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APPROXIMATELY INNER AUTOMORPHISMS OF SEMI-FINITE VON NEUMANN ALGEBRAS

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§1. Introduction and preliminaries.

Throughout this paper M will denote a σ-finite von Neumann algebra for which we will assume that (H, J, P) is a standard form. M' will denote the commutant of M on the Hilbert space H. Also if $\psi \in M_*^+$ then $\xi_\psi \in P$ will denote the unique vector in P implementing ψ. $C^*(M, M')$ will denote the C^*-algebra on H generated by M and M'. In [5, Theorem 3.1] Connes used $C^*(M, M')$ to characterize approximately inner automorphisms of a separable factor of type II_1. As an automorphism ϑ of M is always normal, we obtain by transposition an isometry $\vartheta : M_* \to M_*$. An automorphism is then said to be *approximately inner* if it is the limit in the topology of point-norm convergence on M_* of a net of inner automorphisms. If M is separable, i.e. M_* is a separable Banach space, then we may take this net to be a sequence. Connes' characterization went as follows. If ϑ is an automorphism of a separable II_1 factor M, then as will be explained below, we obtain Θ, an automorphism of $C(M, M')$, the $*$-subalgebra of $\mathscr{L}(H)$ generated by M and M', by setting $\Theta(\sum x_i y_i) = \sum \vartheta(x_i) y_i$. Connes showed that ϑ is approximately inner if and only if Θ extends to an automorphism of $C^*(M, M')$. In this paper we shall give a new proof of this result using a probabilistic technique introduced by Haagerup [12]. In addition to being conceptually simpler the proof has the advantage that it works for any σ-finite algebra of type II_1.

It is clear that any approximately inner automorphism will act trivially on the centre of M so from now on we will tacitly assume that the automorphisms of M under consideration will act trivially on the centre of M. We can now state our main result.

THEOREM 1. *Let M be a semi-finite σ-finite von Neumann algebra and ϑ an automorphism of M which acts trivially on the centre of M. Then ϑ is approximately inner if and only if ϑ fixes some faithful normal semi-finite trace on M and Θ extends to an automorphism of $C^*(M, M')$.*

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Most of the work in this paper is devoted to proving the 'if' part of the Theorem 1. Let us dispense with the "only if" part. So let us suppose that ϑ is the limit in the point norm topology on M_* of a net $\{\vartheta_\alpha\}$ of inner automorphisms. We first want to show that ϑ fixes some faithful normal semi-finite trace. If M is finite $\mathscr{E}_z$ denotes the unique centre valued trace on M (see e.g. [20, Theorem 7.11]), then by uniqueness $\mathscr{E}_z \circ \vartheta = \mathscr{E}_z$ and so for any trace τ, $\tau = \tau \circ \mathscr{E}_z = \tau \circ \mathscr{E}_z \circ \vartheta = \tau \circ \vartheta$, ([20, Exercise E.7.12]).

If M is semi-finite and Tr is a faithful normal semi-finite trace then we shall show that the Radon-Nikodym cocycle $(D\mathrm{Tr} \circ \vartheta^{-1}; D\mathrm{Tr})_t = 1$ for all t. Then we may apply [19, Corollary 3.6]. Let ψ be a faithful normal state, then by the chain rule $(D\mathrm{Tr} \circ \vartheta^{-1}; D\mathrm{Tr})_t = \vartheta((D\mathrm{Tr}; D\psi)_t)(D\psi \circ \vartheta^{-1}; D\psi \circ \vartheta_\alpha^{-1})_t \vartheta_\alpha((D\mathrm{Tr} : D\psi)_{-t})$. For a fixed t, $(D\psi \circ \vartheta^{-1}; D\psi \circ \vartheta_\alpha^{-1})_t \to 1$ strongly ([19, Proposition 7.18]), and for the same t, $\vartheta_\alpha((D\mathrm{Tr}; D\psi)_t) \to \vartheta((D\mathrm{Tr}; D\psi)_t)$ weakly and hence strongly. So $(D\mathrm{Tr} \circ \vartheta^{-1}; D\mathrm{Tr})_t = (D\mathrm{Tr} \circ \vartheta^{-1};\ D\mathrm{Tr} \circ \vartheta_\alpha^{-1})_t \to 1$ as $\alpha \to \infty$. This can only happen if $(D\mathrm{Tr} \circ \vartheta^{-1}; D\mathrm{Tr})_t = 1$, i.e. $\mathrm{Tr} \circ \vartheta^{-1} = \mathrm{Tr}$.

Now choose a finite projection e in M of central support 1 and let $N = eMe$. Write $M \cong N \mathbin{\overline{\otimes}} \mathscr{L}(H)$. By Lemma 3.1 there is a unitary u in M so that $\mathrm{Ad}_u \circ \vartheta = \alpha \otimes \iota$ relative to the tensor product decomposition of M, where α is an automorphism of N. Let φ_o be a faithful normal state on $\mathscr{L}(H)$ and τ be a faithful normal trace on N. Then $\varphi = \tau \otimes \varphi_o$ is a faithful normal state on M which is fixed by $\mathrm{Ad}_u \circ \vartheta$. Hence both in the finite and semi-finite case we may, upon replacing ϑ by $\mathrm{Ad}_u \circ \vartheta$ if necessary, suppose that ϑ fixes some faithful normal state φ. (This is only a special case of a more general result [20,25.1]). Suppose $\vartheta_\alpha = \mathrm{Ad}_{u_\alpha}$, then by the Powers-Størmer inequality $\|[u_\alpha, \xi_\varphi]\|^2 = \|\xi_\varphi - u_\alpha^* \xi_\varphi u_\alpha\|^2 = \|\xi_\varphi - \xi_{u_\alpha^* \varphi u_\alpha}\|^2 \leqq \|\varphi \circ \vartheta - \varphi \circ \vartheta_\alpha\| \to 0$.

Now let us recall the canonical implementation of an automorphism [11, Theorem 2.3]. If ψ is a faithful normal state on M, let $u_\vartheta(x\xi_\psi) = \vartheta(x)\,\xi_{\psi\circ\vartheta^{-1}}$. Then u_ϑ doesn't depend on ψ and for all $x \in M$ $\vartheta(x) = u_\vartheta x u_\vartheta^*$, $Ju_\vartheta = u_\vartheta J$, and $u_\vartheta P = P$. If $\vartheta = \mathrm{Ad}_u$ for $u \in M$, then $u_\vartheta = uJuJ$. If $\varphi \circ \vartheta = \varphi$, then $u_\vartheta \xi_\varphi = \xi_\varphi$.

Now if ψ is any state on M we have $\|u_\vartheta \xi_\psi - u_{\vartheta_\alpha} \xi_\psi\|^2 \leqq \|\psi \circ \vartheta^{-1} - \psi \circ \vartheta_\alpha^{-1}\| \to 0$. So $u_\alpha J u_\alpha J = u_{\vartheta_\alpha} \to u_\vartheta$ strongly, as the linear span of $\{\xi_\psi \mid \psi \in M_*^+\}$ equals H. Thus for all x in M $\|(\vartheta(x) - u_\alpha x u_\alpha^*)\xi_\varphi\| \leqq \|(u_\vartheta - u_\alpha J u_\alpha J)x\xi_\varphi\| + \|x\|\,\|[u_\alpha, \xi_\varphi]\| \to 0$. Hence for all x in M, $u_\alpha x u_\alpha^*$ converges strongly to $\vartheta(x)$.

Let $x_1, \ldots, x_n \in M$, $y_1, \ldots, y_n \in M'$, and $\xi \in H$ be given, then $\|\Theta(\sum x_i y_i)\xi\| = \|\sum \vartheta(x_i) y_i \xi\| = \lim_\alpha \|\sum u_\alpha x_i u_\alpha^* y_i \xi\| \leqq \|\sum x_i y_i\|\,\|u_\alpha^* \xi\| = \|\sum x_i y_i\|\,\|\xi\|$. Hence Θ is bounded on $C^*(M, M')$. By applying the same argument to ϑ^{-1} we may similarly conclude that Θ^{-1} is bounded and thus Θ is an automorphism.

As mentioned above when an automorphism ϑ acts trivially on the centre of M we obtain an automorphism Θ of $\mathrm{C}(M, M')$ by the equation $\Theta(\sum x_i y_i) = \sum \vartheta(x_i) y_i$ for $x_i \in M$ and $y_i \in M'$. This clearly will be an automorphism if we can show that it is well defined; that this is so follows from the following lemma of Murray and von Neumann.

LEMMA 2. *Let* $x_1, \ldots, x_n \in M$ *and* $y_1, \ldots, y_n \in M'$ *be such that* $\sum_i x_i y_i = 0$. *Then there exists a projection* $\{e_{ij}\} \in M_n(Z(M))$ *such that* $\sum_i e_{ij} x_i = 0$ *for all* $1 \leqq j \leqq n$, *and* $\sum_j e_{ij} y_j = y_i$ *for all* $1 \leqq i \leqq n$.

PROOF ([17, Theorem 3 p. 140], and [14, Lemma 3.1.3]). The proof in [17] is nominally for factors but a careful reading shows that it goes through without change in the more general case. The projection e is constructed as follows. Let $H^{(n)}$ denote n copies of the Hilbert space H and K the subspace of $H^{(n)}$ consisting of vectors $(\xi_1, \ldots, \xi_n)$ such that $\sum_i x_i a \xi_i = 0$ for all $a \in M$. Then e is the projection of $H^{(n)}$ onto K.

COROLLARY 3. $\mathrm{C}(M, M')$ *is isomorphic to* $M \otimes_Z M'$ *via* $\sum x_i y_i \mapsto \sum x_i \otimes y_i$ *for* $x_i \in M$ *and* $y_i \in M'$, *(Z denotes the centre of M).*

The main novelty of our approach is the explicit use of correspondences. Let us recall briefly the definitions. Let M be a von Neumann algebra and M^{op} its opposite algebra; M^{op} is isomorphic to M', but from the conceptual point of view it is easier to use M^{op}. Effros and Lance introduced the notion of a binormal representation of the algebraic tensor product $M \otimes M^{\mathrm{op}}$; it is a pair of commuting normal representations on the same Hilbert space: one of M and one of M^{op}. For $x \in M \otimes M^{\mathrm{op}}$ let $\|x\|_{\mathrm{bin}}$ be the supremum of $\|\pi(x)\|$ as π runs over the binormal representations of $M \otimes M^{\mathrm{op}}$. Completing with respect to this norm we obtain the C^*-algebra $M \otimes_{\mathrm{bin}} M^{\mathrm{op}}$. A *correspondence from M to M* is a binormal representation of $M \otimes_{\mathrm{bin}} M^{\mathrm{op}}$. Given a completely positive normal map $\Phi: M \to M$, we associate to Φ a binormal representation (π_Φ, H_Φ) of $M \otimes_{\mathrm{bin}} M^{\mathrm{op}}$ as follows. On $M \otimes H$ define the sesquilinear form generated by $(a \otimes \xi \mid b \otimes \eta) = (\Phi(b^* a)\xi \mid \eta)$. Mod out by the null vectors and complete to obtain a Hilbert space H_Φ. On H_Φ define a representation of $M \otimes M^{\mathrm{op}}$ by $\pi_\Phi(x \otimes y^\circ) a \otimes \xi = xa \otimes Jy^* J\xi$. If M is σ-finite (as it will be throughout this paper) and $\xi \in P$ is a cyclic and separating vector then $\xi_\Phi = 1 \otimes \xi \in H_\Phi$ will be a cyclic vector for the representation π_Φ. Two particular completely positive maps will be of interest to us. The first is the identity map ι. In this case $H_\iota = H$, $\pi_\iota(x \otimes y^\circ) = xJy^* J \in \mathscr{L}(H)$, and $\pi_\iota(M \otimes_{\mathrm{bin}} M^{\mathrm{op}}) = C^*(M, M')$. The second example is when the completely positive map is an automorphism ϑ. In this case $H_\vartheta = H$, $\pi(x \otimes y^\circ) = \vartheta(x) Jy^* J$, and $\pi_\vartheta(M \otimes_{\mathrm{bin}} M^{\mathrm{op}}) = C^*(M, M')$. If ϑ is one of those automorphisms for which Θ extends to an automorphism of $C^*(M, M')$, then $\pi_\vartheta = \Theta \circ \pi_\iota$ and so π_ϑ and π_ι are weakly equivalent in the sense of Fell (see e.g.

Dixmier [9, Definition 2.4.5]), and conversely if π_ϑ and π_ι are weakly equivalent then Θ extends to an automorphism of $C^*(M, M')$. This explains the appearance of the C*-algebra $C^*(M, M')$ in the picture. For further details on correspondences the reader may consult [1], [2], [3], [15], [16] and [18].

Recall that a completely positive map is called *inner* if it is of the form $\Phi(x) = \sum a_i^* x a_i$ with $\{a_i\} \subseteq M$. Φ is *approximately inner* if Φ is the limit in the point-$\sigma(M, M_*)$ topology of a net of inner maps Φ_α with $\|\Phi_\alpha\| \leqq 1$. In [16] and [3] it was shown that ϑ is approximately inner as a completely positive map if and only if π_ϑ is weakly contained in π_ι, i.e. if and only if Θ extends to a bounded homomorphism of $C^*(M, M')$. One may ask if it is possible that for some automorphism ϑ, Θ extends to a bounded homomorphism of $C^*(M, M')$ which is not an automorphism? In §2 (remark following Lemma 1) we shall show that this cannot happen; for this reason we shall just say that for a particular automorphism Θ *is bounded*, but by this mean that Θ extends to an automorphism of $C^*(M, M')$. But another question arises; can an automorphism be approximately inner as a completely positive map but not be an approximately inner automorphism? If M is a finite von Neumann algebra then we shall show in this paper that if an automorphism is approximately inner as a completely positive map then it is approximately inner as an automorphism. Connes had already done most of the work necessary to show this for separable finite *factors*. Moreover he also observed that for every automorphism ϑ of the separable injective factor of type II_∞, Θ is bounded but if ϑ scales the trace then ϑ cannot be an approximately inner automorphism. In §3 we shall show that this is the only obstruction for algebras of type II_∞. In the type III case the analogue of fixing the trace is the condition of having $\mathrm{mod}(\vartheta) = 1$, and Connes has suggested that this should be the only obstruction [8. §4]. We shall conclude this introduction by showing that for an injective semi-finite von Neumann algebra Θ is bounded for any automorphism ϑ which acts trivially on the centre.

By the equivalence of injectivity and semi-discreteness, we know that for an injective von Neumann algebra M there exists a map $\eta\colon M \otimes_{\min} M' \to C^*(M, M')$ (see [10, Proposition 4.5]). When M is a factor η is an isomorphism by the minimality of the spatial norm $\|\cdot\|_{\min}$. If we transfer the automorphism $\vartheta \otimes \iota$ of $M \otimes_{\min} M'$ to $C^*(M, M')$ via η we get that Θ is an automorphism of $C^*(M, M')$. When M is not a factor we need to show that the kernel of η is left invariant by $\vartheta \otimes \iota$. We were able to prove this only by an indirect method; we construct a C*-algebra which we call $M \otimes_{Z\text{-}\min} M'$, which, for any von Neumann algebra M is always a quotient of $M \otimes_{\min} M'$ via a homomorphism we shall call $\pi_{\mathscr{E}}$, such that the automorphism $\vartheta \otimes \iota$ of $M \otimes_{\min} M'$ descends to an automorphism, called $\vartheta \otimes_Z \iota$ provided that ϑ acts trivially on the centre of M. The algebra $M \otimes_{Z\text{-}\min} M'$ is a completion of $M \otimes_Z M'$ (the tensor product over Z, the centre of M), which is minimal in a certain sense. When the centre of M is two dimensional i.e.

$M = M_1 \oplus M_2$ with M_1 and M_2 factors then $M \otimes_{Z\text{-min}} M' \cong (M_1 \otimes_{\min} M'_1) \oplus (M_2 \otimes_{\min} M'_2)$, so heuristically $M \otimes_{Z\text{-min}} M'$ can be regarded as a direct integral of minimal tensor products. The properties of $M \otimes_{Z\text{-min}} M'$ that we shall have to establish are as follows.

THEOREM 4. *For any von Neumann algebra M there is an automorphism* $\vartheta \otimes_Z \iota$ *of* $M \otimes_{Z\text{-min}} M'$ *such that* $\vartheta \otimes_Z \iota \circ \pi_{\mathscr{E}} = \pi_{\mathscr{E}} \circ \vartheta \otimes \iota$.

THEOREM 5. *Let M be a semi-finite von Neumann algebra, then there exists a *-homomorphism* ρ: $C^*(M, M') \to M \otimes_{Z\text{-min}} M'$ *which is onto and extends the isomorphism of Corollary* 3.

THEOREM 6. *Let M be an injective semi-finite von Neumann algebra, then there exists* $\hat{\eta}$: $M \otimes_{Z\text{-min}} M' \to C^*(M, M')$ *such that* $\hat{\eta} = \rho^{-1}$ *and* $\eta = \hat{\eta} \circ \pi_{\mathscr{E}}$.

Let us recall some facts from Takesaki [21, §4] about conditional expectations. Let φ be a faithful normal state on M and $\xi_\varphi \in P$ the implementing vector. Let e_Z be the projection onto $[Z(M)\mathscr{E}_\varphi]$. Then $e_Z M e_Z = e_Z Z(M) e_Z$ and e_Z commutes with J (in this special case a more direct proof is available that avoids most of the technical complications of the general case, see e.g. [15, Lemma 1.1]). Let ε: $Z(M) \to e_Z Z(M) e_Z$ be the isomorphism $\varepsilon(x) = e_Z x e_Z$. Then $\mathscr{E}(x) = \varepsilon^{-1}(e_Z x e_Z)$ is a faithful normal conditional expectation of M onto Z, such that $e_Z \mathscr{E}(x) = e_Z x e_Z$ and $\varphi(\mathscr{E}(x)) = \varphi(x)$ for all x in M.

Now consider the von Neumann subalgebra N of $\mathscr{L}(H)$ generated by M and e_Z. Let $x \in N'$ then $x \in M'$ and $e_Z x = x e_Z$; hence $x = JyJ$ for some $y \in \mathrm{M}$. Thus $JyJ\xi_\varphi = x e_Z \xi_\varphi = e_Z x \xi_\varphi = J e_Z y \xi_\varphi = J\mathscr{E}(y)J\xi_\varphi$, and $x \in Z$ as ξ_φ is separating for M'. Thus $N' = Z$ and so $N = M \vee M'$.

Now N is a type I von Neumann algebra and we shall construct a normal faithful semi-finite trace Tr_N on N as follows. First observe that the central support of e_Z in N is 1, and second that e_Z is a finite (in fact abelian) projection in N; in fact ω_{ξ_φ} is a trace on N_{e_Z}. Thus there is a unique faithful normal semi-finite trace Tr_N on N such that $\mathrm{Tr}_N \mid N_{e_Z} = \omega_{\xi_\varphi}$, (for example one could write N as $N_{e_Z} \mathbin{\bar{\otimes}} \mathscr{L}(l^2(\mathrm{N}))$ and let $\mathrm{Tr}_N = \omega_{\xi_\varphi} \otimes \mathrm{Tr}$, where Tr is the usual trace on $\mathscr{L}(l^2(\mathrm{N}))$.

Now form the Hilbert space $L^2(N, \mathrm{Tr}_N)$. Letting $J_N \eta_{\mathrm{Tr}_N}(x) = \eta_{\mathrm{Tr}_N}(x^*)$ and $P_N = \overline{\eta_{\mathrm{Tr}_N}(N^+)}$ we obtain a standard form for N. By restriction we obtain a correspondence from M to M: for x, $y \in M$ and $\xi \in L^2(N, \mathrm{Tr}_N)$ let $x\xi y = x J_N y^* J_N \xi$. Observe that $\eta_{\mathrm{Tr}_N}(e_Z)$ is a cyclic vector for this correspondence as e_Z has central support 1 and $(x\eta_{\mathrm{Tr}_N}(e_Z)y \mid \eta_{\mathrm{Tr}_N}(e_Z)) = (\eta_{\mathrm{Tr}_N}(x e_Z y) \mid \eta_{\mathrm{Tr}_N}(e_Z)) = \mathrm{Tr}_N(e_Z x e_Z y) = (e_Z x e_Z y \xi_\varphi \mid \xi_\varphi) = (\mathscr{E}(x)\xi_\varphi y \mid \xi_\varphi)$. Thus as a correspondence from M to M $L^2(N, \mathrm{Tr}_N) \cong H_{\mathscr{E}}$. Moreover as $Z(M) = Z(N)$ we have that for all $\xi \in H_{\mathscr{E}}$ and all $z \in Z$ $z\xi = \xi z$. Thus we have a representation of the *-algebra $M \otimes_Z M'$ on $H_{\mathscr{E}}$. This representation is faithful: for if $\sum x_i \xi y_i = 0$ for all ξ in $H_{\mathscr{E}}$ then

$\sum_i x_i(\sum_j a_j e_Z b_j) y_i = 0$ for all a_j, b_j in M, and as e_Z has central support 1 we get that $\sum x_i y_i = 0$ and hence by Corollary 3 $\sum x_i \otimes y_i = 0$ in $M \otimes_Z M'$. Thus $\|\pi_{\mathscr{E}}(\cdot)\|$ is a norm on $M \otimes_Z M'$. Let $M \otimes_{Z\text{-min}} M'$ be the completion of $M \otimes_Z M'$ with respect to this norm.

If ψ is another faithful normal state on M and we let e'_Z be the projection onto $[Z(M)\xi_\psi]$, then $e'_Z \in Z(M)' = N$, and so e'_Z is an abelian projection with central support 1. Thus e'_Z is also a cyclic vector for the M–M correspondence $L^2(N, \mathrm{Tr}_N)$. Hence the norm $\|\pi_{\mathscr{E}}(\cdot)\|$ is independent of the choice of which conditional expectation we choose.

Proof of Theorem 4. Let $u = u_\vartheta$ be the canonical implementation of the automorphism ϑ. For $z \in Z$ and $a \in M$, $uza\xi_\psi = \vartheta(z)\vartheta(a)\xi_\varphi = z\vartheta(a)\xi_\varphi = zua\xi_\psi$. So by the cyclicity of ξ_ψ, $u \in Z' = N$.

Vectors of the form $T = \sum a_i e_Z b_i$, with $a_i, b_i \in M$ form a dense set of vectors in $L^2(N, \mathrm{Tr}_N)$, moreover if $S = u^* T$ then $\|S\|_2^2 = \mathrm{Tr}_N(S^*S) = \mathrm{Tr}_N(T^*T) = \|T\|_2^2$. Hence for $x_1, \ldots, x_n \in M$, and $y_1, \ldots, y_n \in M'$,

$$\begin{aligned} &\|\pi_{\mathscr{E}}(\textstyle\sum \vartheta(x_i) \otimes y_i)\|^2 \\ &= \sup\{\|\textstyle\sum \vartheta(x_i) T y_i\|_2^2 : T = \sum a_i e_Z b_i,\ \|T\|_2 = 1\} \\ &= \sup\{\textstyle\sum_{i,j,k,l} \mathrm{Tr}_N(y_j^* b_k^* e_Z a_k^* u x_j^* x_i u^* a_l e_Z b_l y_i) \mid T = \sum a_i e_Z b_i,\ \|T\|_2 = 1\} \\ &= \sup\{\|\textstyle\sum x_i u^* T y_i\|^2 \mid T = \sum a_i e_Z b_i,\ \|T\|_2 = 1\} \\ &= \|\pi_{\mathscr{E}}(\textstyle\sum x_i \otimes y_i)\|^2. \end{aligned}$$

Proof of Theorem 5. What we must show is that $\pi_{\mathscr{E}}$ is weakly contained in π_ι. By [3, Theorem 2.6] we must show that $\mathscr{E}$ is approximately inner as a completely positive map.

First let us suppose that M is finite, and that $\mathscr{E}$ denotes the centre valued trace of M; $\mathscr{E}$ is the conditional expectation of M onto Z along *any* trace τ. (Observe that if $\mathscr{E}$ is such a conditional expectation then it is tracial: $(\mathscr{E}([x,y])a\xi_\tau \mid b\xi_\tau) = \tau([x,y]\,\mathscr{E}(b^*a)) = 0$.) By the Dixmier averaging theorem for each $x \in M$ and $\varepsilon > 0$ there exists an inner completely positive map Φ of the form $x \mapsto \sum \lambda_i u_i x u_i^*$, with u_i a unitary in M and $0 < \lambda_i < 1$, $\sum \lambda_i = 1$, such that $\|\mathscr{E}(x) - \Phi(x)\| < \varepsilon$.

So let $x_1, \ldots, x_n \in M$ and $\varepsilon > 0$ be given. Choose Φ_1 of the form above such that $\|\mathscr{E}(x_1) - \Phi_1(x_1)\| < \varepsilon$. Choose Φ_2 of the same form again so that $\|\mathscr{E}(\Phi_1(x_2)) - \Phi_2 \circ \Phi_1(x_2)\| < \varepsilon$. Then as $\mathscr{E}(x_i) \in Z$, $\|\mathscr{E}(x_1) - \Phi_2 \circ \Phi_1(x_1)\| = \|\Phi_2(\mathscr{E}(x_1) - \Phi_1(x_1))\| \leqq \|\mathscr{E}(x_1) - \Phi_1(x_1)\| < \varepsilon$, and $\|\mathscr{E}(x_2) - \Phi_2 \circ \Phi_1(x_2)\| = \|\mathscr{E}(\Phi_1(x_2)) - \Phi_2 \circ \Phi_1(x_2)\| < \varepsilon$. Finally choose Φ_n such that $\|\mathscr{E}(x_n) - \Phi_n \circ \ldots \circ \Phi_1(x_n)\| < \varepsilon$, and let $\Phi = \Phi_n \circ \ldots \circ \Phi_1$. Then $\|\mathscr{E}(x_i) - \Phi(x_i)\| < \varepsilon$, for all i. Thus we have shown that the centre valued trace is approximately inner. As observed above $H_{\mathscr{E}}$ is independent of which conditional expectation we choose. Thus every faithful normal conditional expectation onto the centre is approximately inner.

Let us turn to the semi-finite case now. M may be decomposed as $M_1 \oplus M_2$ with M_1 finite and M_2 properly infinite and $\mathscr{E}$ as $\mathscr{E}_1 \oplus \mathscr{E}_2$. So to conclude the proof we shall assume that M is properly infinite. Write $M = N \bar{\otimes} \mathscr{L}(H)$ with N a finite algebra. Let $\varphi = \tau \otimes \varphi_o$ where τ is a faithful normal trace on N and φ_o is a faithful normal state on $\mathscr{L}(H)$ of the form $\varphi_\circ = \mathrm{Tr}(h\,\cdot)$ where h is a positive trace class operator: $h = \mathrm{diag}(h_1, h_2, \ldots)$ which is diagonal with respect to a system of matrix units $\{e_{ij}\}$ of $\mathscr{L}(H)$. Let $\mathscr{E}$ be the conditional expectation of M onto $Z(M)$ along φ; then $\mathscr{E}(\sum_{i,j} a_{ij} \otimes e_{ij}) = \sum h_i \mathscr{E}_\tau(a_{ii})$, where $\mathscr{E}_\tau$ is the conditional expectation of N onto $Z(N)$ along τ. Let $x_1, \ldots, x_m \in M$ and $\varepsilon > 0$ be given. Choose n such that for $p = \sum_{i=1}^{n} 1 \otimes e_{ii}$ one has $\|(x_i - px_ip)\xi_\varphi\| < \varepsilon/2$. Write $px_ip = \sum_{s,t=1}^{n} a_{st}^{(i)} \otimes e_{st}$. So choose an inner map Φ_o given by unitaries in $M_p \otimes 1$ as above so that $\sum_{s=1}^{n} h_s \|\mathscr{E}_\tau(a_{ss}^{(i)}) - \Phi_o(a_{ss}^{(i)})\| < \varepsilon/2$ for all i. Suppose $\Phi_o(x) = \sum \lambda_i x u_i^*$ with u_i unitaries in N. For $x \in N \bar{\otimes} \mathscr{L}(H)$ let $\Phi(x) = \sum_{i,j,k=1}^{n} \lambda_i h_j (u_i \otimes e_{jk}) x (u_i \otimes e_{jk})^*$. If $x = \sum_{i,j=1}^{n} x_{ij} \otimes e_{ij}$ then $\Phi(x) = \sum_i h_i \Phi_o(x_{ii})$. Hence $\|\mathscr{E}(x_i) - \Phi(px_ip)\|_\varphi \leqq \|\mathscr{E}(x_i) - \mathscr{E}(px_ip)\|_\varphi + \|\mathscr{E}(px_ip) - \Phi(px_ip)\| \leqq \|x_i - px_ip\|_\varphi + \|\sum h_j(\mathscr{E}_\tau(a_{jj}^{(i)}) - \Phi_o(a_{jj}^{(n)}))\| < \varepsilon$.

Our proof of Theorem 6 is a paraphrase of the technique of [7]. As before let $N = M \vee M' = \langle M, e_Z \rangle$.

LEMMA 7. *Let M be a finite injective von Neumann algebra with faithful normal trace τ. Let $\varepsilon > 0$ and $x_1, \ldots, x_n \in M$ be given, then there exists a normal state of N, φ, such that $\|[x_i, \varphi]\| \leqq \varepsilon$ for $1 \leqq i \leqq n$, and $\|\tau - \varphi|_M\| \leqq \varepsilon$.*

PROOF. Let E be a projection of norm 1 of N onto M which exists as M is injective, and let $\psi = \tau \circ E$. Then ψ is a state of N, (in general non-normal), and $x\psi = \psi x$ for all $x \in M$. Let $K = \mathrm{co}\{N_+ \cup \{x - \tau(x)1 \mid x = x^*,\ \|x\| \leqq 3\varepsilon^{-1},\ x \in M\}\}$. Then K is a $\sigma(N, N_*)$ closed convex set of the real Banach space $N_{\mathrm{s.a.}}$ (the self-adjoint elements of N). Let $K^\circ = \{f \in N_{\mathrm{s.a.}}^* \mid f(K) > -1\}$ and $K_\circ = K^\circ \cap N_*$. By [4, Théorème 1 of chap. II §2 n°3] $K = (K_\circ)^\circ$. Regarding K_o as a subset of $N_{\mathrm{s.a.}}^*$ we obtain that $((K_\circ)^\circ)^\circ$ is the $\sigma(N^*, N)$ closure of K_o, but $(K_\circ)^\circ = K$. Thus K° is the $\sigma(N^*, N)$ closure of $K_\circ$. Now $\psi|_M = \tau$, so $\psi \in K^\circ$. Hence there exists a net $\{\varphi_\alpha\}$ in K_o such that φ_α converges to ψ in the $\sigma(N^*, N)$-topology. As $\varphi_\alpha(N_+) \geqq -1$, we have that $\varphi_\alpha \geqq 0$, moreover as $\varphi_\alpha(1) \to \tau(1) = 1$ we may suppose that φ_α is a state and $\|\tau - \varphi_\alpha|_M\| \leqq \varepsilon$.

Let $N^{(n)}$ be the von Neumann algebra formed from n copies of N. We have that 0 is in the $\sigma(N_*^{(n)}, N^{(n)})$ closure of $\{([x_1,\omega],\ldots,[x_n,\omega]) \mid \omega \in N_*^+ \ \|\omega\| = 1\}$. As this set is convex we have that 0 is in the norm closure. Thus there exists a state $\varphi \in N_*^+$ such that $\|[x_i,\varphi]\| \leqq \varepsilon$ and $\|\tau - \varphi|_M\| \leqq \varepsilon$.

PROOF OF THEOREM 6. We begin by assuming that M is finite. It is clear that $\hat{\eta}(\sum x_i \otimes y_i) = \sum x_i y_i$, what we must show is that this is bounded. As ξ_τ is a cyclic vector for $\hat{\eta}$ we only have to show that $|(\hat{\eta}(X)\xi_\tau \mid \xi_\tau)| \leqq \|\pi_{\mathscr{E}}(X)\|$ for $X \in M \otimes_Z M'$. Now let $X = \sum_{i=1}^{n} x_i \otimes y_i$ with $\|x_i\|, \|y_i\| \leqq 1$. Let $\varepsilon > 0$ be given. Write $y_i = \sum_j u_{ij}$ with u_{ij} unitaries. Recall that N is the von Neumann subalgebra of $\mathscr{L}(H)$ generated by M and e_Z. By Lemma 7 we may choose a normal state φ of N such that $\sum_j \|[u_{ij},\varphi]\| < \varepsilon^2/4n^2$ and $\|\tau - \varphi|_M\| < \varepsilon/2n$. Let $T_\varphi \in P_N$ be the vector implementing φ. Then $\|[y_i, T_\varphi]\|_2 \leqq \sum_j \|[u_{ij}, T_\varphi]\|_2 = \sum_j \|u_{ij}T_\varphi u_{ij}^* - T_\varphi\|_2$, and so by the Powers-Størmer inequality $\|[y_i, T_\varphi]\|_2 \leqq \sum_j \|[u_{ij},\varphi]\|^{1/2} < \varepsilon/2n$.

Thus $\|\sum x_i T_\varphi y_i\|_2 \geqq |(\sum x_i y_i T_\varphi \mid T_\varphi)| - |(\sum x_i [T_\varphi, y_i] \mid T_\varphi)| > |(\sum x_i y_i T_\varphi \mid T_\varphi)| - \varepsilon/2$. On the other hand $|(\sum x_i y_i T_\varphi \mid T_\varphi) - \tau(\sum x_i y_i)| < \varepsilon/2$. Thus $-\varepsilon/2 < (\sum x_i y_i T_\varphi \mid T_\varphi) - \tau(\sum x_i y_i)$, so $\|\sum x_i T_\varphi y_i\|_2 > |(\sum x_i y_i T_\varphi \mid T_\varphi)| - \varepsilon/2 > \tau(\sum x_i y_i) - \varepsilon = (\hat{\eta}(\sum x_i y_i)\xi_\tau \mid \xi_\tau) - \varepsilon$. Hence $(\hat{\eta}(\sum x_i y_i)\xi_\tau \mid \xi_\tau) \leqq \sup\{\|\sum x_i T y_i\|_2 \mid \|T\|_2 \leqq 1\} = \|\pi_{\mathscr{E}}(\sum x_i y_i)\|$.

Now let us turn to the general case. As in the proof of Theorem 5 we may assume that M is properly infinite. What we have just shown is that when M is a finite injective von Neumann algebra π_ι is weakly contained in $\pi_{\mathscr{E}}$, and we must show the same in the semi-finite case. By [3, Theorem 2.6] we must show that the identity map ι can be approximately factored through $\mathscr{E}$.

So let $x_1,\ldots,x_m \in M$, $\varphi_1,\ldots,\varphi_n \in M_*^+$, be states, and $\varepsilon > 0$ be given. We may find a finite projection e of central support 1 such that $|\varphi_i(x_j - ex_je)| < \varepsilon/2$, for all i and j. Let $y_i = ex_ie$. Write $M \cong M_e \mathbin{\overline{\otimes}} \mathscr{L}(H)$, identify M_e with $M_e \otimes 1$, and let $\varphi = \tau \otimes \varphi_o$ be a faithful normal state on $M_e \mathbin{\overline{\otimes}} \mathscr{L}(H)$ where τ is a trace on M_e and φ_o is a faithful normal state on $\mathscr{L}(H)$. Then $Z(M) = Z(M_e) \otimes 1$ and if $\mathscr{E}$ is the conditional expectation of M onto Z along φ then $\mathscr{E}(exe \otimes y) = \mathscr{E}_\tau(exe)\varphi_o(y)$, where $\mathscr{E}_\tau$ is the conditional expectation of M_e onto Z_e along τ. Hence $\mathscr{E}(y_i) = \mathscr{E}_\tau(y_i)$.

Now by the result in the finite case we know that there exist a_{ij}, b_{ij} in M_e such that $|\varphi_s(y_t - \sum_{i,j,k} b_{ij}^*\mathscr{E}_\tau(a_{ij}^* y_t a_{kj})b_{kj})| < \varepsilon/2$. Thus $|\varphi_s(x_t - \sum_{i,j,k} b_{ij}^*\mathscr{E}(a_{ij}^* x_t a_{kj})b_{kj})| < \varepsilon$. Thus the identity can be approximately factored through $\mathscr{E}$ and so π_ι is weakly contained in $\pi_{\mathscr{E}}$.

§2. The case of an algebra of type II_1.

To begin with let M be a σ-finite von Neumann algebra. In the first two lemmas of this section we have adapted some of the techniques of [3, §2].

On the set CP (M) of completely positive maps from M to M we shall consider two topologies:

1) point-σ-weak
2) point-σ-strong

LEMMA 1. *Suppose $F \subseteq CP(M)$ is convex. Then $\bar{F}^{\mathrm{pt}-\sigma-\mathrm{wk}} = \bar{F}^{\mathrm{pt}-\sigma-\mathrm{strong}}$.*

PROOF. It is clear that $\bar{F}^{\mathrm{pt}-\sigma-\mathrm{strong}} \subseteq \bar{F}^{\mathrm{pt}-\sigma-\mathrm{wk}}$. Let $T \in \bar{F}^{\mathrm{pt}-\sigma-\mathrm{wk}}$, we shall show that $T \in \bar{F}^{\mathrm{pt}-\sigma-\mathrm{strong}}$. So let $x_1, \ldots, x_n \in M$, $\xi_1, \ldots \xi_n \in H$ (unit vectors), and $\varepsilon > 0$ be given. We must show that $\exists\, S \in F$ such that $\|(T(x_i) - S(x_i))\xi_i\| < \varepsilon$. Let $\underset{\sim}{x} = x_1 \oplus x_2 \oplus \ldots \oplus x_n \in M^{(n)} = M \oplus M \oplus \ldots \oplus M$, and $F_{\underset{\sim}{x}} = \{S(x_1) \oplus S(x_2) \oplus \ldots \oplus S(x_n) \mid S \in F\} \subseteq M^{(n)}$. Then $F_{\underset{\sim}{x}}$ is a convex set in the von Neumann algebra $M^{(n)}$. Thus $\bar{F}_{\underset{\sim}{x}}^{\sigma-\mathrm{wk}} = \bar{F}_{\underset{\sim}{x}}^{\sigma-\mathrm{strong}}$ (see e.g. Takesaki [22, Theorem II.2.6]). So $\exists\, S \in F$ such that

$$\begin{aligned}&\|[(T(x_1) - S(x_1)) \oplus (T(x_2) - S(x_2)) \oplus \ldots\\ &\quad \oplus (T(x_n) - S(x_n))](\xi_1 \oplus \xi_2 \oplus \ldots \xi_n)\|\\ &\quad < \varepsilon^2.\end{aligned}$$

Thus $\|(T(x_i) - S(x_i))\xi_i\| < \varepsilon\, \forall\, i$.

Now let us define two states f and f_ϑ on $M \otimes_{\mathrm{bin}} M^{\mathrm{op}}$. Let

$$\begin{aligned}f(\textstyle\sum x_i \otimes y_i^\circ) &= (\pi_\iota(\textstyle\sum x_i \otimes y_i^\circ)\,\xi_\varphi \mid \xi_\varphi) \text{ and}\\ f_\vartheta(\textstyle\sum x_i \otimes y_i^\circ) &= (\pi_\vartheta(\textstyle\sum x_i \otimes y_i^\circ)\,\xi_\varphi \mid \xi_\varphi).\end{aligned}$$

As ξ_φ is a cyclic vector for both π_ι and π_ϑ we see that π_ι and π_ϑ are the GNS representations associated to f and f_ϑ respectively. Let $u = u_\vartheta$ be the canonical implementation of ϑ. For a normal state ψ on M let $\psi_1 = \psi \circ \vartheta$. If $\xi_\psi \in P$ implements ψ then $u^*\xi_\psi$ implements ψ_1 and $(\vartheta(x)\xi_\psi y \mid \xi_\psi) = (x\xi_{\psi_1}\vartheta^{-1}(y) \mid \xi_{\psi_1})$. ψ_1 is faithful whenever ψ is. Thus if ψ is a faithful normal state on M then $f_1(x \otimes y^\circ) = (x\xi_{\psi_1}\vartheta^{-1}(y) \mid \xi_{\psi_1}) = (\vartheta(x)\xi_\psi y \mid \xi_\psi)$ is a state on $M \otimes_{\mathrm{bin}} M^{\mathrm{op}}$ giving a representation equivalent to π_ϑ.

From now on we shall suppose that π_ϑ is weakly contained in π_ι, or equivalently that Θ is a bounded homomorphism on $C^*(M, M')$. This is equivalent by [3, Theorem 2.6] to the condition that ϑ is approximately inner as a completely positive map. Note that Lemma 2 shows that ϑ^{-1} is approximately inner as a completely positive map, which is equivalent to Θ^{-1} being bounded on $C^*(M, M')$. This will then justify the remark made in §1 that when Θ is bounded it is automatically an automorphism for any σ-finite von Neumann algebra. Our hypothesis that π_ϑ is weakly contained in π_ι implies by [9,3.3.4] that f_1 is a weak* limit of finite sums of vector forms associated to π_ι.

LEMMA 2. *Given* $x_1, x_2, \ldots, x_m \in M$, $\varphi_1, \varphi_2, \ldots, \varphi_n \in M_*^+$ *and* $\varepsilon > 0$, $\exists$ $a_1, a_2, \ldots, a_p \in M$ *such that* $\forall\, i,j$ $|\varphi_i(\vartheta^{-1}(x_j) - \sum_k a_k x_j a_k^*)| < \varepsilon$.

PROOF. Choose a faithful $\psi \in M_*^+$ such that $\varphi_i \leqq \psi$. Let $\xi_\psi \in P$ implement ψ. Then $\exists\, y_1, y_2, \ldots, y_n \in M$ such that $\forall\, x \in M$, $\varphi_i(x) = (x\xi_\psi y_i \mid \xi_\psi)$ (see [20,5.19]). Let f_1 be as above. By hypothesis f_1 is a weak* limit of finite sums of vector forms associated to π_ι. As $[\xi_\psi M] = H$ we may choose these vector forms to be of the form $x \otimes y^\circ \mapsto (x\xi_\psi a y \mid \xi_\psi)$, for some $a \in M$. Let $t_{ij} = y_i \otimes x_j^\circ \in M \otimes M^{\mathrm{op}}$. We may then choose $a_1, a_2, \ldots, a_p \in M$ so that $|f_1(t_{ij}) - \sum_k (\pi_\iota(t_{ij})\xi_\psi a_k \mid \xi_\psi a_k)| < \varepsilon$, for all i and j. Hence $|\varphi_i(\vartheta^{-1}(x_j) - \sum_k a_k x_j a_k^*)| = |(\vartheta^{-1}(x_j)\xi_\psi y_i | \xi_\psi) - \sum_k (a_k x_j a_k^* \xi_\psi y_i | \xi_\psi)| = |(y_i \xi_\psi \vartheta^{-1}(x_j)| \,\xi_\psi) - \sum_k (y_i \xi_\psi a_k x_j | \,\xi_\psi a_k)| = |f_1(t_{ij}) - \sum_k (\pi_\iota(t_{ij})\xi_\psi a_k | \xi_\psi a_k)| < \varepsilon$.

COROLLARY 3. ϑ^{-1} *is a* point-σ-strong* *limit of inner completely positive maps.*

PROOF. Lemma 2 shows that ϑ^{-1} is in the point σ-weak closure of the inner completely positive maps. Thus by Lemma 1 it is in the point σ-strong* closure of the inner completely positive maps.

LEMMA 4. *Given* $x_1, x_2, \ldots, x_m \in M_1$, $\varphi_1, \varphi_2, \ldots, \varphi_n \in M_{*1}^+$, *and* $\varepsilon > 0, \exists$ $a_1, a_2, \ldots, a_p \in M$ *such that* $\sum_k a_k a_k^* \leqq 1$, *and* $|\varphi_i(\vartheta^{-1}(x_j) - \sum_k a_k x_j a_k^*)| < \varepsilon$.

PROOF. Choose $b_1, b_2, \ldots, b_p \in M$ such that $|\varphi_i(\vartheta^{-1}(x_j) - \sum_k b_k x_j b_k^*)| < \varepsilon/2\, \forall\, i,j$ and $\|\vartheta^{-1}(1) - \sum_k b_k b_k^*\|_{\varphi_j}^{1/2} < \varepsilon/6$. Let $b = 1 + (\sum_k b_k b_k^* - 1)_+$. Here we are using the notation that for a self-adjoint element x, x_+ and x_- are the positive and negative parts respectively of x; $x = x_+ - x_-$, $x_+x_- = 0$, x_+, $x_- \geqq 0$. Then $b \geqq 1$ and so $(b^{1/2} - 1)^2 \leqq b - 1$. Let $a_k = b^{-1/2} b_k$ and $\Phi(x) = \sum_k a_k x a_k^*$. Then $\Phi(1) = b^{-1/2} \sum_k b_k b_k^* b^{-1/2} = b^{-1/2}(b - (\sum_k b_k b_k^* - 1)_-) b^{-1/2} \leqq 1$. Also $\|b^{1/2} - 1\|_{\varphi_j} = \sqrt{\varphi_j((b^{1/2} - 1)^2)} \leqq \sqrt{\varphi_j(b - 1)} = \sqrt{((\sum_k b_k b_k^* - 1)\xi_{\varphi_j} | e_+ \xi_{\varphi_j})} \leqq \|\sum_k b_k b_k^* - 1\|_{\varphi_j}^{1/2} \leqq \varepsilon/6$, where e_+ is the support projection of $(\sum_k b_k b_k^* - 1)_+$. Hence $|\varphi_j(\Phi(x_i) - \sum_k b_k x_i b_k^*)| = |\varphi_j(\Phi(x_i) - b^{1/2}\Phi(x_i) b^{1/2})| = |\varphi_j((1 - b^{1/2})\Phi(x_i) +$

$b^{1/2}\Phi(x_i)(1 - b^{1/2}))| \leqq \|1 - b^{1/2}\|_{\varphi_j}(\|\Phi(x_i)\| + \|\Phi(x_i^*)b^{1/2}\|_{\varphi_j}) \leqq \|1 - b^{1/2}\|_{\varphi_j}$ $(2 + \|1 - b^{1/2}\|_{\varphi_j}) \leqq \varepsilon/2$. Thus $|\varphi_j(\Phi(x_i) - \vartheta^{-1}(x_i))| \leqq |\varphi_j(\Phi(x_i) - \sum_k b_k x_i b_k^*)| + |\varphi_j(\sum_k b_k x_i b_k^* - \vartheta^{-1}(x_i))| < \varepsilon$.

We have not yet used the fact that M is a finite von Neumann algebra, so far our argument (adapted from Anantharaman-Delaroche and Havet) works for any σ-finite von Neumann algebra. For the rest of this section let M be a *finite* von Neumann algebra and let τ be a faithful finite normal trace on M and $\xi_o \in P$ the implementing vector.

COROLLARY 5. *Given $x_1, x_2, \ldots, x_n \in M$ and $\varepsilon > 0$ there are $a_1, a_2, \ldots, a_k \in M$ such that $\sum_i a_i a_i^* \leqq 1$ and for all $y \in M$ and $1 \leqq i \leqq n$*

$$|\tau([\vartheta(x_i) - \sum_k a_k^* x_i a_k]y)| < \varepsilon \|y\|.$$

PROOF. If $\Phi: M \to M$ is any completely positive map let $T_\Phi: M \to M_*$ be defined by $T_\Phi(x)(y) = \tau(\Phi(x)y)$. We have by Lemma 4 that $T_\vartheta^{-1} \in \overline{\{T_\Phi | \Phi \text{ inner}, \Phi(1) \leqq 1\}}^{\text{wot}}$. As $\tau([\vartheta^{-1}(x) - \sum a_i x a_i^*]y) = \tau(x[\vartheta(y) - \sum a_i^* y a_i])$, we thus have

$$T_\vartheta \in \overline{\{T_\Phi \mid \Phi(x) = \sum a_i^* x a_i \text{ and } \sum a_i a_i^* \leqq 1\}}^{\text{wot}}.$$

Hence by the convexity of $\{T_\Phi \mid \Phi(x) = \sum a_i^* x a_i \text{ and } \sum a_i a_i^* \leqq 1\}$

$$T_\vartheta \in \overline{\{T_\Phi \mid \Phi(x) = \sum a_i^* x a_i \text{ and } \sum a_i a_i^* \leqq 1\}}^{\text{sot}}.$$

exactly as required.

LEMMA 6. *Given $1 = x_o, x_1, x_2, \ldots x_m, y_1, y_2, \ldots, y_n \in M_1$ (the unit ball of M), and $\varepsilon > 0$, $\exists\ a_1, a_2, \ldots, a_p \in M$ such that $\sum a_k a_k^*$, $\sum a_k^* a_k \leqq 1$, and $|\tau((\vartheta(x_i) - \sum a_k^* x_i a_k^*)y_j)| < \varepsilon$.*

PROOF. Suppose $\varepsilon < 1$. By Corollary 5 there are $b_1, b_2, \ldots, b_p \in M$ such that $\sum b_i b_i^* \leqq 1$ and for all $y \in M$

$$|\tau([\vartheta(x_i) - \sum_k b_k^* x_i b_k]y)| < (\varepsilon/4)^2 \|y\|, 0 \leqq i \leqq m.$$

Let $b = 1 + (\sum b_i^* b_i - 1)_+$. Then $\sum b_i^* b_i = 1 + (\sum b_i^* b_i - 1)_+ - (\sum b_i^* b_i - 1)_- = b - (\sum b_i^* b_i - 1)_-$. Let $a_i = b_i b^{-1/2}$. Then $\sum a_i^* a_i = b^{-1/2} \sum b_i^* b_i b^{-1/2} = b^{-1/2}(b - (\sum b_i^* b_i - 1)_-)b^{-1/2} \leqq 1$, and $\sum a_i a_i^* = \sum b_i b^{-1} b_i^* \leqq \sum b_i b_i^* \leqq 1$. Also $\|b^{1/2} - 1\|_2^2 = \tau((b^{1/2} - 1)^2) \leqq \tau(b - 1) = \tau((\sum b_i^* b_i - 1)_+) = \tau([\sum b_i^* x_0 b_i - \vartheta(x_0)]e_+) < (\varepsilon/4)^2$, (where e_+ is the support projection of $(\sum b_i^* b_i - 1)_+$). Then

$$
\begin{aligned}
&|\tau([\vartheta(x_i) - \sum_k a_k^* x_i a_k] y_j)| \\
&\leqq |\tau([\vartheta(x_i) - \sum_k b_k^* x_i b_k] y_j)| + |\tau([b^{1/2} \sum_k a_k^* x_i a_k b^{1/2} - \sum_k a_k^* x_i a_k] y_j)| \\
&\leqq (\varepsilon/4)^2 \|y_j\| + |\tau(\sum_k a_k^* x_i a_k (1 - b^{1/2}) y_j)| \\
&\quad + |\tau((b^{1/2} - 1) \sum_k a_k^* x_i a_k b^{1/2} y_j)| \\
&\leqq \varepsilon/4 + |(y_j \sum_k a_k^* x_i a_k \xi_o | (1 - b^{1/2}) \xi_o)| \\
&\quad + |(\sum_k a_k^* x_i a_k b^{1/2} y_j \xi_o | (1 - b^{1/2}) \xi_o)| \\
&\leqq \varepsilon/4 + \|1 - b^{1/2}\|_2 (1 + \|b^{1/2}\|_2) \\
&\leqq \varepsilon/4 + \|1 - b^{1/2}\|_2 (2 + \|1 - b^{1/2}\|_2) \\
&\leqq \varepsilon/4 + 3\|1 - b^{1/2}\|_2 < \varepsilon.
\end{aligned}
$$

COROLLARY 7. *Given $x_1, \ldots, x_n \in M$, and $\varepsilon > 0$, there exists $a_1, a_2, \ldots, a_p \in M$ such that $\sum a_i^* a_i, \sum a_i a_i^* \leqq 1$ and $\|\sum_i a_i^* x_j a_i - \vartheta(x_j)\|_2 < \varepsilon$.*

PROOF. Apply Lemma 1 and Lemma 6 to the convex set of inner completely positive maps Φ such that $\Phi(x) = \sum a_i^* x a_i$ with $\sum a_i^* a_i, \sum a_i a_i^* \leqq 1$.

Let us recall Haagerup's probabilistic technique [12, §4] and [13, §2]. Suppose we are given $a_1, a_2, \ldots a_p \in M$ and a finite set of unitaries $u_1, u_2, \ldots, u_n \in M$, and $\varepsilon > 0$ such that

$$
\sum a_i^* a_i, \sum a_i a_i^* \leqq 1, \ \|\sum a_i^* a_i - 1\|_2 < \varepsilon \text{ and}
$$
$$
\sum \|a_i \vartheta(u_k) - u_k a_i\|_2^2 < \varepsilon, \ 1 \leqq k \leqq n.
$$

Given an integer $r < \varepsilon^{-1}$ (This implies that $\tau(1 - \sum a_i^* a_i) \leqq \|1 - \sum a_i^* a_i\|_2 < \varepsilon$. So $\tau(\sum a_i^* a_i) > 1 - \varepsilon > 1 - 1/r$, and thus we have exactly the starting condition at the beginning of the proof of [12, Lemma 4.3].) he constructs $b_1, b_2, \ldots, b_r \in M$ such that

$$
\|\sum b_i^* b_i - 1\|_2^2, \ \|\sum b_i b_i^* - 1\|_2^2 < 9/r \text{ and}
$$
$$
\sum \|b_i \vartheta(u_k) - u_k b_i\|_2^2 < 3n\varepsilon, \ 1 \leqq k \leqq n \text{ as follows.}
$$

Let $\Omega = \mathrm{T}^{(n)}$, s_i be the ith coordinate function function on Ω, $d\omega$ be Haar measure on Ω, and $a(\omega) = \sum s_i(\omega) a_i$. Let $\tilde{\omega} = (\omega_1, \omega_2, \ldots, \omega_r) \in \Omega^r$. Then

$$\int_{\Omega^r} \|1/r \sum a(\omega_i)^* a(\omega_i) - 1\|_2^2 d\tilde{\omega} < 3/r$$

$$\int_{\Omega^r} \|1/r \sum a(\omega_i) a(\omega_i)^* - 1\|_2^2 d\tilde{\omega} < 3/r, \text{ and}$$

$$\int_{\Omega^r} 1/r \sum_i \|a(\omega_i) u_k - u_k a(\omega_i)\|_2^2 d\tilde{\omega} < \varepsilon, \ 1 \leqq k \leqq n.$$

Hence $\exists\, \tilde{\omega} = (\omega_1, \ldots, \omega_r) \in \Omega^r$ such that

$$\|1/r \sum a(\omega_i)^* a(\omega_i) - 1\|_2^2, \ \|1/r \sum a(\omega_i) a(\omega_i)^* - 1\|_2^2 < 9/r, \text{ and}$$
$$1/r \sum \|a(\omega_i)\vartheta(u_k) - u_k a(\omega_i)\|_2^2 < 3n\varepsilon, \ 1 \leqq k \leqq n.$$

So let $b_i = 1/\sqrt{r}\, a(\omega_i)$ $1 \leqq i \leqq r$. Letting $c_i = g(b_i b_i^*) b_i$ where

$$g(t) = \begin{cases} 1 & 0 \leqq t \leqq r \\ \sqrt{r/t} & r < t \end{cases}$$

we have the following.

Lemma 8 ([12, Lemma 4.3 and Lemma 4.4]). *Suppose $a_1, a_2, \ldots, a_p \in M$ and unitaries $u_1, u_2, \ldots, u_n \in M$, and $\varepsilon > 0$ are given such that $\sum a_i^* a_i \leqq 1$, $\sum a_i a_i^* \leqq 1$, $\|\sum a_i^* a_i - 1\|_2 < \varepsilon$, and $\sum_i \|a_i \vartheta(u_k) - u_k a_i\|_2^2 < \varepsilon$. Then for any $r < \varepsilon^{-1}$ $\exists\, c_1, c_2, \ldots, c_r \in M$ such that $\|c_i\| \leqq \sqrt{r}$, $\|\sum c_i^* c_i - 1\|_2^2$, $\|\sum c_i c_i^* - 1\|_2^2 < 18/r$, and $\sum_i \|c_i \vartheta(u_k) - u_k c_i\|_2^2 < 3n\varepsilon$, $1 \leqq k \leqq n$.*

Corollary 9. *Let $u_1, u_2, \ldots, u_n \in \mathscr{U}(M)$, $r > 1$ an integer, and $\varepsilon > 0$ be given. Then $\exists\, c_1, c_2, \ldots, c_r \in M$ such that*

$$\|c_i\| \leqq \sqrt{r}$$
$$\|\sum c_i^* c_i - 1\| < 18/r$$
$$\|\sum c_i\, c_i^* - 1\| < 18/r$$
$$\sum_i \|c_i \vartheta(u_j) - u_j c_i\|_2^2 < \varepsilon, \ 1 \leqq j \leqq n.$$

Proof. Choose $\delta > 0$ so that $\delta^2 + 2\delta < 3n\varepsilon$ and $\delta < 1/r$. Let $u_o = 1$. By Corollary 7 $\exists\, a_1, a_2, \ldots, a_p \in M$ such that

$$\sum a_i^* a_i \leqq 1$$
$$\sum a_i a_i^* \leqq 1$$
$$\|\sum a_i^* u_k a_i - \vartheta(u_k)\|_2 < \delta, \ 0 \leqq k \leqq n.$$

Let $\Phi(x) = \sum a_i^* x a_i$. Then $\sum \|a_i \vartheta(u_k) - u_k a_i\|_2^2$

$$\begin{aligned}
&= \sum \tau(2a_i^* a_i - \vartheta(u_k)^* a_i^* u_k a_i - a_i^* u_k^* a_i \vartheta(u_k)) \\
&\leqq \tau(1 + 1 - \vartheta(u_k)^* \Phi(u_k) - \Phi(u_k)^* \vartheta(u_k)) \\
&= \|\Phi(u_k) - \vartheta(u_k)\|_2^2 + \tau((\vartheta(u_k) - \Phi(u_k))^* \vartheta(u_k)) \\
&\qquad\qquad + \tau(\Phi(u_k)^*(\vartheta(u_k) - \Phi(u_k))) \\
&\leqq \|\Phi(u_k) - \vartheta(u_k)\|_2^2 + 2\,\|\Phi(u_k) - \vartheta(u_k)\|_1 \\
&\leqq \delta^2 + 2\delta < \varepsilon. \text{ Now we may apply Lemma 8.}
\end{aligned}$$

THEOREM 10. *Let $u_1, u_2, \ldots, u_n \in \mathcal{U}(M)$ and $\varepsilon > 0$ be given, then there exists a unitary $w \in M$ such that $\|w\vartheta(u_k) - u_k w\|_2 < \varepsilon$, and thus ϑ is approximately inner as an automorphism.*

PROOF (Cf. [13, Proof of Theorem 2.3]). Let $\omega \in \beta\mathsf{N}\backslash\mathsf{N}$ be a free ultrafilter. Let H_ω be the ultraproduct of the Hilbert space H. Let $A = l^\infty(\mathsf{N}) \mathbin{\overline{\otimes}} M$ be the von Neumann tensor product of $l^\infty(\mathsf{N})$ and M. H_ω is a Hilbert A-bimodule in that we have commuting pair of *-homomorphisms: one of A and one of A^{op}. Let us suppose at first that $n = 1$. Let $\xi = (u\xi_o) \in H_\omega$ and $\eta = (\vartheta(u)\xi_o) \in H_\omega$. Let $r > 0$ be given. By Corollary 9 we may for each integer m choose $c_1^{(m)}, c_2^{(m)}, \ldots, c_r^{(m)} \in M$ such that

$$\begin{aligned}
&\|c_i^{(m)}\| \leqq \sqrt{r}, \\
&\|(\textstyle\sum c_i^{(m)*} c_i^{(m)} - 1)\,u\xi_o\|^2 = \|\textstyle\sum c_i^{(m)*} c_i^{(m)} - 1\|_2^2 < 18/r, \\
&\|(\textstyle\sum c_i^{(m)} c_i^{(m)*} - 1)\,\vartheta(u)\xi_o\|^2 = \|\textstyle\sum c_i^{(m)} c_i^{(m)*} - 1\|_2^2 < 18/r, \text{ and} \\
&\sum_{i=1}^r \|c_i^{(m)}\vartheta(u)\xi_o - u\xi_o c_i^{(m)}\|^2 + \sum_{i=1}^r \|c_i^{(m)}\vartheta(u^*)\xi_o - u^*\xi_o c_i^{(m)}\|^2 < 1/m.
\end{aligned}$$

Let $c_i = (c_i^{(1)}, c_i^{(2)}, \ldots) \in A$. Then $\|c_i\| \leqq \sqrt{r}$. Also

$$\begin{aligned}
&\|(\textstyle\sum c_i^* c_i - 1)\,\xi\|^2 = \lim_\omega \|[\textstyle\sum c_i^{(m)*} c_i^{(m)} - 1]u\xi_o\|^2 < 18/r, \\
&\|(\textstyle\sum c_i c_i^* - 1)\,\eta\|^2 = \lim_\omega \|[\textstyle\sum c_i^{(m)} c_i^{(m)*} - 1]\vartheta(u)\xi_o\|^2 < 18/r, \\
&\textstyle\sum \|c_i\eta - \xi c_i\|^2 = \lim_\omega \displaystyle\sum_{i=1}^r \|\, c_i^{(m)}\vartheta(u)\xi_o - u\xi_o c_i^{(m)}\|_2^2 = 0 \\
&\textstyle\sum \|c_i^*\xi - \eta c_i^*\|^2 = \lim_\omega \displaystyle\sum_{i=1}^r \|\, c_i^{(m)}\vartheta(u^*) - u^* c_i^{(m)}\|_2^2 = 0.
\end{aligned}$$

Thus $c_i\eta = \xi c_i$ and $c_i^*\xi = \eta c_i^*$ for $1 \leqq i \leqq r$. So by [13, Lemma 2.6] there is a unitary $\tilde{w}$ in A so that $\|\tilde{w}\eta - \xi\tilde{w}\|_2 < \varepsilon/2$, and thus $\exists$ a unitary $w \in M$ such that $\|w\vartheta(u) - uw\|_2 < \varepsilon$. This finishes the proof when $n = 1$.

When n is arbitrary we replace H by $H^{(n)} = H \oplus H \oplus \ldots \oplus H$, and ξ by $u_1\xi_o \oplus u_2\xi_o \oplus \ldots \oplus u_n\xi_o$, and apply the previous proof as in [13, Proof of Theorem 2.3].

We have now shown that for any finite set of unitaries $u_1, \ldots, u_n$ in M and any $\varepsilon > 0$ we may find a unitary w in M such that $\|\vartheta(u_i) - w^*u_iw\|_2 < \varepsilon$. As the unitaries in M span M we may replace the u_i by an arbitrary finite set $\{x_i\}$ in M and the norm $\|\cdot\|_2$ by the smaller one $\|\cdot\|_1$. Thus ϑ is the limit in the topology of point norm convergence on M_* of inner automorphisms.

3. The case of an algebra of type II_∞.

In this section we shall suppose that M is a σ-finite von Neumann algebra of type II_∞ and ϑ is an automorphism of M which acts trivially on $Z(M)$ and fixes a faithful normal semi-finite trace Tr (and hence by the chain rule for Radon-Nikodym cocycles ϑ must fix every trace). We shall show that if Θ is bounded on $C^*(M, M')$ then ϑ is approximately inner as an automorphism. Our approach will be to reduce to the II_1 case.

LEMMA 1 (cf. [5, Lemma 3.11]. *Let $e \in M$ be a finite projection with central support* 1 *and $N = eMe$. Then there is a unitary u in M such that relative to the tensor product decomposition of $M \cong N \mathbin{\overline{\otimes}} \mathscr{L}(K)$, $\mathrm{Ad}_u \circ \vartheta = \alpha \otimes \iota$ where α is an automorphism of N.*

PROOF. As ϑ preserves Tr and is the identity on the centre, $\vartheta(e) \sim e$. (For if z is a central projection such that $ze \lesssim z\vartheta(e)$ but $ze \not\sim z\vartheta(e)$, then there would exist a partial isometry u such that $u^*u = ze$ and $uu^* < z\vartheta(e)$; and so $\mathrm{Tr}(ze) = \mathrm{Tr}(uu^*) < \mathrm{Tr}(z\vartheta(e)) = \mathrm{Tr}(ze)$, which is impossible.) One can now apply the argument of Connes [5].

So in order to show that ϑ is approximately inner it suffices to show that $\tilde{\vartheta} = \mathrm{Ad}_u \circ \vartheta$ is approximately inner, and so from now on we shall assume that $\vartheta = \alpha \otimes \iota$ with α an automorphism if N. Our strategy will now be to show that α satisfies the condition of Theorem 2.10. That is we must show that if $A: \mathrm{C}(N, N') \to \mathrm{C}(N, N')$ is given by $A(\sum x_iy_i) = \sum \alpha(x_i)y_i$ then A extends to a bounded map on $\mathrm{C}^*(N, N')$.

LEMMA 2. *There is an isomorphism of $C^*(N, N')$ with a subalgebra of $C^*(M, M')$ such that Θ restricted to this subalgebra is A.*

PROOF. Let K^c be the conjugate Hilbert space of K. On $K \otimes K^c$ define a conjugate linear isometry J_K by $J_K(\xi \otimes \eta^c) = \eta \otimes \xi^c$, let P_K be the closure of $\{\sum_{i,j} \eta_i \otimes \eta_j^c \mid \eta_1, \ldots, \eta_n \in K\}$, and let $x \in \mathscr{L}(K)$ act on $K \otimes K^c$ by $x \otimes 1$. Then

$(K \otimes K^c, J_K, P_K)$ is a standard form for $\mathscr{L}(K)$. If $\tilde{P}$ is the closure of $\{\sum_{i,j} x_{ij} \otimes \eta_i \otimes \eta_j^c \mid \eta_1, \ldots, \eta_n \in K, (x_{ij}) \in M_n(N)_+\}$ in $L^2(N) \otimes K \otimes K^c$, and $J = J_N \otimes J_K$ on $L^2(N) \otimes K \otimes K^c$, then $(L^2(N) \otimes K \otimes K^c, J, \tilde{P})$ is a standard form for $N \mathbin{\overline{\otimes}} \mathscr{L}(K)$. Thus $J(N \otimes 1)J = J_N N J_N \otimes 1 = N' \otimes 1$.

Now $C^*(N, N') \otimes 1 = C^*(N \otimes 1,\ J(N \otimes 1)J)) \subseteq C^*(M, JMJ) = C^*(M, M')$, and so $\Theta|_{C^*(N,N') \otimes 1} = A \otimes 1$.

THEOREM 3. *Let M be a von Neumann algebra of type II_∞ and ϑ an automorphism trivial on the centre which fixes a faithful normal semi-finite trace. If Θ extends to an automorphism of $C^*(M, M')$, then ϑ is an approximately inner automorphism.*

PROOF. By Lemmas 1 and 2 we may assume that $M = N \mathbin{\overline{\otimes}} \mathscr{L}(K)$ and $\vartheta = \alpha \otimes \iota$ with A extending to an automorphism of $C^*(N, N')$. By Theorem 2.10 α is approximately inner as an automorphism of N. So there exists a net v_λ of unitaries in N so that $\alpha(x) = \lim_\lambda v_\lambda x v_\lambda^*$ (strongly) for all x in N. Let $u_\lambda = v_\lambda \otimes 1 \in M$. Then $\vartheta(x \otimes y) = \alpha(x) \otimes y = \lim_\lambda v_\lambda x v_\lambda^* \otimes y = \lim_\lambda u_\lambda (x \otimes y) u_\lambda^*$ (strongly) for all $x \in N$ and $y \in \mathscr{L}(K)$. As finite sums of elementary tensors are strongly dense we have for all $x \in N \mathbin{\overline{\otimes}} \mathscr{L}(K)$, $u_\lambda x u_\lambda^*$ converges strongly to $\vartheta(x)$.

Let φ_0 be a faithful normal state on $\mathscr{L}(K)$ and τ be a faithful normal trace on N, and $\varphi = \tau \otimes \varphi_o$. Then $u_\lambda \xi_\varphi = \xi_\varphi u_\lambda$, so for $x \in M$ $\|(u_\vartheta - u_\lambda J u_\lambda J) x \xi_\varphi\| = \|(\vartheta(x) - \mathrm{Ad}_{u_\lambda}(x)) \xi_\varphi\| \to 0$. Hence $u_\lambda J u_\lambda J$ converges strongly to u_ϑ. So for any $\psi \in M_*^+$, $\|\psi \cdot \vartheta^{-1} - \psi \cdot \mathrm{Ad}_{u_\lambda^*}\| \leqq 2\,\|\psi\|\ \|u_\vartheta \xi_\psi - u_\lambda J u_\lambda J \xi_\psi\| \to 0$. Thus ϑ is approximately inner.

NOTE ADDED DECEMBER 1991. Since this paper was submitted we have been able to show that Theorem 1.5 and Theorem 1.6 hold for any σ-finite von Neumann algebra. Also, as suggested by Connes [8, §4] there is an extension of this result to the case of factors of type III, where one assumes that Θ is bounded and $\mathrm{mod}(\vartheta) = 1$. We have been able to show that these two hypotheses imply that ϑ is approximately inner when M is a factor of type III_λ for $0 < \lambda < 1$.

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