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A NUMERICAL INVARIANT FOR FINITELY GENERATED GROUPS VIA ACTIONS ON GRAPHS

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Abstract.

Let G be a group acting with finite stabilizers on a directed graph X which is finite modulo G and connected. Consider the G-invariant space $h(X)$ of square-summable complex functions on the edges of X which have zero directed flow out of each vertex and zero directed circulation around each closed path in X. The main result of this paper is that the von Neumann dimension of $h(X)$, computed as the trace of an appropriate projection in a matrix algebra over the von Neumann algebra of G, depends only on G. The resulting invariant ρ, defined for all finitely generated groups, enjoys several computational properties, e.g. $\rho(H) = (G:H)\ \rho(G)$ when H is a finite-index subgroup of G, and $\rho(G * H) = \rho(G) + \rho(H) + 1$ for infinite, finitely generated groups G and H.

0. Introduction.

Take a locally finite graph, and give each edge an orientation by specifying an initial and a terminal vertex. A complex-valued function on the set of edges will then have a directed flow out of each vertex (sum of values on edges pointing out minus sum on edges pointing in) and a directed circulation along any finite-length path with given direction of traverse (sum of values on edges on the path, with each edge weighted by the number of times the path traverses it positively.) We will call a function on the edges harmonic if it has zero directed flow out of each vertex and zero directed circulation around each closed path. A finite graph is easily seen to admit no non-zero harmonic functions, but examples are readily manufactured in various infinite graphs if no growth constraint is placed on the function. The existence of non-trivial harmonic functions that are square-summable is a more delicate matter. For instance, it follows from Proposition 3.7 below (or see the last example in [8]) that the hexagonal honeycomb graph does not support such functions, but that the analogous k-gonal honeycomb does when $k \geqq 7$. This crucial distinction has to do with groups that act properly with finite quotient on these graphs.

Our approach is to turn things around and start with the group. A finitely

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generated group G acts properly with finite quotient on numerous connected graphs. For each of these, we may consider the orthogonal projection P_h of the space of l^2-functions on the edges onto the subspace of harmonic l^2-functions. The number $\sum |G_y|^{-1}(P_h\delta_y, \delta_y)$, where y runs over a set of orbit representatives for the action of G on the edges, and δ_y and G_y denote respectively the indicator function and stabilizer of the edge y, is independent of the choice of orbit representatives. Our main result below, Theorem 2.1, says that this number in fact depends only on G. The resulting invariant, which we dub $\rho(G)$, rather resembles the l^2-Betti number $b^1_{(2)}(G)$ defined (along with $b^j_{(2)}(G)$ for all non-negative integers j) by Cheeger and Gromov in [3] for arbitrary countable groups; see the end of section 2 below. We observe there that $\rho(G) = b^1_{(2)}(G)$ for many finitely generated groups G, but leave open the question of the coincidence of these two invariants in general.

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1. Preliminaries.

A directed graph X consists of a set V of vertices, a set E of edges, and maps $i, t\colon E \to V$. The edge y joins the initial vertex $i(y)$ to the terminal vertex $t(y)$. We will consider only locally finite graphs, that is graphs for which $\deg(v) \equiv |i^{-1}(v) \cup t^{-1}(v)|$ is finite for every vertex v. A path p in X of length n is a sequence $v_1, y_1, \ldots v_n, y_n, v_{n+1}$, where for each $j = 1, \ldots, n$, the edge y_j joins the vertex v_j to the vertex v_{j+1}. We think of p as having a direction of traverse, from v_1 to v_{n+1}, so each y_j will point either forward or backward along p. To keep track of this, we set

$$\langle y, p\rangle = |\{j\colon y = y_j, i(y) = v_j\}| - |\{k\colon y = y_k, t(y) = v_k\}|,$$

the net number of times the edge y is traversed positively by p. (Notice that $\langle y, p\rangle = 0$ if y does not lie on p.) We say that p is closed if $v_{n+1} = v_1$. The graph X is said to be connected if there is a path from u to v for any two distinct vertices u and v.

1.1. Definition. We denote by $h(X)$ the closed subspace of $l^2(E)$ consisting of all harmonic functions in $l^2(E)$, i.e. those l^2-functions ξ which satisfy

(i)
$$\sum_{i(y)=v} \xi(y) = \sum_{t(x)=v} \xi(x) \quad \forall v \in V, \quad \text{and}$$

(ii)
$$\sum_y \langle y, p\rangle \xi(y) = 0 \quad \text{for all closed paths } p.$$

Notice that for any function ξ on E satisfying (ii), there is a "potential" function φ on V such that $\xi(y) = \varphi(t(y)) - \varphi(i(y))\ \forall y \in E$. If ξ also satisfies (i), then for each vertex v, the values of φ on the immediate neighbors of v must average to $\varphi(v)$, whence it follows that the support of ξ must be either infinite or empty. (In particular, $h(X) = (0)$ if X is finite.) Changing the way in which X is directed changes $h(X)$ by a unitary operator on $l^2(E)$, namely $h(X) = \operatorname{diag}\{r(y)\colon y \in E\}\, h(X')$ if X' is obtained from X by reversing some of the edges, where $r(y)$ is 1 or -1 depending on whether or not y has the same orientation in X' that it has in X.

We are mainly interested in graphs acted on by groups. For a group G, we have the left and right regular representations $g \mapsto L_g$ and $g \mapsto R_g$ of G on $l^2(G)$, defined by $(L_g\psi)(h) = \psi(g^{-1}h)$ and $(R_g\psi)(h) = \psi(hg)$. We write $\mathrm{VN}(R_G)$ for the von Neumann algebra generated by R_G. It is well known that $\mathrm{VN}(R_G)$ is precisely the commutant $(L_G)'$ of L_G in the algebra of bounded operators on $l^2(G)$; further, the functional $\tau\colon \mathrm{VN}(R_G) \to \mathbf{C}$ defined by $\tau(T) = (T\delta_1, \delta_1)$, where δ_1 is the point mass at 1, is a faithful trace, i.e. $\tau(T^*T) > 0$ for all non-zero T in $\mathrm{VN}(R_G)$ and $\tau(ST) = \tau(TS)$ for all S, T in $\mathrm{VN}(R_G)$. On the algebra $\mathrm{VN}(R_G) \otimes M_n$ of $n \times n$ matrices, the functional $\tau \otimes \mathrm{tr}$ which sums the values of τ on the diagonal is likewise a faithful trace. (See chapter 6 of [7] for a thorough treatment of group von Neumann algebras.)

Let C be a set equipped with a (left) G-action $(g, c) \mapsto gc$. Suppose that the action is proper (that is, for each c in C, the stabilizer subgroup $G_c \equiv \{g \in G\colon gc = c\}$ is finite) and that C is G-finite (that is, the set $G\backslash C$ of G-orbits is finite). For c in C, let $W_c\colon l^2(Gc) \to l^2(G)$ be the isometry defined by $(W_c\xi)(g) = |G_c|^{-1/2}\xi(gc)$. Pick a complete set C_0 of orbit representatives. Identifying $l^2(C)$ with $\oplus\{l^2(Gc)\colon c \in C_0\}$, we obtain an isometry $W = \oplus\{W_c\colon c \in C_0\}\colon l^2(C) \to l^2(G)^n$, where $n = |C_0| = |G\backslash C|$. Let $g \mapsto A_g$ be the unitary representation of G on $l^2(C)$ defined by $(A_g\psi)(c) = \psi(g^{-1}c)$. Notice that $WA_g = (L_g \otimes 1)W$ for g in G. Thus, if K is any closed G-invariant subspce of $l^2(C)$, the projection P_{WK} of $l^2(G)^n$ on WK belongs to $(L_G \otimes 1)' = \mathrm{VN}(R_G) \otimes M_n$. We denote the non-negative real number $\tau \otimes \mathrm{tr}(P_{WK})$ by $\dim(K)$ (or $\dim_G(K)$ if more than one group is present); this number is called the von Neumann dimension in [3] and elsewhere. It is independent of the choice of C_0 because changing C_0 just conjugates P_{WK} by a direct sum of n R_g's, which doesn't affect $\tau \otimes \mathrm{tr}$.

1.2. Lemma. *With the notation of the previous paragraph, let P_K be the projection of $l^2(C)$ on K. Then* $\dim(K) = \sum\{|G_c|^{-1}(P_K\delta_c, \delta_c)\colon c \in C_0\}$. *In particular,* $\dim(l^2(C)) = \sum\{|G_c|^{-1}\colon c \in C_0\}$.

Proof. For each c in C_0, let Δ_c be the vector in $l^2(G)^n$ whose entry in the slot corresponding to c is δ_1, with all other entries 0. We then have $(P_{WK}\Delta_c, \Delta_c) = (WP_KW^*\Delta_c, \Delta_c) = (P_KW^*\Delta_c, W^*\Delta_c) = |G_c|^{-1}(P_K\delta_c, \delta_c)$, and the lemma follows by summing over C_0.

1.3. LEMMA. *Let G act properly on C and $C^{\sim}$, with each finite modulo G, and let K and $K^{\sim}$ be closed G-invariant subspaces of $l^2(C)$ and $l^2(C^{\sim})$ respectively. Suppose there is a bounded operator T: $l^2(C) \to l^2(C^{\sim})$ intertwining the associated representations Λ and $\Lambda^{\sim}$ of G, with $TK \subset K^{\sim}$ and* $\ker(T) \cap K = (0)$. *Then* $\dim(K) \leqq \dim(K^{\sim})$, *and if TK is dense in $K^{\sim}$, then* $\dim(K) = \dim(K^{\sim})$.

PROOF. Let $n = |G\backslash C|$ and $m = |G\backslash C^{\sim}|$. Let W and $W^{\sim}$ be the isometric embeddings of $l^2(C)$ and $l^2(C^{\sim})$ into $l^2(G)^n$ and $l^2(G)^m$ obtained as above by choosing orbit representatives. The operator $T_0 = W^{\sim}TW^*P_{WK}$: $l^2(G)^n \to l^2(G)^m$ interwines $\Lambda \otimes 1_n$ and $\Lambda^{\sim} \otimes 1_m$, and so belongs to $\mathrm{VN}(R_G) \otimes M_{m+n}$; further, the closure of its range is contained in $W^{\sim}K^{\sim}$, and its kernel is precisely $(WK)^{\perp}$. It follows that $P_{WK} \oplus 0_m$ is Murray-von Neumann equivalent in $\mathrm{VN}(R_G) \otimes M_{m+n}$ to a subprojection of $0_n \oplus P_{W^{\sim}K^{\sim}}$, so $\tau \otimes \mathrm{tr}(P_{WK}) \leqq \tau \otimes \mathrm{tr}(P_{W^{\sim}K^{\sim}})$. In case TK is dense in $K^{\sim}$, the range of T_0 is dense in $W^{\sim}K^{\sim}$, and equality of the two von Neumann dimensions follows.

The following notation and terminology, for a connected directed graph X with vertices V and edges E, will be used in the sequel. Let S: $l^2(V) \to l^2(E)$ be the bounded operator defined by $(S\varphi)(y) = \varphi(i(y)) - \varphi(t(y))$. For ξ in $l^2(E)$ and v in V, we call the number $(S^*\xi)(v)$ the directed flow of ξ out of v. Let P_s be the projection of $l^2(E)$ on the closure of the range of S. For a path p in X, let $p^{\wedge}$ be the (finitely supported) function on E defined by $p^{\wedge}(y) = \langle y, p\rangle$. If p is closed, we call the number $(\xi, p^{\wedge})$ the directed circulation of ξ around p. Let P_c be the projection of $l^2(E)$ onto the closed linear span of $\{p^{\wedge}$: p a closed path in $X\}$. The projection P_h of $l^2(E)$ onto $h(X)$ is then $1 - P_s - P_c$. (Notice that the ranges of P_s and P_c are orthogonal.) For ξ in $\ker(P_c)$ (which includes $h(X)$), and vertices v, w, we write $I(\xi; v, w) = (\xi, p^{\wedge})$, where p is a path in X from v to w. The number $I(\xi; v, w)$ is independent of the choice of p because ξ is orthogonal to $q^{\wedge}$ for all closed paths q. Notice that $I(S\varphi; v, w) = \varphi(v) - \varphi(w)$ for φ in $l^2(V)$.

When another graph $X^{\sim}$ is in play, we use $S^{\sim}$, $P_s^{\sim}$, etc. to denote the corresponding apparatus there.

2. Dim($h(X)$) when G acts on X.

To say that the group G acts on the directed graph X means that G acts on V and on E, and that these actions satisfy $gi(y) = i(gy)$ and $gt(y) = t(gy)$ for all g in G and y in E. We say that the action is proper if G acts properly on V and on E, and that X is G-finite if E and V are. In the presence of these assumptions, X is necessarily locally finite, so we may consider the closed G-invariant subspace $h(X)$ and its von Neumann dimension.

2.1. THEOREM. *Let the group G act properly on the connected directed graphs X_1 and X_2, both finite modulo G. Then* $\dim(h(X_1)) = \dim(h(X_2))$.

The main thing is to prove that $\dim(h(X))$ is unchanged by connecting on a single new edge orbit, either with or without the addition of a new vertex orbit; these special cases are taken care of in the next two lemmas.

2.2. LEMMA. *Let G act properly on the connected directed graph X, with X finite modulo G. Pick a vertex u in V and a finite subgroup B of G. Let $C = G_u \cap B$, $V^\sim = V \cup (G/B)$, and $E^\sim = E \cup (G/C)$. Extend the orienting maps to $i^\sim, t^\sim: E^\sim \to V^\sim$ by setting $i^\sim(g(C)) = gu$ and $t^\sim(gC) = gB$ to obtain a connected proper G-finite directed G-graph $X^\sim$. In this situation,* $\dim(h(X)) = \dim(h(X^\sim))$.

PROOF. For each bC in B/C, choose a path p_{bC} in X from u to bu. Define $L: l^2(E) \to l^2(E^\sim)$ by setting $(L\xi)(y) = 0$ for y in E and

$$(L\xi)(gC) = |B|^{-1} \sum \{(\xi, \Lambda_{gc}(p_{bC})^\wedge) \colon c \in C, bC \in B/C\}.$$

[The right-hand side defines a bounded operator from $l^2(E)$ to $l^2(G)$ that is a finite linear combination of operators $R_c W_y F_y$ (c in C, y an edge on one or another of the p_{bC}'s), where F_y is the projection of $l^2(E)$ on $l^2(Gy)$ and W_y is as in section 1 above. The averaging makes the resulting $l^2(G)$ functions R_C-invariant, so in $l^2(G/C)$.] Notice that L intertwines Λ and $\Lambda^\sim$. Denoting by J: $1^2(\mathrm{E}) \to 1^2(\mathrm{E}^\sim)$ the natural inclusion, we set $K = J + L$. Let $T = P_h^\sim K$: $l^2(E) \to l^2(E^\sim)$. Clearly T intertwines Λ and $\Lambda^\sim$. We claim that the restriction of T to $h(X)$ is bounded below, so in particular $\ker(T) \cap h(X) = (0)$. [To see this, take ξ in $h(X)$, so $P_s\xi = 0 = P_c\xi$. We have $(K\xi)(y) = \xi(y)$ for y in E and

$$(K\xi)(gC) = |B|^{-1} |C| \sum \{I(\xi; gu, gbu) \colon bC \in B/C\}.$$

For b_0 in B, g in G, let p be a path from gu to gb_0u in X, and let q be the closed path in $X^\sim$ consisting of p followed by a (positive) traverse of gb_0C to gB and then a (negative) traverse of gC to gu. We have

$$(K\xi, q^\wedge) = I(\xi; gu, gb_0u) + (K\xi)(gb_0C) - (K\xi)(gC)$$

$$= |B|^{-1} |C| \sum \{I(\xi; gu, gb_0u) + I(\xi; gb_0u, gb_0bu) - I(\xi; gu, gbu) \colon bC \in B/C\}.$$

$$= |B|^{-1} |C| \sum \{I(\xi; gu, gb_0bu) - I(\xi; gu, gbu) \colon bC \in B/C\} = 0,$$

because summing over bC yields the same result as summing over $b_0^{-1}bC$. This is enough to show that $(K\xi, r^\wedge) = 0$ for any closed path r in $X^\sim$, since $K\xi$ agrees with ξ on E, and $P_c\xi = 0$. Thus $T\xi = (1 - P_s^\sim)K\xi$. Since $P_s\xi = 0$, and since $J\xi$ vanishes on all the edges that have been added to make $X^\sim$ from X, we have $P_s^\sim J\xi = 0$. Thus $T\xi = J\xi + (1 - P_s^\sim)L\xi$, with the two summands orthogonal: $(J\xi, (1 - P_s^\sim)L\xi) = ((1 - P_s^\sim)J\xi, L\xi) = (J\xi, L\xi) = 0$. Hence $\|T\xi\| \geqq \|\xi\|$. This proves the claim.] We conclude from lemma 1.3 that $\dim(h(X)) \leqq \dim(h(X^\sim))$.

For the reverse inequality, we apply lemma 1.3 to the inertwining operator P_hJ^*: $l^2(E^\sim) \to l^2(E)$, which restricts to E and then projects on $h(X)$. Take ψ in

$h(X^\sim)$ with $P_h J^* \psi = 0$; we must show that $\psi = 0$. We have $P_c J^* \psi = 0$ because $P_c^\sim \psi = 0$, so $J^* \psi = P_s J^* \psi$. There is thus a sequence $\{\varphi_n\}$ in $l^2(V)$ with $S\varphi_n \to J^* \psi$. Extend each φ_n to $\varphi_n^\sim$ in $l^2(V^\sim)$ by setting $\varphi_n^\sim(gB) = |B|^{-1}|C| \sum \{\varphi_n(gbu)\colon bC \in B/C\}$. Consider $S^\sim \varphi_n^\sim$. It agrees with $S\varphi_n$ on E, and on the rest of $E^\sim$ we have

$$(S^\sim \varphi_n^\sim)(gC) = \varphi_n^\sim(gu) - \varphi_n^\sim(gB)$$
$$= |B|^{-1}|C| \sum \{\varphi_n(gu) - \varphi_n(gbu)\colon bC \in B/C\}$$
$$= |B|^{-1}|C| \sum \{I(S\varphi_n; gu, gbu)\colon bC \in B/C\}$$
$$= (KS\varphi_n)(gC).$$

Thus $S^\sim \varphi_n^\sim = KS\varphi_n$, which means that $(1 - P_s^\sim)KJ^* \psi = 0$. But $KJ^* \psi = \psi$: these two functions agree on E, and on the rest of $E^\sim$ we have

$$(KJ^* \psi)(gC) = |B|^{-1}|C| \sum \{I(J^* \psi; gu, gbu)\colon bc \in B/C\}$$
$$= |B|^{-1}|C| \sum \{\psi(gC) - \psi(gbC)\colon bC \in B/C\}$$
$$= \psi(gC) - |B|^{-1}|C| \text{ (directed flow of } \psi \text{ into } gB) = \psi(gC),$$

where the second equality follows from the vanishing of the directed circulation of ψ around a closed path that goes in X from gu to gbu, then traverses gbC positively from gbu to gB, then traverses gC negatively to gu. We thus have $\psi = (1 - P_s^\sim - P_c^\sim)\psi = 0$, as required.

2.3. LEMMA. *Let G act properly on the connected directed graph X, with X finite modulo G. Pick vertices u, w in V. Let $C = G_u \cap G_w$, $V^\sim = V$, and $E^\sim = E \cup (G/C)$. Extend the orienting maps to $i^\sim$, $t^\sim\colon E^\sim \to V^\sim$ by setting $i^\sim(gC) = gu$ and $t^\sim(gC) = gw$ to obtain a connected proper G-finite directed G-graph $X^\sim$. In this situation,* $\dim(h(X)) = \dim(h(X^\sim))$.

PROOF. Choose a path p in X from u to w, and define $L\colon l^2(E) \to l^2(E^\sim)$ by setting $(L\xi)(y) = 0$ for y in E and $(L\xi)(gC) = |C|^{-1} \sum \{(\xi, \Lambda_{gc} p^\wedge)\colon c \in C\}$. As with its counterpart in the proof of the preceding lemma, one checks that L is a bounded operator intertwining Λ and $\Lambda^\sim$. Letting $J\colon l^2(E) \to l^2(E^\sim)$ denote the natural inclusion, we write $K = J + L\colon l^2(E) \to l^2(E^\sim)$. Take ξ in $h(X)$. Notice that $(K\xi)(gC) = I(\xi; gu, gw)$, so ξ has zero circulation around any closed path in $X^\sim$ that traverses gC positively and then winds through X from gw to gu. We claim that $P_c^\sim K\xi = 0$. [This is shown as follows. For h in G_u, g in G, let r be a closed path that starts at gu, traverses gC positively to gw, joins gw in X to ghw, and then returns to u by traversing ghC negatively. We have $(K\xi, r^\wedge) = (K\xi)(gC) + I(\xi; gw, ghw) - (K\xi)(ghC) = I(\xi; gu, gw) + I(\xi; gw, ghw) - I(\xi; ghw, gu) = 0$. The same is true for paths with two successive edges in G/C followed by a path in X from gu to gku, where $k \in G_v$. The cases considered so far are enough to show $(K\xi, q^\wedge) = 0$ for any closed path q in $X^\sim$. This proves the claim.] We thus have $P_h^\sim K\xi = (1 - P_s^\sim)K\xi$, and, just as in the proof of the preceding lemma, $\|P_h^\sim K\xi\| \geqq K\|\xi\|$. It now follows from lemma 1.3 that $\dim(h(X)) \leqq \dim(h(X^\sim))$.

To go the other way, we use $P_h J^*$: $l^2(E^\sim) \to l^2(E)$. We will show that the restriction of this operator to $h(X^\sim)$ is injective. Take ψ in $h(X^\sim)$ with $P_h J^*\psi = 0$. We have $P_c J^*\psi = 0$ because $P_c^\sim \psi = 0$, so $J^*\psi = P_s J^*\psi$. There is thus a sequence $\{\phi_n\}$ in $l^2(V)$ with $S\varphi_n \to J^*\psi$. Since $V^\sim = V$, we have $S^\sim$: $l^2(V) \to l^2(E^\sim)$, with $(S^\sim \varphi_n)(y) = (S\varphi_n)(y) = (KS\varphi_n)(y)$ for y in E, and $(S^\sim \varphi_n)(gC) = \varphi_n(gu) - \varphi_n(gw) = I(S\varphi_n; gu, gw) = (KS\varphi_n)(gC)$. Thus $S^\sim \varphi_n = KS\varphi_n \to KJ^*\psi$. But $KJ^*\psi = \psi$ – these two functions agree on E, and furthermore $(KJ^*\psi)(gC) = I(J^*\psi; gu, gw) = \psi(gC)$. We have $P_s\psi = \psi$ because $S^\sim \varphi_n \to \psi$, but also $P_s\psi = 0$, because $\psi \in h(X^\sim)$. Thus $\ker(P_h J^*) \cap h(X^\sim) = (0)$, and lemma 1.3 gives us the desired inequality.

PROOF OF THEOREM 2.1. Let X_j have vertices V_j, edges E_j, and orienting maps i_j, t_j. Pick vertices v_1 in V_1 and v_2 in V_2, and let C be the intersection of their respective stabilizers in G. Let V be the disjoint union of V_1 and V_2, and let E be the disjoint union of E_1, E_2, and G/C. Define i, t: $E \to V$ by setting $i(y) = i_j(y)$ and $t(y) = t_j(y)$ for y in E_j, and $i(gC) = gv_1$, $t(gC) = gv_2$. The resulting directed graph X is connected, and G acts on it properly with finite quotient. We can obtain X from X_1 by performing several times the constructions covered by lemmas 2.2 and 2.3, so $\dim(h(X_1)) = \dim(h(X))$, and likewise $\dim(h(X)) = \dim(h(X_2))$.

2.4. REMARK. The groups that act properly with finite quotient on connected directed graphs are precisely the finitely generated groups. [If G is generated by $a_1, \ldots, a_n$, let X be the corresponding Cayley graph with $V = G, E = G \times \{1, \ldots, n\}$, $i(g,j) = g$, $t(g,j) = ga_j$. Then X is connected, and all stabilizers are trivial. Given, conversely, a connected X on which G acts properly with finite quotient, let V_0 be a complete set of vertex-orbit representatives, and let V_1 be the finite subset of V consisting of V_0 and all the immediate neighbors of vertices in V_0. Let $F = \{h \in G: hV_1 \cap V_1 \neq \phi\}$. We claim that every g in G is a product of elements in F. To see this, pick w in V_0. If gw is either w or an immediate neighbor of w, then $g \in F$. Otherwise, consider a path in X from w to gw passing succesively through vertices $w = w_0, w_1, \ldots, w_{r-1}, w_r = gw$. Write each w_j as $g_j u_j$, where $u_j \in V_0$ (and $g_0 = 1, g_r = g$). Set $h_j = g_j^{-1} g_{j+1}$ for $j = 0, \ldots, r-1$. We have $h_j u_{j+1} = g_j^{-1} w_{j+1}$, which is an immediate neighbor of $g_j^{-1} w_j = u_j$, so each $h_j \in F$, and of course $g = h_0 h_1 \ldots h_{r-1}$.]

2.5. DEFINITION. Let G be a finitely generated group. We set $\rho(G) = \dim(h(X))$, where X is any connected directed graph on which G acts properly with finite quotient.

We end this section with a brief description of the l^2-Betti numbers $b^j_{(2)}(G)$ defined in [3] for arbitrary countable groups G. Fix a countable set V and consider the family $\mathbf{C}$ of simplicial complexes with vertices contained in V on which G acts freely and simplicially with compact quotient. For X in $\mathbf{C}$, write $C^j_{(2)}(X) = l^2(S_j)$, where S_j is the set of j-simplices of X, $j = 0, 1, 2, \ldots$. Each S_j is finite modulo G, and we have coboundary maps d_j: $C^j_{(2)}(X) \to C^{j+1}_{(2)}(X)$ which are bounded operators intertwining the respective actions of G. Let $H^j_{(2)}(X\colon G) = \ker d_j \cap \ker d^*_{j-1}$, a subspace of $C^j_{(2)}(X)$. If X_1 is a G-subcomplex of X_2 in $\mathbf{C}$ (in which case we write $X_2 \geqq X_1$), restriction gives a map $r_{X_1, X_2}\colon H^j_{(2)}(X_2\colon G) \to H^j_{(2)}(X_1\colon G)$. Take the inverse limit of this system of maps, i.e. form the vector space E^j of $\mathbf{C}$-tuples (ξ_X) where each ξ_X belongs to the corresponding $H^j_{(2)}(X\colon G)$ and $r_{X_1, X_2}(\xi_{X_2}) = \xi_{X_1}$ whenever $X_2 \geqq X_1$. For each X in $\mathbf{C}$, let E^j_X be the closure of the image of the projection from E^j to $H^j_{(2)}(X\colon G)$. One then defines $b^j_{(2)}(G)$ in $[0, \infty]$ as $\sup\{\dim_G(E^j_X)\colon X \in \mathbf{C}\}$.

In [3], this definition is embedded in a larger context that includes an arbitrary topological space Y on which G acts. There are, correspondingly, l^2-Betti numbers $b^j_{(2)}(Y\colon G)$, with $b^j_{(2)}(G) = b^j_{(2)}(\{\mathrm{pt}\}\colon G)$. If K_G is a contractible cell complex on which G acts freely, a significant feature of this theory is that $b^j_{(2)}(G) = b^j_{(2)}(K_G\colon G)$. Suppose that such a K_G can be found in the family $\mathbf{C}$ considered in the previous paragraph. (There seems to be no concise characterization of the groups G for which this is possible; see [6]. Suffice it here to remark that there are many such groups.) Let X be the 1-skeleton of K_G, so X is a connected graph on which G acts freely with finite quotient. On the one hand, $h(X) = H^1_{(2)}(K_G\colon G)$ because every closed path in X can be gotten by adding up boundaries of 2-simplices in K_G. On the other hand, the inverse limit in the previous paragraph turns out in this case to be simply $H^1_{(2)}(K_G\colon G)$. We thus have $\rho(G) = b^1_{(2)}(G)$ for finitely generated groups G which act freely, simplicially, and cocompactly on a contractible simplicial complex. By our Proposition 3.6 below and the corresponding result in [3] for their Betti numbers, the same is true if G has a finite-index subgroup which so acts.

Properties of ρ.

All groups considered in this section are assumed to be finitely generated.

3.1. LEMMA. *Let G act properly with finite quotient on the connected directed graph X. Then*

$$\rho(g) = \sum |G_y|^{-1} - \sum |G_v|^{-1} - \sum |G_y|^{-1}(P_c\delta_y, \delta_y),$$

where the sums extend over complete sets of edge-orbit representatives y and vertex-orbit representatives v, and P_c is the projection described at the end of section 1 above.

PROOF. The range of $I - P_s$ is the orthogonal complement in $l^2(E)$ of the range of the operator S: $l^2(V) \to l^2(E)$, which is injective because X is connected. Thus by lemma 1.2,

$$\dim(\text{range}(I - P_s)) = \dim(l^2(E)) - \dim(l^2(V)) = \sum |G_y|^{-1} - \sum |G_v|^{-1}.$$

Now use $P_h = 1 - P_s - P_c$ and lemma 1.2 again.

3.2. PROPOSITION. (a) *Suppose that G acts properly on a directed tree with finite quotient. Then $\rho(G) = \sum |G_y|^{-1} - \sum |G_v|^{-1}$, where the sums extend over complete sets of edge-orbit representatives y and vertex-orbit representatives v.*

(b) *In particular, if $H_1, \ldots, H_n$ are finite groups with H_j and H_{j+1} sharing a subgroup A_j for $j = 1, \ldots, n-1$, then*

$$\rho(H_1 *_{A_1} H_2 *_{A_2} * \ldots *_{A_{n-1}} H_n) = \sum |A_j|^{-1} - \sum |H_k|^{-1}.$$

(c) *For the free group F_n on n generators, $\rho(F_n) = n - 1$.*

PROOF. For a directed tree X, we have $P_c = 0$, so part (a) of the proposition follows immediately from lemma 3.1. The amalgamated product $G = H_1 *_{A_1} H_2 *_{A_2} * \ldots *_{A_{n-1}} H_n$ acts on the directed graph X whose set of edges is the disjoint union of G/A_j $(j = 1, \ldots, n-1)$ and whose set of vertices is the disjoint union of G/H_k $(k = 1, \ldots, n)$, with $i(gA_j) = gH_j$ and $t(gA_j) = H_{j+1}$. This X is in fact a tree [9], so part (b) follows from part (a). If $a_1, \ldots, a_n$ freely generate F_n, the corresponding Cayley graph is a tree on which F_n acts freely with one vertex orbit and n edge orbits; part (a) then implies that $\rho(F_n) = n - 1$.

For more about how ρ treats amalgamated products over finite groups, we will need the following graph-theoretic lemma.

3.3. LEMMA. *Let X (with vertices V and edges E) be a directed graph, and X_1 (with vertices V_1 and edges E_1) be a directed subgraph, that is, E_1 and V_1 are subsets of E and V respectively, with $i(E_1)$, $t(E_1) \subset V_1$. Suppose that X_1 has the following property:*

(*) *no path in X that begins at v in V_1 and then leaves X_1 on an edge in $E \backslash E_1$ joining v to a vertex in $V \backslash V_1$ can re-enter X_1 at any vertex in V_1 except v.*

Let P (resp. P_1) be the projection of $l^2(E)$ on the closed linear span of $\{p^\wedge : p$ a closed path in X (resp. X_1)$\}$, and let Q be the projection of $l^2(E)$ on $l^2(E_1)$. Then $QP = PQ = P_1$, and in particular, $P_1\delta_y = P\delta_y$ for all y in E_1.

PROOF. Consider a closed path p in X that traverses at least one edge in E_1, and at least one edge in $E \backslash E_1$. By (*) there is a sequence of paths $p_1, \ldots, p_k$ (some possibly empty) in X_1 and a sequence $r_1, \ldots, r_{k-1}$ of closed paths in X that use only edges in $E \backslash E_1$ such that p traverses p_1, then r_1, then p_2, and so on, finally traversing r_{k-1}, and then p_k back to the vertex from which p_1 started. Putting the p_j's together, without the interpolating r's, yields a closed path q in X_1 such that $p^\wedge - q^\wedge = \sum r_j^\wedge$. We then have $P_1 p^\wedge = P_1 q^\wedge = q^\wedge = Pq^\wedge = PQp^\wedge$. For closed paths p in X that traverse only edges in E_1, or only edges in $E \backslash E_1$, it is clear that $P_1 p^\wedge = PQp^\wedge$. We have so far shown that $P_1 P = PQP$, but P_1 projects onto a subspace of the range of P, so $PP_1 = P_1 P = P_1 = PQP$. Since also $QP_1 = P_1 = P_1 Q$, the selfadjoint element $QPQ - P_1$ has square zero, so $P_1 = QPQ$. At this point, we have $(QP - P_1)(QP - P_1)^* = QPQ - P_1 - P_1 + P_1 = 0$, and the lemma is proved.

3.4. PROPOSITION. *Let $\Gamma = G *_F H$, where F is a finite subgroup of G and H.*
(a) *If G is infinite and H is finite, then $\rho(\Gamma) = \rho(G) + |F|^{-1} - |H|^{-1}$.*
(b) *If G and H are both infinite, $\rho(\Gamma) = \rho(G) + \rho(H) + |F|^{-1}$.*

PROOF. (a) Let $\{a_1, \ldots, a_n\}$ be a set of generators for G. Consider the directed graph X with vertex set $V = \Gamma/H$, edge set $E = (\Gamma/F) \times \{1, \ldots, n\}$, and orienting maps defined by $i(sF, j) = sH$, $t(sF, j) = sa_j H$ $(s \in \Gamma)$. This graph is connected because Γ is generated by H and the a_j's. The subgraph X_1 with $V_1 = \{gH: g \in G\}$ and $E_1 = (G/F) \times \{1, \ldots, n\}$ satisfies condition (*) of lemma 3.3. Lemmas 3.1 and 3.3 give

$$\begin{aligned} \rho(\Gamma) &= n|F|^{-1} - |H|^{-1} - \sum |F|^{-1}(P_c \delta_{(F,j)}, \delta_{(F,j)}) \\ &= n|F|^{-1} - |H|^{-1} - \sum |F|^{-1}(P_1 \delta_{(F,j)}, \delta_{(F,j)}), \end{aligned}$$

where P_1 is the counterpart of P_c for X_1. On the other hand, X_1 is a connected graph on which G acts properly with finite quotient, so by lemma 3.1 again,

$$\rho(G) = n|F|^{-1} - |F|^{-1} - \sum |F|^{-1}(P_1 \delta_{(F,j)}, \delta_{(F,j)}).$$

This proves part (a).

(b) Let $\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_m\}$ generate G and H, respectively, and form the graph X with vertex set $V = \Gamma/F$, edge set $E = (\Gamma/F) \times \{1, \ldots, n, 1^*, \ldots, m^*\}$, and orienting maps defined by $i(sF, j) = sF = i(sF, k^*)$, $t(sF, j) = sa_j F$, $t(sF, k^*) = sb_k F$ $(s \in \Gamma)$. The subgraphs X_1 and X_2 with $V_1 = G/F$, $V_2 = H/F$, $E_1 = (G/F) \times \{1, \ldots, n\}$, and $E_2 = (H/F) \times \{1^*, \ldots, m^*\}$ satisfy condition (*) of lemma 3.3. By lemmas 3.1 and 3.2, we have

$$\begin{aligned} \rho(G) &= (n-1)|F|^{-1} - \sum |F|^{-1}(P_c \delta_{(F,j)}, \delta_{(F,j)}), \text{ and} \\ \rho(H) &= (m-1)|F|^{-1} - \sum |F|^{-1}(P_c \delta_{(F,k^*)}, \delta_{(F,k^*)}), \end{aligned}$$

while for Γ, lemma 3.1 says

$$\rho(\Gamma) = (m+n-1)|F|^{-1} - \sum |F|^{-1}(P_c \delta_{(F,j)}, \delta_{(F,j)}) - \sum |F|^{-1}(P_c \delta_{(F,k^*)}, \delta_{(F,k^*)}).$$

This proves part (b).

3.5. Remark. If we define ρ^* for finitely generated groups by $\rho^*(G) = \rho(G) - |G|^{-1}$, then propositions 3.2(b) and 3.4 say that $\rho^*(G *_F H) = \rho^*(G) + p^*(H) - \rho^*(F)$ for all G and H sharing a finite subgroup F.

3.6. Proposition. *Let H be a subgroup of G of finite index. Then $\rho(H) = \rho(G)(G:H)$.*

Proof. Let X be a Cayley graph for G, so all stabilizers are trivial. Let G_0 be a complete set of right H-coset representatives in G, and let E_0 be a complete set of orbit representatives for the action of G on E. Lemma 1.2 then gives

$$\rho(H) = \sum \{(P_h\delta_{gy}, \delta_{gy}) \colon y \in E_0,\, g \in G_0\} = |G_0| \sum \{(P_h\delta_y, \delta_y) \colon y \in E_0\} = \rho(G)(G:H),$$

where the second equality comes from the G-invariance of $h(X)$.

3.7. Proposition. *Suppose that G acts with finite stabilizers and compact quotient on a two-dimensional cell complex K homeomorphic to $\mathbb{R}^2$ by orientation-preserving cellular maps. Then*

$$\rho(g) = -\sum |G_t|^{-1} + \sum |G_c|^{-1} - \sum |G_s|^{-1},$$

where t, c, and s range over complete sets of orbit representatives for the action of G on the 0-, 1-, *and* 2-*cells, respectively.*

Proof. For $j = 0, 1, 2$, let $C^{(j)}$ be the set of j-cells of K. Let $V = C^{(0)} \cup C^{(1)}$ and $E = C^{(1)} \times \{1, 2\}$. Define orienting maps by $i(c, 1) = i(c, 2) = c$ for c in $C^{(1)}$, and $t(c, 1) =$ one end of c, $t(c, 2) =$ the other end of c. The resulting directed graph X is obtained pictorially from K by drawing arrows from the midpoint of each 1-cell to the 0-cells at either end of it. The actions of G on V and E come from the action on K; in particular, $g(c, j) = (gc, k)$, where k is such that $t(gc, k) = gt(c, j)$. For each s in $C^{(2)}$, let $p(s)$ be the closed path in X obtained by traversing ∂s once counterclockwise. For each y in E, let $r(y)$ and $l(y)$ be the 2-cells in $C^{(2)}$ that lie on the right and left of y as y is traversed positively; we then have $\langle y, p(l(y))\rangle = 1 = -\langle y, p(r(y))\rangle$. Define $T\colon l^2(C^{(2)}) \to l^2(E)$ by $(T\eta)(y) = \eta(l(y)) - \eta(r(y))$. Then $T\delta_s = p(s)^\wedge$ for all s in $C^{(2)}$. Notice that for any closed path q in X, we have

$$q^\wedge = \sum \{n_s p(s)^\wedge \colon s \text{ in } C^{(2)} \text{ inside } q\},$$

where n_s is the number of times q winds counterclockwise around s. It follows that the range of T is dense in the range of the projection P_c. The operator T is injective – a function in its kernel would have to be constant – and appropriately respectful of the actions of G, so by lemma 1.3 we have $\dim(\mathrm{range}(P_c)) = \dim(l^2(C^{(2)}))$; the latter quantity is $\sum |G_s|^{-1}$, where s runs over a complete set of representatives for the action of G on $C^{(2)}$. For each c in $C^{(1)}$, either every g in G_c fixes both endpoints of c, in which case $G_c = G_{(c,1)} = G_{(c,2)}$ and the edges $(c, 1)$ and

$(c, 2)$ lie in different edge orbits, or some g in G_c interchanges the endpoints of c, in which case $(c, 1)$ and $(c, 2)$ lie the in same edge orbit and $|G_c| = 2|G_{(c,1)}|$; thus $2\sum |G_c|^{-1}$, where the sum extends over orbit representatives for the action of G on $C^{(1)}$, coincides with $\sum |G_y|^{-1}$, where the sum extends over orbit representatives for the action of G on E. By lemma 3.1, we then have

$$\rho(G) = 2\sum |G_c|^{-1} - \left(\sum |G_t|^{-1} + \sum |G_c|^{-1}\right) - \sum |G_s|^{-1}.$$

3.8. PROPOSITION. *For infinite groups G_1 and G_2, we have $\rho(G_1 \times G_2) = 0$.*

PROOF. Let G_i act properly with finite quotient on the connected directed graph X_i, with vertices V_i and edges E_i $(i = 1, 2)$. Let $V = V_1 \times V_2$, $E = (V_1 \times E_2) \cup (E_1 \times V_2)$, and define orienting maps i, t: $E \to V$ by

$$\begin{cases} i \\ t \end{cases} (v_1, y_2) = \left(v_1, \begin{cases} i \\ t \end{cases} (y_2)\right) \quad \text{and} \quad \begin{cases} i \\ t \end{cases} (y_1, y_2) = \left(\begin{cases} i \\ t \end{cases} (y_1), v_2\right).$$

One checks easily that the resulting directed graph X, with the product action of $G_1 \times G_2$, is suitable for the computation of $\rho(G_1 \times G_2)$. Given edges y_i in E_i $(i = 1, 2)$, with $i(y_i) = u_i$ and $t(y_i) = v_i$, let $p(y_1, y_2)$ be the following closed path in X:

$$(u_1, u_2), (y_1, u_2), (v_1, u_2), (v_1, y_2), (v_1, v_2), (y_1, v_2), (u_1, v_2), (u_1, y_2), (u_1, u_2).$$

The first two edges are traversed positively, the second two negatively. Identifying $l^2(E)$ with $(l^2(V_1) \otimes l^2(E_2)) \oplus (l^2(E_1) \otimes l^2(V_2))$, we have

$$p(y_1, y_2)^{\wedge} = ((-S_1^*\delta_{y_1}) \otimes \delta_{y_2}, \delta_{y_1} \otimes S_2^*\delta_{y_2}).$$

Thus the range of the map

$$\begin{aligned} T = (-S_1^* \otimes I, I \otimes S_2^*)\colon\ & l^2(E_1) \otimes l^2(E_2) \\ & \to (l^2(V_1) \otimes l^2(E_2)) \oplus (l^2(E_1) \otimes l^2(V_2)) = l^2(E) \end{aligned}$$

is contained in the range of P_c. Notice that $\ker(T) = \ker(S_1^*) \otimes \ker(S_2^*)$. Applying lemma 1.3 to the restriction of T to the orthogonal complement of $\ker(T)$, we have

$$\begin{aligned} \dim(\text{range } P_c) &\geqq \dim(l^2(E_1) \otimes l^2(E_2)) - \dim(\ker(S_1^*) \otimes \ker(S_2^*)) \\ &= \dim_{G_1}(l^2(E_1)) \dim_{G_2}(l^2(E_2)) - \dim_{G_1}(\ker(S_1^*)) \dim_{G_2}(\ker(S_2^*)). \end{aligned}$$

(In the last line we have taken advantage of the fact that the natural trace on $\mathrm{VN}(R_{G_1 \times G_2})$, which acts on $l^2(G_1) \otimes l^2(G_2)$, is the tensor product of the traces on $\mathrm{VN}(R_{G_i})$.) Write $a_1 = \dim_{G_i}(l^2(E_i))$ and $b_i = \dim_{G_i}(l^2(V_i))$. Then $\dim_{G_i}(\ker(S_i^*)) = a_i - b_i$, and the previous inequality says

$$\dim(\text{range } P_c) \geqq a_2 b_1 + a_1 b_2 - b_1 b_2.$$

As in lemma 3.1 and its proof, we have on the other hand

$$\begin{aligned}\rho(G_1 \times G_2) &= \dim(l^2(E)) - \dim(l^2(V)) - \dim(\text{range } P_c) \\ &= a_2 b_1 + a_1 b_2 - b_1 b_2 - \dim(\text{range } P_c),\end{aligned}$$

so $\rho(G_1 \times G_2) \leqq 0$, so $\rho(G_1 \times G_2) = 0$.

3.9. CONCLUDING REMARKS. Proposition 3.6 above says that ρ is an Euler characteristic in the generous sense of Chiswell [4]. It coincides with (minus) the famous rational Euler characteristic (see [1], [2], [9]) on the groups covered by Propositions 3.2 and 3.7. (For instance, when G is a Fuchsian group having a Dirichlet region with finite hyperbolic area A (see [5]), we have $A = 2\pi\rho(G)$.) Proposition 3.8 indicates that in general ρ captures at best only a low-dimensional piece of the rational Euler characteristic.

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