial irreducible unitary representations of SU(1, 1) have three classes: principal series, discrete series and complementary series:

First (even) principal series	π^e_{is}, $s \in \mathrm{R}$, $\pi^e_{is} \sim \pi^e_{-is}$
Second (odd) principal series	π^o_{is}, $s \in \mathrm{R}\backslash\{0\}$, $\pi^o_{is} \sim \pi^o_{-is}$
Holomorphic discrete series	π^+_n, $n \in \mathrm{N}^+$, n is the lowest weight,
Conjugate holomorphic discrete series	π^-_{-n}, $n \in \mathrm{N}^+$, $-n$ is the highest weight,
Complementary series	π^e_s, $s \in (-1, 1)\backslash\{0\}$,

where π^+_1 and π^-_{-1} are limits of discrete series representations. They and the complementary series do not appear in the irreducible decomposition of the left regular representation.

The results of this paper (see §4) show that for the different ν the irreducible representations of the irreducible decomposition of T^ν are as follows.

$\nu = 0$:	π^e_{is}, $s \in \mathrm{R}$,	
$\nu = 2, 4, \ldots$:	π^e_{is}, $s \in \mathrm{R}$,	$\pi^+_2, \pi^+_4, \ldots, \pi^+_\nu$;
$\nu = 1, 3, \ldots$:	π^o_{is}, $s \in \mathrm{R}\backslash\{0\}$,	$\pi^+_1, \pi^+_3, \ldots, \pi^+_\nu$;
$\nu = -2, -4, \ldots$:	π^e_{is}, $s \in \mathrm{R}$,	$\pi^-_{-2}, \pi^-_{-4}, \ldots, \pi^-_\nu$;
$\nu = -1, -3, \ldots$:	π^o_{is}, $s \in \mathrm{R}\backslash\{0\}$,	$\pi^-_{-1}, \pi^-_{-3}, \ldots, \pi^-_\nu$.

The result of the case $\nu = 0$ is due to Helgason [4], the result of the case $\nu = 2, 4, \ldots$ is due to Peetre-Peng-Zhang [5], the others are new. It is clear that they are very different. So it is necessary to study the cases of different ν. Moreover, when ν is odd, a limit of discrete series representation appears in the decomposition. This is a phenomenon showed by Bargmann [1], i.e. the limits of discrete series embed in the principal series. And we will show that (see §3) the different ν determines the irreducibility of eigenspace representations.

§3. Weighted Plancherel formulas.

One finds in [5] the family of eigenfunctions of $\Box_\nu$

$$e^\nu_{\lambda,b}(z) = \frac{(1 - |z|^2)^{\frac{-\nu+1+i\lambda}{2}}}{(1 - z\bar{b})^{\frac{\nu+1+i\lambda}{2}}(1 - \bar{z}b)^{\frac{-\nu+1+i\lambda}{2}}}, \qquad \lambda \in \mathrm{C}, |b| = 1. \tag{3}$$

The corresponding eigenvalues are $1 + \lambda^2$. We can also write $e^\nu_{\lambda,b}(z)$ as

$$e^{\nu}_{\lambda,b}(z) = \frac{1}{(1 - z\bar{b})^{\nu}} e^{(-\nu+1+i\lambda)\langle z,b\rangle},$$

where $\langle z, b\rangle$ is the hyperbolic distance from 0 to the horocycle through z and b. Denote $B = \partial D$ the unit circle and db be the normalized Lebesgue measure on B. Let

$$\varphi^{\nu}_{\lambda}(z) = \int_B e^{\nu}_{\lambda,b}(z)\,db; \tag{4}$$

then $\varphi^{\nu}_{\lambda}(z)$ is the radial eigenfunction of $\Box_{\nu}$, and any radial eigenfunction with the eigenvalue $1 + \lambda^2$ is $C\varphi^{\nu}_{\lambda}(z)$. This implies that $\varphi^{\nu}_{\lambda}(z) = \varphi^{\nu}_{-\lambda}(z)$, and $\varphi^{\nu}_{\lambda}(z)$ is real if $\lambda \in \mathrm{R}$ or $i\lambda \in \mathrm{R}$ (see [5]).

Let $\mathscr{D}(D)$ be the space of C^{∞}-functions on D having compact supports, and $\mathscr{D}^{\#}(D)$ be the space of radial functions in $\mathscr{D}(D)$.

For $f \in \mathscr{D}^{\#}(D)$, define the spherical transform $\hat{f}(\lambda)$ by

$$\hat{f}(\lambda) = \int_D f(z)\varphi^{\nu}_{-\lambda}(z)\,d\mu_{\nu}(z), \qquad \lambda \in \mathrm{R}. \tag{5}$$

For $f \in \mathscr{D}(D)$, define the generalized Fourier transform $\hat{f}(\lambda, b)$ by

$$\hat{f}(\lambda, b) = \int_D f(z) e^{\nu}_{-\lambda,\bar{b}}(\bar{z})\,d\mu_{\nu}(z), \qquad \lambda \in \mathrm{R},\ b \in B. \tag{6}$$

Then we have following results.

THEOREM 1. *Assume that* $\nu \in \mathrm{R}$, $k = \max\{j \in \mathrm{Z}: j < \frac{|\nu|-1}{2}\}$. *Then for* $f \in \mathscr{D}^{\#}(D)$, *we have*

(i) *the inversion formula*

$$f(z) = \int_{\mathrm{R}} \hat{f}(\lambda)\varphi^{\nu}_{\lambda}(z)\rho_{\nu}(\lambda)\,d\lambda + \sum_{l=0}^{k} \frac{(|\nu| - 1 - 2l)}{\pi} \hat{f}(-i(|\nu| - 1 - 2l))\varphi^{\nu}_{i(|\nu|-1-2l)}(z),$$

where the density ρ_{ν} *is given by*

$$\rho_{\nu}(\lambda) = \frac{1}{4\pi}\frac{\lambda\sinh(\pi\lambda)}{\cosh(\pi\lambda) + \cos(\pi\nu)},$$

and (ii) *the Plancherel formula*

$$\int_D |f(z)|^2 d\mu_{\nu}(z) = \int_{\mathrm{R}} |\hat{f}(\lambda)|^2 \rho_{\nu}(\lambda)\,d\lambda + \sum_{l=0}^{k} \frac{(|\nu| - 1 - 2l)}{\pi} |\hat{f}(-i(|\nu| - 1 - 2l))|^2.$$

THEOREM 2. *Assume that* $\nu \in \mathrm{R}$, $k = \max\{j \in \mathrm{Z}: j < \frac{|\nu|-1}{2}\}$. *Then for* $f \in \mathscr{D}(D)$, *we have*

(i) *the inversion formula*

$$f(z) = \int_B \int_{\mathsf{R}} \tilde{f}(\lambda, b) e^{\nu}_{\lambda, b}(z) \rho_\nu(\lambda) d\lambda db$$
$$+ \sum_{l=0}^{k} \frac{(|\nu| - 1 - 2l)}{\pi} \int_B \tilde{f}(-i(|\nu| - 1 - 2l), b) e^{\nu}_{-i(|\nu|-1-2l), b}(z) db,$$

(ii) *the Plancherel formula*

$$\int_D |f(z)|^2 d\mu_\nu(z) = \int_B \int_{\mathsf{R}} |\tilde{f}(\lambda, b)|^2 \rho_\nu(\lambda) d\lambda db + \sum_{l=0}^{k} \frac{(|\nu| - 1 - 2l)}{\pi} \cdot$$
$$\int_B \tilde{f}(-i(|\nu| - 1 - 2l), b) \overline{\tilde{f}(i(|\nu| - 1 - 2l), b)} db,$$

and the integral

$$\int_B \tilde{f}(-i(|\nu| - 1 - 2l), b) \overline{\tilde{f}(i(|\nu| - 1 - 2l), b)} db$$

is nonnegative,
(iii) *the operators P_l defined by*

$$P_l f(z) = \frac{(|\nu| - 1 - 2l)}{\pi} \int_B \tilde{f}(-i(|\nu| - 1 - 2l), b) e^{\nu}_{-i(|\nu|-1-2l), b}(z) db$$

are jointly orthogonal projections,
and (iv) *denote $A^\nu_l(D) = P_l L^2(D, d\mu_\nu)$, then the map $f(z) \mapsto \tilde{f}(\lambda, b)$ extends to an unitary isometry from*

$$A^\nu_\omega(D) = L^2(D, d\mu_\nu) \ominus \sum_{l=0}^{k} {}^{\oplus} A^\nu_l(D)$$

onto

$$L^2(\mathsf{R}^+ \times B, 2\rho_\nu(\lambda) d\lambda db).$$

The proofs of Theorem 1 and 2. If $\nu = 0$, these are the results of Helgason [4]. If $\nu > 1$ and is not an odd integer, these are the results of Peetre-Peng-Zhang [5] (except (iv) of Theorem 2), and their proof holds also for the case $0 < \nu < 1$. Now we consider the other cases, and we still use the methods of [4] and [5], so we give the details only for the points that differ.

Suppose that $f \in \mathscr{D}^{\#}(D)$. As in [5], there exists $g \in C_0^\infty(\mathsf{R})$ such that

$$\tilde{f}(\lambda) = \int_{\mathsf{R}} e^{-i\lambda t} g(t) dt, \tag{7}$$

and

$$f(0) = -\frac{1}{2\pi}\int_{\mathsf{R}} \frac{g'(t)}{\sinh t} e^{-\nu t}\,dt. \tag{8}$$

(It is not hard to check that the derivation of (7) and (8) in [5] holds for all $\nu \in \mathsf{R}$.)
For $\nu > 1$,

$$\begin{aligned}
&\int_{\mathsf{R}} \frac{g'(t)}{\sinh t} e^{-\nu t}\,dt - \int_{\mathsf{R}} \frac{g'(t)}{\sinh t} e^{(-\nu+2)t}\,dt \\
&= -2\int_{\mathsf{R}} g'(t) e^{(-\nu+1)t}\,dt \\
&= -2(\nu-1)\int_{\mathsf{R}} g(t) e^{(-\nu+1)t}\,dt \\
&= -2(\nu-1)\hat{f}(-i(\nu-1)).
\end{aligned}$$

Thus

$$\hat{f}(0) = -\frac{1}{2\pi}\int_{\mathsf{R}} \frac{g'(t)}{\sinh t} e^{(-\nu+2)t}\,dt + \frac{\nu-1}{\pi}\hat{f}(-i(\nu-1)).$$

Repeating this argument, we get

$$\begin{aligned}
\hat{f}(0) = &-\frac{1}{2\pi}\int_{\mathsf{R}} \frac{g'(t)}{\sinh t} e^{(-\nu+2k+2)}\,dt \\
&+ \sum_{l=0}^{k} \frac{|\nu|-1-2l}{\pi}\hat{f}(-i(\nu-1-2l)).
\end{aligned} \tag{9}$$

For $\nu < -1$, notice that

$$\begin{aligned}
&\int_{\mathsf{R}} \frac{g'(t)}{\sinh t} e^{-\nu t}\,dt - \int_{\mathsf{R}} \frac{g'(t)}{\sinh t} e^{(-\nu-2)t}\,dt \\
&= 2\int_{\mathsf{R}} g'(t) e^{(-\nu-1)t}\,dt \\
&= 2(\nu+1)\int_{\mathsf{R}} g(t) e^{(-\nu-1)t}\,dt \\
&= -2(|\nu|-1)\hat{f}(-i(|\nu|-1)),
\end{aligned}$$

we have

(10)
$$\hat{f}(0) = -\frac{1}{2\pi}\int_{\mathrm{R}} \frac{g'(t)}{\sinh t} e^{(-\nu-2)t}\,dt + \frac{|\nu|-1}{\pi}\hat{f}(-i(|\nu|-1))$$
$$= -\frac{1}{2\pi}\int_{\mathrm{R}} \frac{g'(t)}{\sinh t} e^{(-\nu-2k-2)t}\,dt$$
$$+ \sum_{l=0}^{k} \frac{|\nu|-1-2l}{\pi}\hat{f}(-i(|\nu|-1-2l)).$$

Since $\hat{f}(\lambda) = \hat{f}(-\lambda)$, we have

(11)
$$g(t) = \frac{1}{2\pi}\int_{\mathrm{R}} \hat{f}(\lambda)\cos\lambda t\,dt,$$

and

(12)
$$g'(t) = -\frac{1}{2\pi}\int_{\mathrm{R}} \lambda\hat{f}(\lambda)\sin\lambda t\,dt.$$

If $\nu < 0$ and $\nu \neq -1, -3\ldots$, then $0 \leqq -\nu - 2k - 2 < 1$, the integral

$$\int_{\mathrm{R}}\int_{\mathrm{R}} \frac{e^{(-\nu-2k-2)t}}{\sinh t}\lambda\hat{f}(\lambda)\sin\lambda t\,dt\,d\lambda$$

is absolutely convergent. By the formula of integral transform (see [2], P. 88)

$$\int_{\mathrm{R}} \frac{e^{(-\nu-2k-2)t}}{\sinh t}\sin\lambda t\,dt = \frac{\pi\sinh(\pi\lambda)}{\cosh(\pi\lambda)+\cos(\pi\nu)},$$

we then get

(13)
$$\hat{f}(0) = \frac{1}{4\pi^2}\int_{\mathrm{R}} \lambda\hat{f}(\lambda)\int_{\mathrm{R}} \frac{e^{(-\nu-2k-2)t}}{\sinh t}\sin\lambda t\,dt\,d\lambda$$
$$+ \sum_{l=0}^{k} \frac{|\nu|-1-2l}{\pi}\hat{f}(-i(|\nu|-1-2l))$$
$$= \int_{\mathrm{R}} \hat{f}(\lambda)\rho_\nu(\lambda)\,d\lambda + \sum_{l=0}^{k} \frac{|\nu|-1-2l}{\pi}\hat{f}(-i(|\nu|-1-2l)).$$

For $\nu = \pm 1, \pm 3, \ldots$, if we still follow the above argument, we have to deal with the integrals

$$\int_{\mathrm{R}} \frac{e^{\pm t}}{\sinh t}\sin\lambda t\,dt.$$

It is obvious that

$$\int_{\mathsf{R}} \frac{e^{-t}}{\sinh t} \sin \lambda t \, dt = \int_{\mathsf{R}} \frac{e^{t}}{\sinh t} \sin \lambda t \, dt.$$

This integral is divergent. So we can not use the same argument as above. This is a reason that [5] removes the case $v = 1, 3, \ldots$. We have to deal with the oscillatory factor in the integral (9). If $v = 1, 3, \ldots$, then (9) become

$$\hat{f}(0) = \frac{1}{4\pi^2} \int_{\mathsf{R}} \frac{e^{-t}}{\sinh t} \left(\int_{\mathsf{R}} \lambda \hat{f}(\lambda) \sin \lambda t \, d\lambda \right) dt + \sum_{l=0}^{k} \frac{v-1-2l}{\pi} \hat{f}(-i(v-1-2l)). \tag{14}$$

By (7), we know that $\hat{f}(\lambda) \in S(\mathsf{R})$ for $\lambda \in \mathsf{R}$. For any $N > 0$, the integral

$$\int_{-N}^{N} \frac{e^{-t}}{\sinh t} \int_{\mathsf{R}} \lambda \hat{f}(\lambda) \sin \lambda t \, d\lambda \, dt$$

is absolutely convergent. Hence

$$\int_{\mathsf{R}} \frac{e^{-t}}{\sinh t} \left(\int_{\mathsf{R}} \lambda \hat{f}(\lambda) \sin \lambda t \, d\lambda \right) dt = \lim_{N \to \infty} \int_{-N}^{N} \frac{e^{-t}}{\sinh t} \int_{\mathsf{R}} \lambda \hat{f}(\lambda) \sin \lambda t \, d\lambda \, dt \tag{15}$$

$$= \lim_{N \to \infty} \int_{\mathsf{R}} \lambda \hat{f}(\lambda) \left(\int_{-N}^{N} \frac{e^{-t}}{\sinh t} \sin \lambda t \, dt \right) d\lambda,$$

and

$$\int_{-N}^{N} \frac{e^{-t}}{\sinh t} \sin \lambda t \, dt = 2 \int_{0}^{N} \coth t \sin \lambda t \, dt$$

$$= 2 \int_{0}^{N} (\coth t - 1) \sin \lambda t \, dt + 2 \int_{0}^{N} \sin \lambda t \, dt = 2 \int_{0}^{N} \frac{e^{-t}}{\sinh t} \sin \lambda t \, dt + \frac{2}{\lambda} - \frac{2}{\lambda} \cos Nt,$$

we get

$$\lim_{N \to \infty} \int_{\mathsf{R}} \lambda \hat{f}(\lambda) \left(\int_{-N}^{N} \frac{e^{-t}}{\sinh t} \sin \lambda t \, dt \right) d\lambda \tag{16}$$

$$= \int_{\mathsf{R}} \lambda \hat{f}(\lambda) \left(\int_{0}^{\infty} \frac{e^{-t}}{\sinh t} \sin \lambda t \, dt + \frac{1}{\lambda} \right) d\lambda - 2 \lim_{N \to \infty} \int_{\mathsf{R}} \hat{f}(\lambda) \cos N\lambda \, d\lambda.$$

By Riemann-Lebesgue's Lemma we have

$$\lim_{N \to \infty} \int_{\mathsf{R}} \hat{f}(\lambda) \cos N\lambda \, d\lambda = 0. \tag{17}$$

By the formula of integral transform (see [2], P. 91)

$$\int_0^\infty \frac{e^{-t}}{\sinh y} \sin \lambda t \, dt = \frac{\pi}{2} \coth \frac{\pi\lambda}{2} - \frac{1}{\lambda}.$$

Finally, we get

$$\lim_{N\to\infty} \int_{\mathbb{R}} \lambda \hat{f}(\lambda) \left(\int_{-N}^{N} \frac{e^{-t}}{\sinh t} \sin \lambda t \, dt \right) d\lambda = \pi \int_{\mathbb{R}} \hat{f}(\lambda) \lambda \coth \frac{\pi\lambda}{2} \, d\lambda. \tag{18}$$

Together, (14) through (18) give us

$$\hat{f}(0) = \int_{\mathbb{R}} \hat{f}(\lambda) \rho_\nu(\lambda) \, d\lambda + \sum_{l=0}^{k} \frac{\nu - 1 - 2l}{\pi} \hat{f}(-i(\nu - 1 - 2l)). \tag{19}$$

If $\nu = -1, -3, \ldots$, the same argument gives

$$\hat{f}(0) = \int_{\mathbb{R}} \hat{f}(\lambda) \rho_\nu(\lambda) \, d\lambda + \sum_{l=0}^{k} \frac{|\nu| - 1 - 2l}{\pi} \hat{f}(-i(|\nu| - 1 - 2l)). \tag{20}$$

Thus (13), (19) and (20) show that (i) of Theorem 1 holds for the special case $z = 0$. The same arguments in [5] give the proofs of Theorem 1 and (i)–(iii) of Theorem 2. Now we turn to the proof of (iv) of Theorem 2.

Notice that if $f \in \mathscr{D}^{\#}(D)$, then

$$\hat{f}(\lambda, b) = \hat{f}(\lambda), \qquad \forall b \in B. \tag{21}$$

Let dg be the Haar measure of SU(1, 1), and transfer $d\mu_\nu(z)$ to a measure (also denoted by $d\mu_\nu$ on G defined by $d\mu_\nu(g) = (1 - |g \cdot 0|^2)^\nu dg$. It satisfies

$$\int_{\mathrm{SU}(1,1)} f(g0) \, d\mu_\nu(g) = \int_D f(z) \, d\mu_\nu(z).$$

Assume that $f_1, f_2 \in \mathscr{D}(D)$, we define $f_1 * f_2$ by

$$f_1 * f_2(z) = \int_G f_1(g0) f_2(g^{-1}z) \{(g^{-1})'(0)\}^{-\frac{\nu}{2}} \{(g^{-1})'(z)\}^{\frac{\nu}{2}} \, d\mu_\nu(g).$$

For $f_1 \in \mathscr{D}(D)$, $f_2 \in \mathscr{D}^{\#}(D)$, we have

$$\begin{aligned} (22) \qquad & (f_1 * f_2)^\wedge(\lambda, b) \\ &= \int_D \int_{\mathrm{SU}(1,1)} f_1(g0) f_2(g^{-1}z) \{(g^{-1})'(0)\}^{-\frac{\nu}{2}} \\ &\quad \cdot \{(g^{-1})'(z)\}^{\frac{\nu}{2}} \frac{1}{(1 - \bar{z}b)^\nu} e^{(-\nu + 1 - i\lambda)\langle z, b\rangle} \, d\mu_\nu(g) \, d\mu_\nu(z), \end{aligned}$$

and

(23) $$\int_D f_2(g^{-1}z)\{(g^{-1})'(z)\}^{\frac{\nu}{2}}\frac{1}{(1-\bar{z}b)^\nu}e^{(-\nu+1-i\lambda)\langle z,b\rangle}d\mu_\nu(z)$$

$$=\int_D f_2(z)\{g'(z)\}^{-\frac{\nu}{2}}\frac{1}{(1-\overline{gz}b)^\nu}e^{(-\nu+1-i\lambda)\langle gz,b\rangle}|g'(z)|^\nu d\mu_\nu(z).$$

If $g=\begin{pmatrix}\alpha & \beta\\ \bar{\beta} & \bar{\alpha}\end{pmatrix}$, we have

(24) $$\frac{1}{(1-\overline{gz}b)^\nu}=\frac{1}{(1-\bar{z}g^{-1}b)^\nu}\frac{1}{(1-\overline{g0}b)^\nu}\{(g^{-1})'(0)\}^{\frac{\nu}{2}}\{\overline{g'(z)}\}^{-\frac{\nu}{2}}.$$

Additionally we have (see [4], P. 83)

(25) $$\langle gz,b\rangle=\langle z,g^{-1}b\rangle+\langle g0,b\rangle.$$

Using Fubini's theorem in (22), (23)–(25) and (21) then yield

(26) $$(f_1*f_2)^\wedge(\lambda,b)$$

$$=\int_{SU(1,1)} f_1(g0)e^\nu_{-\lambda,\bar{b}}(\overline{g0})\int_D f_2(z)e^\nu_{-\lambda,\overline{g^{-1}b}}(\bar{z})\,d\mu_\nu(z)\,d\mu_\nu(g)$$

$$=\hat{f}_1(\lambda,b)\hat{f}_2(\lambda).$$

In particular if $f_1, f_2\in\mathscr{D}^\#(D)$, then

(27) $$(f_1*f_2)^\wedge(\lambda)=\hat{f}_1(\lambda)\hat{f}_2(\lambda).$$

Since φ_λ is real for $\lambda\in\mathsf{R}$, if $f\in\mathscr{D}^\#(D)$, then

(28) $$\overline{\hat{f}(\lambda)}=\hat{\bar{f}}(\lambda).$$

Also for any $\lambda_0\in\mathsf{R}$, there exists $f_0\in\mathscr{D}^\#(D)$, such that

(29) $$\hat{f}_0(\lambda_0)\neq 0$$

and for any $\lambda_1,\lambda_2\in\mathsf{R}$, $\lambda_1\neq\lambda_2$, there exists $f\in\mathscr{D}^\#(D)$, such that

(30) $$\hat{f}(\lambda_1)\neq\hat{f}(\lambda_2).$$

Using Weierstrass-Stone's theorem, (27)–(30) imply that every $h(\lambda)\in C_0^\infty(\mathsf{R}^+)$ can be uniformly approximated by the spherical transforms of elements of $\mathscr{D}^\#(D)$.

Now we give the proof of Theorem 2 (iv) as follows.

For $f\in A^\nu_\omega(D)$, we have (see [5])

(31) $$f*\varphi^\nu_\lambda(z)=\int_B e^\nu_{\lambda,b}(z)\hat{f}(\lambda,b)\,db.$$

Thus

$$\int_D f * \varphi_\lambda^\nu(z)\overline{f(z)}\,d\mu_\nu(z) = \int_B |\hat{f}(\lambda, b)|^2\,db = \int_B |\hat{f}(-\lambda, b)|^2\,db, \qquad \lambda \in \mathsf{R}. \tag{32}$$

Then Plancherel formula yields

$$\int_D |f(z)|^2\,d\mu_\nu(z) = \int_{\mathsf{R}}\int_B |\hat{f}(\lambda, b)|^2 \rho_\nu(\lambda)\,d\lambda\,db = 2\int_{\mathsf{R}+}\int_B |\hat{f}(\lambda, b)|^2 \rho_\nu(\lambda)\,d\lambda\,db, \tag{33}$$

Suppose that $F \in L^2(\mathsf{R}^+ \times B, 2\rho_\nu(\lambda)\,d\lambda\,db)$ and F is orthogonal to the generalized Fourier transforms of $A_l^\nu(D)$, then for every $f \in \mathscr{D}(D)$ and $h \in \mathscr{D}^{\#}(D)$,

$$2\int_{\mathsf{R}}\int_B \hat{f}(\lambda, b)\tilde{h}(\lambda)F(\lambda, b)\rho_\nu(\lambda)\,d\lambda\,db = 0. \tag{34}$$

This implies that

$$\int_B \hat{f}(\lambda, b)F(\lambda, b)\,db = 0, \qquad \text{a.e.} \quad \lambda \in \mathsf{R}^+.$$

Since $\{\hat{f}(\lambda, \cdot) : f \in \mathscr{D}(D)\}$ is dense in $L^2(B)$ (for the proof, see §3), we deduce that

$$F(\lambda, b) = 0, \qquad \text{a.e.} \quad (\lambda, b) \in \mathsf{R}^+ \times B. \tag{35}$$

This completes the proof of Theorem 2 (iv).

We conclude this section by the Paley-Wiener type theorem, which characterize the spherical transforms of $\mathscr{D}^{\#}(D)$ and the generalized Fourier transforms of $\mathscr{D}(D)$. Using the terminology of [4], a holomorphic function $F(\lambda)$ on C is called an entire function of exponential type R, if for any $N \in \mathsf{Z}^+$,

$$\sup_{\lambda \in \mathsf{C}} e^{-R|\mathrm{Im}\,\lambda|}(1 + |\lambda|)^N |F(\lambda)| < \infty.$$

A C^∞-function $F(\lambda, b)$ on $\mathsf{C} \times B$ is called an entire function of uniform exponential type R, if $F(\lambda, b)$ is holomorphic to λ, and for any $N \in \mathsf{Z}^+$,

$$\sup_{\lambda \in \mathsf{C},\, b \in B} e^{-R|\mathrm{Im}\,\lambda|}(1 + |\lambda|)^N |F(\lambda, b)| < \infty.$$

THEOREM 3. *Let* $\nu \in \mathsf{R}$. (i) *$F(\lambda)$ is the spherical transform of $f \in \mathscr{D}^{\#}(D)$ if and only if $F(\lambda)$ is an entire function of exponential type, and satisfies*

$$F(\lambda) = F(-\lambda).$$

(ii) *$F(\lambda, b)$ is the generalized Fourier transform of $f \in \mathscr{D}(D)$ if and only if $F(\lambda, b)$ is an entire function of uniform exponential type and satisfies*

$$\int_B e_{\lambda, b}^\nu(z)F(\lambda, b)\,db = \int_B e_{-\lambda, b}^\nu(z)F(-\lambda, b)\,db.$$

The proof, being similar to ones given in [4] and [7], is omitted.

§3. Eigenspace representations.

For $\lambda \in \mathbf{C}$, we denote the eigenspace of $\Box_\nu$ by $\varepsilon_\lambda^\nu(D)$:

$$\varepsilon_\lambda^\nu(D) = \{f \in C^\infty(D) \colon \Box_\nu f = (1 + \lambda^2) f\}.$$

The topology of $\varepsilon_\lambda^\nu(D)$ is the reduced topology of $C^\infty(D)$. Then we define the eigenspace representation:

$$\tilde{T}^{\nu,\lambda}(g) \colon f(z) \mapsto f(gz)\{g'(z)\}^{\frac{\nu}{2}}, \qquad f \in \varepsilon_\lambda^\nu(D).$$

The aim of this section is to study the reducibility of $\tilde{T}^{\nu,\lambda}$.

DEFINITION. For $\nu \in \mathbf{R}$, $\lambda \in \mathbf{C}$ is called *simple*, if the map from $L^2(B)$ to $C^\infty(D)$ given by

$$F(b) \mapsto f(z) = \int_B e_{\lambda,b}^\nu(z) F(b)\, db$$

is one to one.

LEMMA 1. *$\lambda \in \mathbf{C}$ is simple if and only if*

$$\lambda \neq i(\pm\nu + 1 + 2k), \qquad k \in \mathbf{Z}^+.$$

PROOF. Suppose that $z = \tanh r e^{i\varphi}$, $b = e^{i\theta}$, then

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} (\cosh r)^{-\nu+1+i\lambda} (1 - \tanh r e^{-i\theta})^{\frac{-\nu-1-i\lambda}{2}} \cdot \tag{38}$$

$$(1 - \tanh r e^{i\theta})^{\frac{\nu-1-i\lambda}{2}} F(\theta + \varphi)\, d\theta.$$

If $\lambda = i(\nu + 1 + 2k)$, then

$$\frac{1}{2\pi} \int_0^{2\pi} e_{\lambda,e^{i\theta}}^\nu(z) e^{i(k+1)\theta}\, d\theta = 0.$$

If $\lambda = i(-\nu + 1 + 2k)$, then

$$\frac{1}{2\pi} \int_0^{2\pi} e_{\lambda,e^{i\theta}}^\nu(z) e^{-i(k+1)\theta}\, d\theta = 0.$$

Hence λ is not simple. Let $\lambda \neq (\pm\nu + 1 + 2k)$. Expanding $F(\theta)$ into Fourier series, we have

$$F(\theta) = \sum_{n=-\infty}^{\infty} a_n e^{in\theta}.$$

If

(39)
$$\frac{1}{2\pi}\int_0^{2\pi}(\cosh r)^{-\nu+1+i\lambda}(1-\tanh re^{-i\theta})^{\frac{-\nu-1-i\lambda}{2}}\cdot$$
$$(1-\tanh re^{i\theta})^{\frac{\nu-1-i\lambda}{2}}F(\theta+\varphi)\,d\theta\equiv 0,$$

let $r=0$, we get

(40)
$$a_0=0.$$

Differentiating with respect to r in (39), and let $r=0$, we then get

$$\frac{\nu+1+i\lambda}{2}a_1e^{i\varphi}+\frac{-\nu+1+i\lambda}{2}a_{-1}e^{i\varphi}\equiv 0.$$

Hence

$$a_1=a_{-1}=0.$$

By induction, differentiating n times, we get

$$a_n=a_{-n}=0.$$

Therefore $F\equiv 0$, so λ is simple. This completes the proof of Lemma 1.

Note. If $\bar{\lambda}$ is simple, then $\{\hat{f}(\lambda,\cdot)\colon f\in\mathcal{D}(D)\}$ is dense in $L^2(B)$, because if $F(b)$ satisfies

$$\int_B\hat{f}(\lambda,b)F(b)\,db=0,\qquad\text{for any } f\in\mathcal{D}(D),$$

then

$$\int_B e^{\nu}_{-\lambda,\bar{b}}(z)F(b)\,db=\overline{\int_B e^{\nu}_{\bar{\lambda},b}(z)\overline{F(b)}\,db}\equiv 0,\qquad\text{for any } z\in D,$$

and $F\equiv 0$. This fact had been used in the proof of (iv) of Theorem 2.

For any $\lambda\in\mathrm{C}$ and $m\in\mathrm{Z}$, we define $\varphi^{\nu}_{\lambda,m}(z)$ by

(42)
$$\varphi^{\nu}_{\lambda,m}(z)=\int_B e^{\nu}_{\lambda,b}(z)\chi_m(b)\,db,$$

where $\chi_m(e^{i\theta})=e^{im\theta}$. It is easy to see $\varphi^{\nu}_{\lambda,0}(z)=\varphi^{\nu}_{\lambda}(z)$. We can give the explicit expression for $\varphi^{\nu}_{\lambda,m}(z)$:

(43)
$$\varphi^{\nu}_{\lambda,m}(e^{i\theta}z)=e^{im\theta}\varphi^{\nu}_{\lambda,m}(z),$$

(44)
$$\begin{aligned}\varphi^{\nu}_{\lambda,m}(|z|) &= (1-|z|^2)^{\frac{-\nu+1+i\lambda}{2}} \int_B (1-|z|\bar{b})^{\frac{-\nu-1-i\lambda}{2}} (1-|z|\,b)^{\frac{\nu-1-i\lambda}{2}} \chi_m(b)\,db \\ &= (1-|z|^2)^{\frac{-\nu+1+i\lambda}{2}} |z|^{|m|} \frac{\Gamma\left(\frac{1+i\lambda}{2}+\varepsilon(m)\left(\frac{\nu}{2}+m\right)\right)}{\Gamma\left(\frac{1+i\lambda}{2}+\varepsilon(m)\frac{\nu}{2}\right)|m|!}. \\ &\quad F\left(\frac{1+i\lambda}{2}+\varepsilon(m)\left(\frac{\nu}{2}+m\right), \frac{1+i\lambda}{2}-\varepsilon(m)\frac{\nu}{2}; |m|+1; |z|^2\right),\end{aligned}$$

where

$$\varepsilon(m) = \begin{cases} 1, & m \geqq 0, \\ -1 & m < 0. \end{cases}$$

(44) follows from an expansion of the integrand and term by term integration. Clearly, $\varphi^{\nu}_{\lambda,m}(z) \in \varepsilon^{\nu}_{\lambda}(D)$. Note that $\varphi^{\nu}_{\lambda,m}(z)$ satisfies (43). The following fact is very useful: if $f \in \varepsilon^{\nu}_{\lambda}(D)$ satisfies

(45)
$$f(e^{i\theta}z) = e^{im\theta} f(z),$$

and

$$\varphi^{\nu}_{\lambda,m}(z) \neq 0,$$

then $f(z) = C\varphi^{\nu}_{\lambda,m}(z)$ for some constant C. In fact, for $z \in D$, $z = \tanh r e^{i\theta}$, the Casimir operator becomes

(46)
$$\begin{aligned}\square_\nu = &-\frac{\partial^2}{\partial r^2} - 2\coth 2r \frac{\partial}{\partial r} - 4\sinh^{-2} 2r \frac{\partial^2}{\partial \theta^2} \\ &+ 2\nu \tanh r \frac{\partial}{\partial r} + 2\nu i \cosh^{-2} r \frac{\partial}{\partial \theta} - \nu^2 + 2\nu\end{aligned}$$

(see [5]). If $f \in \varepsilon^{\nu}_{\lambda}(D)$ satisfies (45), let $F(r) = f(\tanh r)$. Then (46) implies that $F(r)$ satisfies

(47)
$$\begin{aligned}\frac{d^2F}{dr^2} &+ (2\coth 2r - 2\nu \tanh r)\frac{dF}{dr} + (-4m^2 \sin^{-2} 2r \\ &+ 2m\nu \cosh^{-2} r + (\nu-1)^2 + \lambda^2)F = 0.\end{aligned}$$

Expanding $F(r) = \sum_{n=0}^{\infty} a_n (\sinh r)^n$, substituting this into (47), we obtain a recurrence formula

(48)
$$((n+2)^2 - m^2)a_{n+2} = -((n+1)^2 + m^2 + \nu^2 + \lambda^2 + 2\nu(m-n-1))a_n + \sum_{k=0}^{[\frac{n-2}{2}]} (-1)^k (m^2 + 2\nu m)a_{n-2-2k}, \qquad \text{for} \quad n \geqq 0,$$

and

(49)
$$m^2 a_0 = 0, \qquad (1 - m^2)a_1 = 0.$$

This means that

$$a_n = 0, \qquad \text{for} \quad 0 \leqq n < |m|,$$
$$a_{|m|+1} = a_{|m|+3} = \ldots = 0,$$

and

$$a_{|m|+2}, a_{|m|+4}, \ldots \text{ are determined uniquely by } a_{|m|}.$$

The above argument also tells us that there exists $0 \neq f(z) \in \varepsilon_\lambda^\nu(D)$ satisfying (45).

We consider the map from $L^2(B)$ to $\varepsilon_\lambda^\nu(D)$:

$$F(b) \mapsto f(z) = \int_B e_{\lambda,b}^\nu(z) F(b)\, db.$$

It is easy to see that this map is continuous. We denote its kernel by K_λ^ν, which is a closed subspace of $L^2(B)$, and denote its image by H_λ^ν. Then H_λ^ν can be given a norm such that H_λ^ν and $\{K_\lambda^\nu\}^\perp$ are isometric, in particular, if λ is simple, H_λ^ν is isometric to $L^2(B)$. Thus H_λ^ν becomes a Hilbert space with an orthogonal basis $\{\varphi_{\lambda,m}^\nu(z)\colon \varphi_{\lambda,m}^\nu(z) \neq 0\}$. We denote the restriction of $\tilde{T}^{\nu,\lambda}$ to H_λ^ν by $T^{\nu,\lambda}$. It is easy to check that $T^{\nu,\lambda}$ is a representation and $T^{\nu,\lambda}$ is unitary for $\lambda \in \mathbb{R}$.

The following theorem gives the total characterization of the irreducibility of the eigenspace representation $\tilde{T}^{\nu,\lambda}$.

THEOREM 4. *The eigenspace representation $\tilde{T}^{\nu,\lambda}$ is irreducible if and only if*

$$\lambda \neq \pm i(\pm\nu + 1 + 2k), \qquad \text{for} \quad k \in \mathbb{Z}^+.$$

In other words, $\tilde{T}^{\nu,\lambda}$ is irreducible if and only if both λ and $\bar{\lambda}$ are simple.

The proof of Theorem 4 can be obtained from the following three lemmas.

LEMMA 2. *$\tilde{T}^{\nu,\lambda}$ is irreducible if and only if H_λ^ν is dense in $\varepsilon_\lambda^\nu(D)$ and $T^{\nu,\lambda}$ is irreducible.*

LEMMA 3. *H_λ^ν is dense in $\varepsilon_\lambda^\nu(D)$ if and only if λ is simple.*

LEMMA 4. *If λ is simple, then $T^{\nu,\lambda}$ is irreducible if and only if $\bar{\lambda}$ is simple.*

THE PROOF OF LEMMA 2. Suppose that H_λ^ν is dense in $\varepsilon_\lambda^\nu(D)$ and $T^{\nu,\lambda}$ is irreducible. If E is a non-zero invariant closed subspace of $\tilde{T}^{\nu,\lambda}$, then $E \cap H_\lambda^\nu \neq \emptyset$, because there exists $f \in E$ such that $f(0) \neq 0$, and thus

$$\frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta}z)\,d\theta = f(0)\varphi_\lambda^\nu(z) \in E \cap H_\lambda^\nu.$$

This means that $E \cap H_\lambda^\nu$ is a non-zero invariant closed subspace of $T^{\nu,\lambda}$. Since $T^{\nu,\lambda}$ is irreducible, $E \cap H_\lambda^\nu = H_\lambda^\nu$. Because H_λ^ν is dense in $\varepsilon_\lambda^\nu(D)$, $E = \varepsilon_\lambda^\nu(D)$, i.e. $\tilde{T}^{\nu,\lambda}$ is irreducible.

Suppose that $\tilde{T}^{\nu,\lambda}$ is irreducible. If V is a non-zero invariant subspace of $T^{\nu,\lambda}$, then V is a non-zero invariant subspace of $\tilde{T}^{\nu,\lambda}$, so V is dense in $\varepsilon_\lambda^\nu(D)$. Thus H_λ^ν is also dense in $\varepsilon_\lambda^\nu(D)$. Let

$$f * \chi_m(z) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{-i\theta}z)e^{im\theta}\,d\theta;$$

then $V * \chi_m \subset V$. Since V is dense in $\varepsilon_\lambda^\nu(D)$ and the map $f \mapsto f * \chi_m$ is continuous, $V * \chi_m$ is dense in $\varepsilon_\lambda^\nu(D) * \chi_m$. Because $\varepsilon_\lambda^\nu(D) * \chi_m = C\varphi_{\lambda,-m}^\nu(z)$, and $\varphi_{\lambda,-m}^\nu(z) \in V$, so $V = H_\lambda^\nu$, i.e. $T^{\nu,\lambda}$ is irreducible.

THE PROOF OF LEMMA 3. Suppose that λ is simple. This means that for all $m \in Z$, $\varphi_{\lambda,m}^\nu(z) \neq 0$. Let $f \in \varepsilon_\lambda^\nu(D)$. Expanding $f(e^{i\theta}z)$ into Fourier series with respect to θ,

$$f(e^{i\theta}z) = \sum_{-\infty}^{\infty} C_m(z)e^{im\theta}, \tag{50}$$

we know that (50) is absolutely convergent in the topology of $C^\infty(D)$. Then we have

$$C_m(z) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta}z)e^{-im\theta}\,d\theta. \tag{51}$$

It is clear that $C_m(z) \in \varepsilon_\lambda^\nu(D)$ and satisfies

$$C_m(e^{i\theta}z) = e^{im\theta}C_m(z).$$

Thus $C_m(z) = a_m\varphi_{\lambda,m}^\nu(z)$. This means that

$$f(z) = \lim_{N\to\infty}\sum_{-N}^{N} a_m\varphi_{\lambda,m}^\nu(z).$$

Conversely, suppose that λ is not simple. From the expression of $\varphi_{\lambda,m}^\nu(z)$ we know that there exists $m \in Z$ such that $\varphi_{\lambda,m}^\nu(z) = 0$. Because that there exists $f(z) \neq 0$ in $\varepsilon_\lambda^\nu(D)$ satisfying $f(e^{i\theta}z) = e^{im\theta}f(z)$, we know that H_λ^ν is not dense in $\varepsilon_\lambda^\nu(D)$.

THE PROOFOF LEMMA 4. It follows from the definition that λ is simple if and only if the set

$$\{F(b) = \sum_k a_k e_\lambda^{\nu,b}(z_k)\colon a_k \in \mathbf{C},\ z_k \in D\}$$

is dense in $L^2(B)$. Suppose that $\bar{\lambda}$ is simple and V is a non-zero invariant closed space of $T^{\nu,\lambda}$, From the proof of Lemma 2 we know that $\varphi_\lambda^\nu(z) \in V$.
Notice that (see [5])

$$\varphi_\lambda^\nu(z) = \{(g^{-1})'(0)\}^{\frac{\nu}{2}}\{(g^{-1})'(z)\}^{-\frac{\nu}{2}} \int_B e_{\lambda,b}^\nu(z) e_{-\lambda,b}^\nu(\overline{g0})\, db. \tag{52}$$

We get

$$\begin{aligned} &\sum_k a_k \varphi_\lambda^\nu(z)\{(g_k^{-1})'(z)\}^{\frac{\nu}{2}} \\ &= \int_B e_{\lambda,b}^\nu(z) \sum_k a_k \{g_k^{-1})'(0)\}^{\frac{\nu}{2}} e_{-\lambda,\bar{b}}^\nu(\overline{g_k 0})\, db \\ &= \int_B e_{\lambda,b}^\nu(z) \sum_k a_k \{(g_k^{-1})'(0)\}^{\frac{\nu}{2}} \overline{e_{\bar{\lambda},b}^{\nu}(g_k 0)}\, db. \end{aligned} \tag{53}$$

Since $\bar{\lambda}$ is simple, $\sum_k a_k \{(g_k^{-1})'(0)\}^{\frac{\nu}{2}} e_{\lambda,b}^\nu(z)$ is dense in $L^2(B)$. Thus V is dense in H_λ^ν, *and* $\bar{V} = H_\lambda^\nu$. So $T^{\nu,\lambda}$ is irreducible. Conversely, if $T^{\nu,\lambda}$ is irreducible, notice that λ is simple and

$$\{\sum_k a_k \varphi_\lambda^\nu(z)\{(g_k^{-1})'(z)\}^{\frac{\nu}{2}}\colon a_k \in \mathbf{C},\ g_k \in \mathrm{SU}(1,1)\}$$

as a non-zero invariant subspace of $T^{\nu,\lambda}$ is dense in H_λ^ν. We get

$$\{\sum_k a_k \{(g_k^{-1})'(0)\}^{\frac{\nu}{2}} \overline{e_{\bar{\lambda},b}^{\nu}(g_k 0)}\colon a_k \in \mathbf{C},\ g_k \in \mathrm{SU}(1,1)\}$$

is dense in $L^2(B)$. Thus $\bar{\lambda}$ is simple.

As in Theorem 4.3 of [4], we can also give the integral representation of eigenspace $\varepsilon_\lambda^\nu(D)$. Let $A(B)$ denote the holomorphic function space on B, $A'(B)$ its dual space. The element T of $A'(B)$ is called a holomorphic functional (or hyperfunction). For $f \in A(B)$ and $T \in A'(B)$, we define formally

$$T(f) = \int_B f(b)\, dT(b)$$

THEOREM 5. *Let λ be simple, then the map*

$$T \mapsto f(z) = \int_B e^{\nu}_{\lambda,b}(z)\,dT(b)$$

is a bijection from $A'(B)$ onto $\varepsilon^{\nu}_{\lambda}(D)$.

We omit the proof.

§4. Irreducible decomposition of T^{ν}.

In §2, we establish the Plancherel formula which is equivalent to the irreducible decomposition of the unitary representation T^{ν}. Now we give some further discussion.

THEOREM 6. *For $\nu \in \mathbf{Z}$, the unitary representation T^{ν} of* SU(1, 1) *is decomposed uniquely into the sum of the irreducible representations as follows*

(i) *if* $\nu = 0$, $$T^{\nu} \sim \int_{-\infty}^{\infty} \pi^{e}_{i\lambda}\rho_{\nu}(\lambda)\,d\lambda,$$

(ii) *if* $\nu = 2, 4, \ldots$, $$T^{\nu} \sim \int_{-\infty}^{\infty} \pi^{e}_{i\lambda}\rho_{\nu}(\lambda)\,d\lambda \oplus \pi^{+}_{2} \oplus \pi^{+}_{4} \ldots \oplus \pi^{+}_{\nu},$$

(iii) *if* $\nu = -2, -4, \ldots$, $$T^{\nu} \sim \int_{-\infty}^{\infty} \pi^{e}_{i\lambda}\rho_{\nu}(\lambda)\,d\lambda \oplus \pi^{-}_{-2} \oplus \pi^{-}_{-4} \ldots \oplus \pi^{-}_{\nu},$$

(iv) *if* $\nu = 1, 3, \ldots$, $$T^{\nu} \sim \int_{-\infty}^{\infty} \pi^{o}_{i\lambda}\rho_{\nu}(\lambda)\,d\lambda \oplus \pi^{+}_{3} \oplus \pi^{+}_{5} \ldots \oplus \pi^{+}_{\nu},$$

(v) *if* $\nu = 1, -3, \ldots$, $$T^{\nu} \sim \int_{-\infty}^{\infty} \pi^{o}_{i\lambda}\rho_{\nu}(\lambda)\,d\lambda \oplus \pi^{-}_{-3} \oplus \pi^{-}_{-5} \ldots \oplus \pi^{-}_{\nu},$$

where π^{o}_{0} should be replaced by π^{+}_{1} in (iv) *and π^{-}_{-1} in* (v) *respectively.*

PROOF. First let us describe the discrete parts. As in [5], we can write out the reproducing kernel $K^{\nu}_{l}(z, w)$ of $A^{\nu}_{l}(D)$

$$K^{\nu}_{l}(z, w) = \frac{|\nu| - 1 - 2l}{\pi} \int_B e^{\nu}_{-i(|\nu|-1-2l),b}(z)\overline{e^{\nu}_{i(|\nu|-1-2l),b}(w)}\,db. \tag{54}$$

It is easy to verify that

$$K^{\nu}_{l}(gz, gw) = K^{\nu}_{l}(z, w)\{g'(z)\}^{-\frac{\nu}{2}}\{\overline{g'(w)}\}^{-\frac{\nu}{2}}. \tag{55}$$

If $\nu > 1$, we have (see [5])

$$K^{\nu}_{l}(z, w) = \frac{\nu - 1 - 2l}{\pi}\frac{1}{(1 - z\bar{w})^{\nu}} \cdot F\left(-l, l - \nu + 1; 1; -\frac{|z - w|^2}{(1 - |z|^2)(1 - |w|^2)}\right), \tag{56}$$

in particular,

$$K_0^\nu(z, w) = \frac{\nu - 1}{\pi} \frac{1}{(1 - z\bar{w})^\nu}$$

is just the reproducing kernel of the weighted Bergman space $A^\alpha(D)$, ($\alpha = \nu - 2$), i.e. $A_0^\nu(D) = A^\alpha(D)$. And $A_l^\nu(D)$ has an orthogonal basis (again see [5])

$$(57)\quad \varepsilon_{l,n}^\nu(z) = \frac{(l+n)!}{n!} F\left(-l, l - \nu + 1; n + 1; -\frac{|z|^2}{1 - |z|^2}\right) z^n, \qquad n \geqq -l.$$

Let $T^{\nu,l}$ denote the restriction of T^ν to $A_l^\nu(D)$. Then for $k_\theta = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$, we have

$$(58)\qquad T^{\nu,l}(k_\theta)\varepsilon_{l,n}^\nu(z) = e^{i(\nu + 2n)\theta}\varepsilon_{l,n}^\nu(z).$$

This shows that $T^{\nu,l}$ has the lowest weight $\nu - 2l$ for $\nu = 2, 3, 4, \ldots$. Therefore

$$T^{\nu,l} \sim \pi_{\nu-2l}^+, \qquad \nu = 2, 3, 4, \ldots$$

(see [6]). And it is clear that $A_l^\nu(D)$ and $A^{\nu-2l}(D)$, as SU(1, 1)-modules, are unitarily equivalent.

If $\nu < 0$, let $g = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}$ satisfy $gw = 0$, then (55) yields

$$(59)\quad K_l^\nu(z, w) = K_l^\nu(gz, 0)\{g'(z)\}^{\frac{\nu}{2}}\overline{\{g'(w)\}}^{\frac{\nu}{2}} = (1 - z\bar{w})^{|\nu|}K_l^\nu(gz, 0)$$

$$= (1 - z\bar{w})^{|\nu|}\frac{|\nu| - 1 - 2l}{\pi}(1 - |gz|^2)^{|\nu| - 1}$$

$$\int_B (1 - gz\bar{b})^l(1 - \overline{gz}b)^{-|\nu| + l}db$$

$$= \frac{|\nu| - 1 - 2l}{\pi}\frac{(1 - |z|^2)^{|\nu|}(1 - |w|^2)^{|\nu|}}{(1 - \bar{z}w)^{|\nu|}}\cdot$$

$$\left\{\frac{(1 - |z|^2)(1 - |w|^2)}{|1 - z\bar{w}|^2}\right\}^{-1} F\left(-l, |\nu| - l; 1; \frac{|z - w|^2}{| - z\bar{w}|^2}\right)$$

$$= \frac{|\nu| - 1 - 2l}{\pi}\frac{(1 - |z|^2)^{|\nu|}(1 - |w|^2)^{|\nu|}}{(1 - \bar{z}w)^{|\nu|}}\cdot$$

$$F\left(-l, l + 1 - |\nu|; 1; -\frac{|z - w|^2}{(1 - |z|^2)(1 - |w|^2)}\right).$$

Here we have used the formula (see [3], P. 64)

$$F(a,b;c;z) = (1-z)^{-a} F\left(a, c-b; c; \frac{z}{z-1}\right).$$

Comparing (56) with (59), it is easy to see that the map

$$U^\nu\colon f(z) \mapsto (1-|z|^2)^\nu f(z)$$

is an unitary isometry from $A_l^\nu(D)$ to $\bar{A}_l^{|\nu|}(D)$, where

$$\bar{A}_l^{|\nu|}(D) = \{f(z)\colon \overline{f(z)} \in A_l^{|\nu|}(D)\}.$$

So $A_l^\nu(D)$ has orthogonal basis:

$$\text{(60)} \qquad \varepsilon_{l,n}^\nu(z) = (1-|z|^2)^{|\nu|} \varepsilon_{l,n}^{|\nu|}(\bar{z}), \qquad n \geqq -l.$$

Since

$$\text{(61)} \qquad T^{\nu,l}(k_\theta)\varepsilon_{l,n}^\nu(z) = e^{i(\nu-2n)\theta}\varepsilon_{l,n}^\nu(z),$$

we see that $T^{\nu,l}$ has the highest weight $\nu + 2l$ for $\nu = -2, -3, -4, \ldots$, and

$$T^{\nu,l} \sim \pi_{\nu+2l}^-, \nu = -2, -3, -4, \ldots.$$

Clearly $A_l^\nu(D)$ and $A^{\nu+2l}(D)$, as SU(1, 1)-modules, are unitarily equivalent.

Now we turn to the continuous part. Let $\nu = \pm 1, \pm 3, \ldots$ and $\lambda \in \mathrm{R}\backslash\{0\}$ or $\nu = \pm 2, \pm 4, \ldots$ and $\lambda \in \mathrm{R}$. We have seen that $\{\hat{f}(\lambda, \cdot)\colon f \in \mathscr{D}(D)\}$ is dense in $L^2(B)$, H_λ^ν is a Hilbert space with an orthogonal basis $\{\varphi_{\lambda,m}^\nu(z)\colon m \in \mathrm{Z}\}$ and $T^{\nu,\lambda}$ is an irreducible unitary representation. Since

$$\text{(62)} \qquad T^{\nu,\lambda}(k_\theta)\varphi_{\lambda,m}^\nu(z) = e^{i(\nu+2m)}\varphi_{\lambda,m}^\nu(z),$$

we know that

$$T^{\nu,\lambda} \sim \pi_{i\lambda}^e, \qquad \text{for} \quad \nu = 0, \pm 2, \pm 4, \ldots,$$

and

$$T^{\nu,\lambda} \sim \pi_{i\lambda}^o, \qquad \text{for} \quad \nu = \pm 1, \pm 3, \ldots.$$

Finally let us look at the point $\lambda = 0$ for $\nu = \pm 1, \pm 3, \ldots$. Notice that

$$\rho_\nu(0) = \frac{1}{2\pi^2} \neq 0, \qquad \text{as} \quad \nu = \pm 1, \pm 3, \ldots.$$

These cases should not be ignored. Suppose $\nu = 1, 3, \ldots,$. From the expression (44) of $\varphi_{\lambda,m}^\nu(z)$, we see that $\varphi_{0,m}^\nu(z) = 0$ provided $m < \frac{1-\nu}{2}$. So H_0^ν has an orthogonal basis $\{\varphi_{0,m}^\nu(z)\colon m = \frac{1-\nu}{2}, \frac{3-\nu}{2}, \ldots\}$. Let L be the subspace of $L^2(B)$ spanned by $\{\chi_m(b)\colon m = \frac{1-\nu}{2}, \frac{3-\nu}{2} \ldots\}$. Then $\{\hat{f}(0,b)\colon f \in \mathscr{D}(D)\} \cap L$ is dense in L. This can be proved by the argument found in the note after Lemma 1. Thus (62) tells us that $T^{\nu,0}$ has the lowest weight $(\nu + 2\cdot\frac{1-\nu}{2}) = 1$ and

$$T^{\nu,0} \sim \pi_1^+, \qquad \nu = 1, 3, \ldots.$$

The same argument shows that

$$T^{\nu,0} \sim \pi_{-1}^-, \qquad \nu = -1, -3, \ldots.$$

REMARK. In particular, in the case of $\nu = 1$, $\varphi_{0,m}^1(z) = z^m$ for $m \geqq 0$ and $\varphi_{0,m}^1(z) = 0$ for $m < 0$. Then H_0^1 is nothing but the Hardy space

$$H^2(D) = \left\{ f(z): \quad f \text{ is holomorphic in } D \text{ and } \|f\|_{H^2} = \sup_{0<r<1} \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 \, d\theta \right)^{\frac{1}{2}} < \infty \right\}.$$

Now H_0^ν $(\nu = 3, 5, \ldots)$ and H_0^1, as $SU(1,1)$-modules, are unitarily equivalent, thus they are unitarily equivalent to the Hardy space $H^2(D)$. And we know that if $f \in H_0^1$, $f \neq 0$, then $f \notin L^2(D, d\mu_1)$. This gives an explanation for the reason that one considers the Hardy space $H^2(D)$ as the limit of the weighted Bergman spaces $A^\alpha(D)$ as $\alpha \to -1$.

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