MATH. SCAND. 72 (1993), 161–190

PENROSE'S TENSORS ON SUPERGRASSMANNIANS

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Introduction.

The main object of the study of Riemannian geometry is the properties of the Riemann tensor, which in turn splits into the Weyl tensor, the traceless Ricci tensor, and the scalar curvature. All these tensors are obstructions to the possibility of "flattening" the manifold on which they are considered. The word "splits" above means that at every point of the Riemannian manifold M^n for $n \neq 4$ the space of values of the Riemann tensor constitutes an $O(n)$-module which splits into the sum of three irreducible components (for $n = 4$ there are four of them, because the Weyl tensor splits additionally in this case) [ALV, Ko].

More generally, let G be any group, not necessarily $O(n)$. If the principle $\mathrm{GL}(n)$-bundle on an n-dimensional manifold M can be reduced to a principle G-bundle we say that M is endowed with a *G-structure.* Later we give the definition of *structure functions,* shortly referred to as SFs, which constitute the complete set of obstructions to "flattening" a manifold with a G-structure. If $G = O(n)$, then the Riemann tensor is an example of SFs. Among the most known other examples of SFs are the following ones:

- a *conformal structure,* $G = O(n) \times \mathrm{R}^+$, SFs are called the Weyl tensor;
- Penrose's twistor theory, $G = S(\mathrm{GL}(2, \mathrm{C}) \times \mathrm{GL}(2, \mathrm{C}))$, SFs-Penrose's tensors-split into two components called the "*α-forms*" and the "*β-forms*";
- an almost *complex structure,* $G = \mathrm{GL}(n; \mathrm{C}) \subset \mathrm{GL}(2n; \mathrm{R})$, SFs are called the *Nijenhuis tensor.*

In several versions of a very lucid paper [G1, G2] A. Goncharov calculated all structure functions for the conformal structures of type X, where X is a "classical space", i.e., an irreducible compact Hermitian symmetric space (CHSS); in his examples G is the reductive part of the stabilizer of a point of the space. The usual conformal structure is the one that corresponds to $X = Q_n$, a quadric in the projective space. The complex grassmannian $X = \mathrm{Gr}_2^4$ corresponds to Penrose's twistors.

Received March 28, 1990; in revised form April 26, 1991.

Here I generalize Goncharov's result to all general and queer supergrassmannians, i.e., I calculate possible values for the super analogues of Penrose's "α-forms" and "β-forms". I will also restate the main geometric theorem from [G2].

SFs for the corresponding reduced structures, Riemannian structures of type Gr and QGr, will be considered in a following paper.

As in the classical case (Lie theory), computation of structure functions reduces to certain problems of representation theory. However, in the super case computations become much more complicated, because of the absence of complete reducibility. I could not directly apply the usual tools for computing (co)homology (spectral sequences and the Borel-Weil-Bott theorem) to superalgebras and had to retreat a step and apply these tools to the even parts of the considered Lie superalgebras. Then, using certain necessary conditions, I verified whether two modules over a Lie superalgebra that could be glued into an indecomposable module actually were glued or not (there are no sufficient conditions at all [Pe]).

Since our results seem to be of interest in relation to problems from various branches of mathematics and physics (non-holonomic mechanics and supergravity), I will explain the major points and the key cases in detail to enable the reader to reproduce the results or get similar ones for other G.

Novel features here are as follows:

– there are *two types of supergrassmannians*, since $GL(n)$ has two superanalogues: the general, $GL(m|n)$, and the "queer", $GQ(n)$. SFs for the queer supergrassmannians constitute a module looking exactly the same as that for grassmannians of generic dimensions, whereas SFs for the general supergrassmannians behave quite differently:

– since the underlying manifold of a supergrassmannian is generally the product of two grassmannians (unless one or both of them degenerate into a projective space or a point), our a priori estimate of *the number of irreducible components of the superspace of* SFs should *vary*. It certainly does vary, but differently than might be expected; besides the *superspace of* SFs *can be not completely reducible.*

Our calculations may contribute to a yet non-existent generalized superanalogue of the BWB theorem for superalgebras. By the BWB theorem we usually mean the case of the complete flags or cohomology of the maximal nilpotent subsuperalgebra, cf. the review [Pe].

As to the case of a non-maximal nilpotent subalgebras, our result is so far the only completely computed case of this yet non-existing BWB theorem in the particular case of the second cohomology of the Lie superalgebra complementary to a minimal parabolic subsuperalgebra with coefficients in the whole superalgebra of types $\mathfrak{sl}$ or $\mathfrak{psq}$. Our answer shows that, while for the general queer supergroup $GQ(n)$ there still is a hope to get an analogue of the BWB theorem, for $GL(m|n)$ even the formulation of such an analogue, if any, will be extremely complicated.

Since our supergrassmannians are complex supermanifolds, we consider everything over C. In particular we represent the supergrassmannians not as coset supermanifolds of unitary supergroups modulo unitary subsupergroups, but as cosets of SL or SQ over C modulo a parabolic subsupergroup.

In this paper we deal with linear algebra: at a point, for global geometry see [MV] and [M1], where some of our tensors are used.

REMARKS. 1) SFs for the majority of other classical superspaces (which are all supergrassmannians with an additional structure), i.e., the superanalogues of other cases considered by Goncharov, are listed in [P1, P2, P3, P4], see the review [LP1] for other ramifications and new problems. (For new problems see also [LSV].)

2) This paper may be considered as introductory to [LPS1, LPS2], where some problems of interest in modern and classical theoretical physics, raised in [L1, LP1, V, VF, VG, LSV], are solved.

3) In [LPS1] we discuss analogues of the Einstein equation for manifolds with G-structures other than $G = O(n)$, and their "superizations".

4) *Warning.* From the literature [Sch, RSh, M1] one might get the wrong impression that the SFs I compute here, and which are discussed in [RSh], pertain to supergravity or supertwistors. This is *not* the case: *"superization" of gravity and twistors leads to contact-type structures.*

The machinery to study such structures and the corresponding calculations are given in [LPS2], where we show how to calculate SFs for the N-extended Minkowski superspaces (understood broadly, as in [GIOS], as arbitrary quotients of SL $(4|N)$ modulo a parabolic subsupergroup) for an arbitrary N as well, and in [LP2], where the case of manifolds is treated.

Note that so far the only cases supposed to be understood by physicists (according to physicists themselves [OS1, OS2, GIOS] are: $N = 1$ (completely understood), and $N = 2, 3$ (partly understood).

Definition of a classical superspace see in [L1], where the problem I solve here was raised and the way to treat mathematically physical models of supersymmetry and supergravity via *contact-type* G-structures (rather than as in [Sch, RSh, M1]) was indicated, or in [S]. Some particular cases of our Main Theorem were announced in [P1, P2, P3]. Here we give the proof for the most difficult generic case, the details for the other cases will be published in further issues.

ACKNOWLEDGEMENTS. I am thankful to D. Leites, A. Onishchik, V. Serganova (they also verified the calculations) and A. Goncharov for their help; to IAS, Princeton, for hospitally during 1989, to an NSF grant through Pennsylvania State University, and SFB-170, Göttingen, for financial support at the final stages of the calculations.

0. Preliminaries.

Terminological conventions. 1) The $\mathfrak{g}$-module V with the highest weight ξ and an even highest vector will be denoted by V_ξ or $R(\xi)$. In what follows $R(\sum_i a_i\pi_i)$ denotes the irreducible $\mathfrak{g}_0$-module with highest weight $\sum_i a_i\pi_i$, where π_i is the i-th fundamental weight, cf. [OV], Reference Chapter. We will sometimes denote the highest weight by its numerical labels $R(a_i; a)$, where after the semicolon we write the component of the highest weight with respect to the center of $\mathfrak{g}_0$. The weights with respect to $\mathfrak{gl}(n)$, however, are most convenient to denote differently: with respect to the standard basis of the dual space to the maximal torus of $\mathfrak{gl}(n)$.

Let $\mathfrak{cg}$ denote the trivial extension of a Lie (super)algebra $\mathfrak{g}$. Recall that a $\mathbb{Z}$-graded Lie (super)algebra of the form $\oplus_{-a\leqq i}\mathfrak{g}_i$ is said to be of *depth d*.

0.1. *G-structures and their structure functions.* We recall the necessary definitions [St].

Let M be a manifold of dimension n over a field $\mathbb{K}$. Let $F(M)$ be the frame bundle over M, i.e., the principal $\mathrm{GL}(n, \mathbb{K})$-bundle. Let $G \subset \mathrm{GL}(n; \mathbb{K})$ be a Lie group. A *G-structure* on M is a reduction of the principal $\mathrm{GL}(n; \mathbb{K})$-bundle to the principal G-bundle.

The simplest G-structure is the *flat* G-structure defined as follows. Let V be $\mathbb{K}^n$ with a fixed frame. The flat structure is the bundle over V whose fiber over $v \in V$ consists of all frames obtained from the fixed one under the G-action, V being identified with $T_v V$.

The obstructions to identification of the kth infinitesimal neighbourhood of a point $m \in M$ on a manifold M with G-structure and that of a point of the flat manifold V with the above G-structure are what we call *structure functions of order k* [St, Gu].

Such an identification is only possible provided all structure functions of lesser orders vanish (see [G2] for details).

PROPOSITION ([St]). SFs constitute the space of the $(k, 2)$th *Spencer cohomology*.

The Spencer cochain complex whose cohomology is mentioned in the Proposition above is defined as follows. Let $S^i V$ denote the operator of the ith symmetric power of a vector space V. Set

$\mathfrak{g}_{-1} = T_m M$, $\mathfrak{g}_0 = \mathfrak{g} = \mathrm{Lie}(G)$ and for $i > 0$ put:

$$\mathfrak{g}_i = \{X \in \mathrm{Hom}(\mathfrak{g}_{-1}, \mathfrak{g}_{i-1}) : X(v)(w, \ldots) = X(w)(v, \ldots) \text{ for any } v, w \in \mathfrak{g}_{-1}\}$$
$$= (\mathfrak{g}_0 \otimes S^i(g_{-1})^*) \cap (\mathfrak{g}_{-1} \otimes S^{i+1}(\mathfrak{g}_{-1})^*).$$

Now set $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_* = \oplus_{i \geqq -1} \mathfrak{g}_i$. Suppose that

(1) the $\mathfrak{g}_0$-module $\mathfrak{g}_{-1}$ is faithful.

Then $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_* \subset \mathfrak{vect}(n) = \mathrm{der}\, \mathrm{K}[[x_1, \ldots, x_n]]$, where $n = \dim \mathfrak{g}_{-1}$. It can be verified that the Lie algebra structure on $\mathfrak{vect}(n)$ induces such a structure on $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_*$. The Lie algebra $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_*$, usually abbreviated $\mathfrak{g}_*$, will be called the *Cartan prolongation* of the pair $(\mathfrak{g}_{-1}, \mathfrak{g}_0)$.

Let E^iV be the ith exterior power of a vector space V. Set $C^{k,s}_{\mathfrak{g}_0} = \mathfrak{g}_{k-2} \otimes E^s(\mathfrak{g}^*_{-1})$. Define the differentials $\partial^{k,s}_{\mathfrak{g}_0} : C^{k,s}_{\mathfrak{g}_0} \to C^{k-1,s+1}_{\mathfrak{g}_0}$ as follows: for any $g_1, \ldots, g_{s+1} \in \mathfrak{g}_{-1}$, $f \in C^{k,s}_{\mathfrak{g}_0}$

$$(2) \quad (\partial^{k,s}_{\mathfrak{g}_0} f)(g_1, \ldots, g_{s+1}) = \sum_i (-1)^{(s+1-i)} [f(g_1, \ldots, \widehat{g_{s+1-i}}, \ldots, g_{s+1}), g_{s+1-i}]$$

As expected, $\partial^{k,s}_{\mathfrak{g}_0} \partial^{k-1,s+1}_{\mathfrak{g}_0} = 0$. The cohomology of bidegree (k, s) of this complex is called the (k, s)th Spencer cohomology group $H^{k,s}_{\mathfrak{g}_0}$.

0.2. *Goncharov's results: recapitulation.* In the main text we will literally "superize" some of Goncharov's ideas and results [G2]. This is possible thanks to the lucid algebraicity of his approach to Riemannian geometry and Penrose's theory of twistors which now attracts ever greater interest of mathematicians and physicists, cf. [Pe, M1], and performed with the use of the point functor, see the Appendix in [LS] (also referred to in some works as the *Grassmann envelope*, cf. [B]).

Recall that Penrose's idea is to embed the Minkowski space M^4 into the complex Grassmann manifold $\mathrm{Gr}_2(\mathbf{C}^4)$ of planes in $\mathbf{C}^4$ (or straight lines in $\mathbf{CP}^3$) and to express the conformal structure on M^4 in terms of the incidence relation of the straight lines in $\mathbf{CP}^3$.

The conformal structure on M^4 is given by a field of quadratic cones in the tangent spaces to the points of M^4. In Penrose's case these cones possess two families of two-dimensional flat generators, the so-called "α-planes" and "β-planes." The geometry of these families is vital for Penrose's considerations. In particular, the Weyl tensor has a lucid description in terms of these families.

It is interesting to include 4-dimensional Penrose theory into a more general theory of geometric structures. Goncharov has shown that there is an analogous field of quadratic cones for any irreducible compact Hermitian symmetric space X of rank greater then one. Here we will generalize his result to the case of supermanifolds. As an aside note of the importance of quadratic conditions in this approach.

PROBLEM. What is the relation of these quadratic conditions with Manin's non-commutative projective geometry constructed for quadratic algebras [M2]?

0.2.1. *F-structures and their structure functions.* Recall that the notion of an F-structure is a generalization of the notion of distribution, i.e., a subbundle in the TM and the SFs of an F-structure generalize the notion of the Frobenius form [G2].

Let $\dim V = \dim T_m M$, $F \subset \mathrm{Gr}_k(V)$ be a manifold with a transitive action of a subgroup of $\mathrm{GL}(V)$, and $\mathfrak{F}(M)$ be a subbundle of $\mathrm{Gr}_k(TM)$, where the fiber of $\mathrm{Gr}_k(TM)$ is $\mathrm{Gr}_k(T_m M)$. The bundle $\mathfrak{F}(M) \to M$ is called an *F-structure* on M if for any point m of M there is a linear isomorphism $I_m: V \to T_m M$, which induces a diffeomorphism $I_m(F) = \mathfrak{F}(m)$. A submanifold $Z \subset M$ of dimension k such that $T_z Z \subset \mathfrak{F}(z)$ for any $z \in Z$ is called an *integral* submanifold. An F-structure is *completely integrable* if for any $m \in M$ and for any subspace $V(m) \subset \mathfrak{F}(m)$ there is an integral manifold Z with $T_m Z = V(m)$.

1) Under what conditions is an F-structure completely integrable?

2) Given a completely integrable F-structure, how many integral submanifolds tangent to it are there?

Goncharov answered both questions. To do so he introduced SFs of an F-structure as follows. For $f \in \mathfrak{F}(m)$ let $V_f \subset T_m M$ be the subspace corresponding to f. Set
$(T_f)_{-1} = T_m M/V_f$, $(T_f)_0 = T_f \mathfrak{F}(m) \subset \mathrm{Hom}(V_f, T_m M/V_f)$. Define
$(T_f)_s = ((T_f)_{s-1} \otimes V_f^*) \cap ((T_f)_{s-2} \otimes S^2 V_f^*)$ for $s > 0$, and
$C^{k,s}(V_f) = (T_f)_{k-2} \otimes E^s V_f^*$.
Define the differentials as in (2). Then the cohomology groups $H^{k,s}(V_f)$ are naturally defined. The elements of $H^{k,2}(V_f)$ are called SFs of order k.

THEOREM ([G2]. 1) *The elements of $H^{k,2}(V_f)$ are obstructions to integrability of the F-structure up to the* kth *infinitesimal neighbourhood, provided the* SFs *of lesser order vanish.*

2) *For the completely integrable F-structure (all* SFs *vanish) the family of integrable manifolds tangent to V_f is of dimension $\sum_k \dim H^{k,1}(V_f)$.*

The super version of this theorem is absolutely similar and the proof is the same.

0.2.2. *Relation between F-structures and G-structures.* Let $G_F \subset \mathrm{GL}(V)$ be the group of all transformations preserving F and $\mathfrak{g}_F = \mathrm{Lie}(G_F)$. Then there exists a projection $H^{k,s}_{\mathfrak{g}_F} \to H^{k,s}(V_f)$. Explicit formulas may be found in [G2].

The other way around the F-structure corresponding to the G-structure is also of interest. It is studied in detail for so-called generalized conformal structures.

0.2.3. *The cone $C(X)$ associated with a* CHSS X. *Definition of generalized conformal structures* ([G2]). Let S be a simple Lie group, P its parabolic subgroup with the Levi decomposition $P = GN$, i.e., G is reductive and N is the radical of P. As one knows (see [He]), N is Abelian if and only if $X = S/P$ is a CHSS, and in this case $G = G_0 \times \mathbf{C}^*$, where G_0 is semisimple. Let N_- be the subgroup opposite to N. Denote by $\mathfrak{s}$, $\mathfrak{p}$, $\mathfrak{g}$, $\mathfrak{g}_0$, $\mathfrak{n}$, $\mathfrak{n}_-$ the corresponding Lie algebras.

Let $P_x = G_x N_x$ be the Levi decomposition of the stabilizer of $x \in X$ in S. Denote by C_x the cone of highest weight vectors in the G_x-module $T_x X$, i.e., each element in C_x is highest with respect to a Borel subgroup in G_x. Since $s \in S$ transforms C_x to C_{sx}, then with X there is associated the cone $C(X) \subset \mathfrak{n}_-$ linearly equivalent to all the C_x. It will be convenient to identify $\mathfrak{n}_-$ with $T_{\bar{e}}X$ and $C(X)$ with $C_{\bar{e}}$, where $\bar{e}$ is the image of the unit $e \in G$ in X.

Let $\mathrm{rk}(X) > 1$, i.e., $X \neq \mathbf{C}\mathrm{P}^n$. Then on a manifold Y a *generalized conformal structure* (GSC) *of type* X is given if Y is endowed with a family of cones C_y and $\mathbf{C}$-linear isomorphisms $A_y : \mathfrak{n}_- \to T_y Y$ such that $A_y(C(X)) = C_y$. The generalized conformal structures of the same type on manifolds Y and Z are *equivalent* if there is a diffeomorphism $f: Y \to Z$ such that $df(C_y) = C_{f(y)}$.

EXAMPLE (The conformal structure). Let X be a non-degenerate quadric in $\mathbf{C}\mathrm{P}^{n+1}$. Then C_x is the non-degenerate quadratic cone consisting of straight lines in $\mathbf{C}\mathrm{P}^{n+1}$ passing through x and belonging to X. The family of cones C_x defines a conformal structure on X. C_x is what is called the *light*, or *zero*, or *null* cone.

Among all manifolds with a GSC of type X the CHSS X is distinguished as the one which has the flat C_x-structure. In other words, a local diffeomorphism, which preserves the family of cones C_x's on any subdomain on X can be extended to a holomorphic automorphism of X. This gives us an infinitesimal characterization of CHSSs of rank greater than 1. In fact, we can recover a CHSS of rank > 1 from any simply connected domain with a flat family of cones [G2].

This property might be useful in the theory of analytic functions on bounded complex symmetric domains of rank > 1, and for us gives a hope to understand how to "superize" Berezin's mysterious quantization, cf. [Pee] and references therein.

Part 2) of Theorem 0.2.1 and the BWB theorem give an estimate from above for the dimension of a family of integrable manifolds. The functor of points translates the definition and results of this section to supermanifolds.

0.2.4. *Generalized conformal and Riemannian structures as G-structures* ([G2]). Goncharov has shown that a manifold Y with generalized conformal structure of type $X = S/P$, where $P = GN$, is a manifold with a G-structure. We will continue his studies and explicitly calculate all the SFs of this G-structure. For a generalized conformal structure of type X the group G is reductive and its center is one-dimensional. To reduce the structure group G to its semisimple part $\hat{G}$ is an action similar to distinguishing a metric from a conformal class on a conformal manifold.

The structure functions of the $\hat{G}$-structures form an analogue of the Riemann tensor for the metric. They include the structure functions of the G-structure and several other irreducible components, some of which are analogues of the traceless Ricci tensor or of the scalar curvature.

More precisely, the structure functions of the G-structure are defined as the part of the structure functions of the $\hat{G}$-structure obtained by a reduction of the G-structure that does not depend on the choice of reduction. In other words, this is a generalized conformally invariant part of the structure of the $\hat{G}$-structure.

There is a striking similarity between generalized conformal structures and the classical geometry of conformal manifolds.

The algebraic reason for this similarity is the fact that generalized conformal structures are G-structures of order 2. In the flat case this means that an automorphism of a CHSS X is not defined by its differential at a point: in an appropriate coordinate system the infinitesimal automorphisms of X are vector fields whose coefficients are polynomials of degree $\leqq 2$ [G2].

Geometrically speaking, the G-structure is determined by the cones C_x.

For example, there are often many "isotropic" linear subspaces in the tangent space to a point of a manifold with a GSC of type X, i.e., subspaces that belong to $C_x \cup 0$. Therefore, on a generalized conformal manifold an F-structure can be introduced. This F-structure is completely integrable if and only if the Weyl tensor (or its corresponding part, of it splits) vanishes. For $X = \mathrm{Gr}_2^4$ this result is due to R. Pensore. This is a non-standard geometric interpretation of the SFs of generalized conformal manifolds due to Goncharov.

His results on the relations of the structure functions of the G_F-structure with obstructions to the integrability of the F-structure generalize a theorem by Penrose, which states that the self-dual part of the Weyl tensor on a 4-dimensional manifold with a conformal structure vanishes if and only if α-surfaces exist.

The space of SFs for a (generalized) Riemannian structure contains that for a conformal one and has, as is shown in [G2] (the proof in [LP2] is more general), an additional summand, $S^2(\mathfrak{g}_{-1})^*$. The case of superalgebras is similar, so we only have to calculate SFs for the conformal structures. This is precisely what we do here. Some of the geometric interpretations of this calculations are given in [LPS1] and below.

0.2.5. *Laplace operators for generalized conformal structures* ([G2]). Since the cone C_x is singled out in T_xX by a system of quadratic equations, the G_0-structure may be considered as a "metric" with values in the bundle E^*, where the fiber of E over x is the linear space of quadratic equations that singles out $C_x \subset T_xX$.

The G_0-structure on X defines a second order differential operator Δ acting on the space of functions into the space $\Gamma(E)$ of sections of E. If $\mathfrak{g}_0 = \mathfrak{o}(n)$, and E is the trivial bundle of rank 1, the operator Δ is the classical Laplace operator.

The characteristic manifold of the Laplace operator Δ is $\cup_{x \in X} C_x^*$, where $C_x^* \subset T_x^* X$ is the cone linearly equivalent to C_x and in a sense dual to C_x. Similarly, on a generalized Riemannian manifold an analogue of the Laplace operator is defined.

0.2.6. *Real forms of* GSC ([G2]). Until now the ground field was C. Let us consider real forms of CHSSs, i.e., what are called R-symmetric Nagano spaces. For example, the real forms of Gr_{2k}^{2n} are RGR_{4k}^{4n} and HGr_k^n, and the real forms of the quadric Q_{n-1} in CP^n are quadrics in RP^n. For us it is only important that each R-symmetric Nagano space is representable in the form $S_{\mathrm{R}}/P_{\mathrm{R}}$, where the Lie algebras of S_{R} and P_{R} are of the form $\mathfrak{s} = \mathfrak{s}_{-1} \oplus \mathfrak{s}_0 \oplus \mathfrak{s}_1$ and $\mathfrak{p} = \mathfrak{s}_0 \oplus \mathfrak{s}_1$, respectively.

To each of these R-symmetric spaces we will assign a geometric structure which is a real form of the corresponding conformal structure. A way of obtaining a (pseudo)Riemannian metric on a real manifold M of dimension n is to consider a G-structure of type X on M, where X is a quadric in RP^{n+1}.

All the results obtained for generalized conformal structures can be easily extended to their real forms. Moreover, several real forms have some very interesting properties. A typical example is HP^n and the corresponding theory of quaternionic manifolds. Indeed, the complexification of HP^n is Gr_2^{2n+2}. Therefore, if X is a quaternionic manifold, then in $T_xX \otimes_{\mathrm{R}} \mathrm{C}$ a cone C_x linearly equivalent to the cone $C(\mathrm{Gr}_2^{2n+2})$ is canonically constructed.

Let $\pi : P \to X$ be the bundle whose fiber over x is the set of all "α-subspaces" in C_x, i.e., the set of isotropic subspaces of C-dimension $2n$ in C_x.

After Goncharov, we say that a unitary connection in a bundle on X is *self-dual* if its curvature form vanishes on α-subspaces. Assigning to a bundle on X with a self-dual connection a certain holomorphic bundle on P we establish an equivalence of the category of such bundles on X with a certain category of holomorphic bundles on P.

By Theorem 4.9 from [G2] each α-subspace has a unique lift to the corresponding point of P. Since $\pi^{-1}(x) \cong \mathrm{CP}^1$, we obtain an almost complex structure on P. This structure is integrable if and only if the anti-self-dual part of the SFs vanishes. To a bundle with a self-dual connection on such X there corresponds a holomorphic bundle on P. This fact is a multi-dimensional generalization of the Atiyah-Ward-Belavin-Zakharov construction, cf. references in [G2].

0.3. *Classical results: the case of a simple Lie algebra* $\mathfrak{g}_*$ *over* C.

0.3.1. The following remarkable fact, though known to experts, is seldom formulated explicitly, cf. [L3, #32]:

PROPOSITION. *Let* $\mathrm{K} = \mathrm{C}$, *and let* $\mathfrak{g}_* = (\mathfrak{g}_{-1}, \mathfrak{g}_0)_*$ *be simple. Then only the following cases are possible:*

1) $\mathfrak{g}_2 \neq 0$, *then* g_* *is either* $\mathfrak{vect}(n)$ *or its special subalgebra* $\mathfrak{svect}(n)$ *of divergence-free vector fields, or its subalgebra* $\mathfrak{h}(2n)$ *of Hamiltonian vector fields.*

2) $\mathfrak{g}_2 = 0$, $\mathfrak{g}_1 \neq 0$, *then* $\mathfrak{g}_*$ *is the Lie algebra of the complex Lie group of automorphisms of a* CHSS *(see above).*

The Proposition explains the reason for imposing the restriction (1). Otherwise, we can not interpret the Cartan prolongation as a Lie subalgebra of $\mathfrak{vect}(\dim \mathfrak{g}_{-1})$. The fact that the Cartan prolongation is nevertheless a Lie algebra can then be proved directly, as in [St].

When $\mathfrak{g}_*$ is a simple finite dimensional Lie algebra over C computation of SFs becomes an easy corollary of the Borel-Weil-Bott (BWB) theorem in a form due to W. Shmid, cf. [G2]. Indeed, by definition,

$$\oplus_k H^{k,2}_{\mathfrak{g}_0} = H^2(\mathfrak{g}_{-1}, \mathfrak{g}_*).$$

The BWB theorem implies that, as a $\mathfrak{g}_0$-module, $H^2(\mathfrak{g}_{-1}, \mathfrak{g}_*)$ has as many components as $H^2(\mathfrak{g}_{-1})$ which, thanks to commutativity of $\mathfrak{g}_{-1}$, is $E^2\mathfrak{g}^*_{-1}$. The BWB theorem also gives the formula for the highest weights of these components. However, Goncharov did not write these weights explicitly and therefore failed to observe a possibility to write Einstein-type equations as is done in [LPS1].

0.3.2. The following theorem is given for comparison with our results as a point of reference. Besides, I never saw (except in [LP1]) the *explicit* calculations of SFs implicitly described by Serre for certain grassmannians (projective spaces); I believe that the experts know the result (Rh. Nelson is among them). The equations for various degrees of non-flatness of an almost symplectic structure, however, seem to be new, cf. [P4] and references therein.

Recall that all Z-gradings of depth 1 of $\mathfrak{sl}(m)$ are of the form $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, where $\mathfrak{g}_1 = \mathfrak{g}^*_{-1}$, with $\mathfrak{g}_0 = \mathfrak{c}(\mathfrak{sl}(p) \oplus \mathfrak{sl}(m-p))$.

THEOREM. 1) (Serre [St]). *In case* 1) *of the Proposition,* i.e., *when* $(p-1)(m-p-1) = 0$, *structure functions can only be of order* 1. *More precisely: for* $\mathfrak{g}_* = \mathfrak{vect}(n)$ *and* $\mathfrak{svect}(n)$ SFs *vanish;*
for $\mathfrak{g}_* = \mathfrak{h}(2n)$ *nonzero* SFs *are* $R(\pi_1)$ *for* $n = 2$, *and* $R(\pi_1) \oplus R(\pi_3)$ *for* $n > 2$.

2) (Goncharov [G2]). SFs *for the grassmannian* Gr^{m+n}_m *(when neither m nor n is equal to* 1, i.e., *the grassmannian is not a projective space) are the direct sum of two components whose weights and orders are as follows:*

Let $A = R(2,0,\ldots,0,-1) \otimes R(1,0,\ldots,0,-1,-1)$, $B = R(1,1,0,,\ldots,0,-1) \otimes R(1,0,\ldots,0,-2)$,
$A' = R(3,0,\ldots,0,-1) \otimes R(0,\ldots,0,-1,-1)$, $B' = R(1,1,0,,\ldots,0,0) \otimes R(1,0,\ldots,0,-3)$. Then

m, n	*order 1*	*order 2*
$m \neq 2, n \neq 2$	A and B	–
$m = 2, n \neq 2$	A	B'
$n = 2, m \neq 2$	B	A'
$n = m = 2$	–	A' and B'

0.4. SF*s on supermanifolds.*

0.4.1. *Recapitulation.* The necessary background on Lie superalgebras and supermanifolds is gathered in [L2]. Recall that $\mathfrak{gl}(m|n)$ is the Lie superalgebra (with respect to the supercommutator) of block matrices of size (m, n). Its "queer" analogue, which preserves the odd complex structure over the ground field K is $\mathfrak{q}(n) = \{X \in \mathfrak{gl}(n|n) \mid [X, J_{2n}] = 0\}$ for an odd nondegenerate form J_{2n}, such that $J_{2n}^2 = -1_{2n}$. The usual choice for J_{2n} is $J_{2n} = \text{antidiag}(1_n, -1_n)$. Then we have: $\mathfrak{q}(n) = \{X \in \mathfrak{gl}(n|n) \mid X = \text{diag}(A, A) + \text{antidiag}(B, B)$, where $A, B \in \mathfrak{gl}(n)\}$. It is known [L1] that the supertrace and the queertrace defined on $\mathfrak{gl}(n|n)$ and $\mathfrak{q}(n)$, respectively, by the formulas:

$$\text{str}(\text{diag}(A, D) + \text{antidiag}(B, C)) = \text{tr}\, A - \text{tr}\, D \text{ and}$$

$$\text{qtr}(\text{diag}(A, A) + \text{antidiag}(B, B)) = \text{tr}\, B$$

are analogues of the trace for $\mathfrak{gl}$ and $\mathfrak{q}$, respectively. Let $\mathfrak{s}$ distinguish traceless subalgebra and $\mathfrak{p}$ stand for projectivization, e.g., $\mathfrak{sq}(n)$, $\mathfrak{psl}(n|n) = \mathfrak{sl}(n|n)/\langle 1_{2n}\rangle$, $\mathfrak{psq}(n)$, etc.

All Z-gradings, in particular of depth 1, of Lie superalgebras of series $\mathfrak{sl}$, $\mathfrak{psl}$, and $\mathfrak{psq}$ are calculated in [K].

0.4.2. *How to "superize".* The above definitions of SFs are generalized to Lie superalgebras via the sign rule. However, in the super case new phenomena appear, which have no analogues in the classical case:

- Cartan prolongations of $(\mathfrak{g}_{-1}, \mathfrak{g}_0)$ and of $(\Pi\mathfrak{g}_{-1}, \mathfrak{g}_0)$, where Π is the functor of the change of parity, are essentially different;
- faithfulness of the $\mathfrak{g}_0$-action on $\mathfrak{g}_{-1}$ is violated in natural examples of supergrassmannians of subsuperspaces in an (n, n)-dimensional superspace when the center z of $\mathfrak{g}_0$ acts trivially. If we retain the same definition of the Cartan prolongation, then it has the form of the semidirect sum $(\mathfrak{g}_{-1}, \mathfrak{g}_0/z)_* \ltimes S^*(\mathfrak{g}_{-1}^*)$ (the ideal is $S^*(\mathfrak{g}_{-1}^*)$) with the natural Z-grading and Lie superalgebra structure, but this Lie superalgebra is not a subsuperalgebra of $\mathfrak{vect}(\dim \mathfrak{g}_{-1})$ anymore.

• the formulation of Proposition (0.3.1) and of Serre's theorem (0.3.2) fails to be literally true for Lie superalgebras.

1. Spencer cohomology of $\mathfrak{g}_* = \mathfrak{psq}(n)$.

The Spencer cohomology of the Lie superalgebras of this series resemble those of $\mathfrak{sl}(n)$ much more than those of $\mathfrak{sl}(m|n)$.

PROPOSITION. [K]. *All* Z*-gradings of depth* 1 *of* $\mathfrak{g} = \mathfrak{psq}(n)$ *are of the form* $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ *with* $\mathfrak{g}_1 = \mathfrak{g}^*_{-1}$ *and* $\mathfrak{g}_0 = \mathfrak{ps}(\mathfrak{q}(p) \oplus \mathfrak{q}(n-p))$, *where* $p \neq 0$, *whereas* $\mathfrak{g}_{-1}$ *is either one of the two irreducible* $\mathfrak{g}_0$*-modules in* $\mathrm{id}^*_p \otimes \mathrm{id}_{n-p}$, *where* id_k *denotes the standard (identity) representation of the "summand" of* $\mathfrak{g}_0$ *isomorphic to* $\mathfrak{q}(k)$.

Explicitly:

$$\mathfrak{g}_{-1} = \langle (x \pm \Pi(x)) \otimes (y \pm \Pi(y)) \rangle, \text{ where } x \in \mathrm{id}^*_p,\ y \in \mathrm{id}_{n-p}.$$

Let $\varepsilon_1, \ldots, \varepsilon_p$ and $\delta_1, \ldots, \delta_{n-p}$ be the standard bases of the dual spaces to the spaces of diagonal matrices in $\mathfrak{q}(p)$ and $\mathfrak{q}(n-p)$, respectively.

MAIN THEOREM (case Q). A) $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_* = \mathfrak{g}$,

B) *all* SF*s are of order* 1 *and split into the direct sum of two irreducible* $\mathfrak{g}_0$*-submodules*: $V_{2\varepsilon_1 - \varepsilon_p + \delta_1 - 2\delta_{n-p}} \oplus V_{\varepsilon_1 - \delta_{n-p}}$.

2. Spencer cohomology of $\mathfrak{g} = \mathfrak{sl}(m|n)$ and $\mathfrak{psl}(n|n)$.

2.1. Z*-gradings of depth* 1 *of* $\mathfrak{g} = \mathfrak{sl}(m|n)$ *and* $\mathfrak{psl}(n|n)$. All these gradings are of the form $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ and $\mathfrak{g}_1 = \mathfrak{g}^*_{-1} = \mathrm{id} \otimes \mathrm{id}^*$.

PROPOSITION. A) *There are the following possible values of* $\mathfrak{g}_0$ *for the* Z*-gradings of* $\mathfrak{g}$ *of depth* 1:

a) $\mathfrak{c}(\mathfrak{sl}(m-p|q) \oplus \mathfrak{sl}(p|n-q))$ *if* $pq \neq 0$;

b) $\mathfrak{c}(\mathfrak{sl}(m|q) \oplus \mathfrak{sl}(n-q))$ *if* $p = 0$, $q(n-q) \neq 0$;

c) $\mathfrak{c}(\mathfrak{sl}(m-p) \oplus \mathfrak{sl}(p|n))$ *if* $q = 0$, $p(m-p) \neq 0$;

d) $\mathfrak{c}(\mathfrak{sl}(m) \oplus \mathfrak{sl}(n))$ *if* $p = q = 0$ *or* $m - p = n - q = 0$.

B) *For* $\mathfrak{sl}(n|n)$ *there are the following possible values of* $\mathfrak{g}_0$ *for the* Z*-gradings of depth* 1:

a) $\mathfrak{c}(\mathfrak{sl}(n-p|q) \oplus \mathfrak{sl}(p|n-q))$ *if* $pq \neq 0$;

b) $\mathfrak{c}(\mathfrak{sl}(n|q) \oplus \mathfrak{sl}(n-q))$ *if* $p = 0$, $q(n-q) \neq 0$;

c) $\mathfrak{c}(\mathfrak{sl}(n-p) \oplus \mathfrak{sl}(p|n))$ *if* $q = 0$, $p(n-p) \neq 0$;

d) $\mathfrak{c}(\mathfrak{sl}(n) \oplus \mathfrak{sl}(n))$ *if* $p = q = 0$ *or* $n - p = n - q = 0$.

C) *The* Z*-gradings of* $\mathfrak{psl}(n|n)$ *are similar to those of* $\mathfrak{sl}(n|n)$ *only* $\mathfrak{g}_0$ *is centerless.*

Let us denote the reductive part of $\mathfrak{g}_0$ by $\hat{\mathfrak{g}}_0$.

2.2. *Cartan prolongation of* $(\mathfrak{g}_{-1}, \mathfrak{g}_0)$.

THEOREM. *For the cases of Proposition* 2.1 *we have* $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_* = \mathfrak{g}$ *except for the cases:*

A) $\mathfrak{g} = \mathfrak{sl}(m|n)$, $m \neq n$. *Then if for the above cases*
a) $n = q$, $p = 1$, *then* $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_* = \mathfrak{vect}(m-1|n)$,
$m = p$, $q = 1$, *then* $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_* = \mathfrak{vect}(n-1|m)$;
b) $n - q = 1$, *or* $m = 0$, $q = 1$, *then* $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_* = \mathfrak{vect}(n-1|m)$;
c) $m - p = 1$, *or* $n = 0$, $p = 1$, *then* $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_* = \mathfrak{vect}(m-1|n)$;
d) $n = 1$ or $m = 1$, then $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_* = \mathfrak{vect}(0|m)$ or $\mathfrak{vect}(0|n)$, *respectively*.
B) $\mathfrak{g} = \mathfrak{sl}(n|n)$. *Then* $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_* = \mathfrak{psl}(n|n) \ltimes S^*(\mathfrak{g}^*_{-1})$ *except for the following cases:*
a) $n = q$, $p = 1$ *or* $n = p$, $q = 1$, b) $n - q = 1$, c) $n - p = 1$, *then* $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_* = \mathfrak{vect}(n-1|n)$;
d) $n = 2$, *then* $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_* = \mathfrak{h}(0|4) \ltimes S^*(\mathfrak{g}^*_{-1})$,
where $\mathfrak{h}(0|n) = \{D \in \mathfrak{vect}(0|n)\colon Dw = 0$ *for the Hamiltonian form* $w = \sum_{i=1,n}(d\xi_i)^2\}$.
C) $\mathfrak{g} = \mathfrak{psl}(n|n)$. *Then* $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_* = \mathfrak{psl}(n|n)$ *except for the following cases:*
a) $n = q$, $p = 1$ *or* $n = p$, $q = 1$, b) $n - q = 1$, c) $n - p = 1$, *then* $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_* = \mathfrak{svect}(n-1|n)$, *where* $\mathfrak{svect}(n-1|n)$ *is the Lie superalgebra of divergence-free vector fields.*
d) $n = 2$, *then* $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_* = \mathfrak{h}(0|4)$.

2.3. MAIN THEOREM (case SL). Let us consider cases of Proposition 2.1. First consider the cases easiest to formulate.

2.3.1. THEOREM. 1) *For* $\mathfrak{g}_* = \mathfrak{vect}(m|n)$, $\mathfrak{svect}(m|n)$ SF*s vanish except for* $\mathfrak{svect}(0|n)$, *when* SF*s are of order n and constitute the* $\mathfrak{g}_0$*-module* $\Pi^n(\langle 1 \rangle)$.

2) *For* $\mathfrak{g}_*$ *of series* $\mathfrak{h}(0|n)$ *nonzero* SF*s are of order* 1*: for* $g_* = \mathfrak{h}(0|n)$, $n > 3$, SF*s are* $\Pi(R(3\pi_1) \oplus R(\pi_1))$.

3) *For* $\mathfrak{g}_* = \mathfrak{sh}(0|n)$, $n > 3$, *nonzero* SF*s are the same as for* $\mathfrak{h}(0|n)$ *and additionally* $\Pi^{n-1}(R(\pi_1))$ *of order* $n - 1$.

Clearly, case c) of Proposition 2.1. is obtained from b) with obvious replacements. In what follows we will consider the values of parameters for which $\mathfrak{g}_*$ is not $\mathfrak{vect}$, $\mathfrak{svect}$, or $\mathfrak{h}$.

2.3.2. THEOREM, case 2.1.d). *The nonzero* SF*s are of orders* 1 *and* 2. *The* $\mathfrak{g}_0$*-modules* $H^{1,2}_{\mathfrak{g}_0}$ *and* $H^{2,2}_{\mathfrak{g}_0}$ *are completely reducible. The highest weights (with respect to* $\varepsilon_1, \ldots, \varepsilon_m$ *and* $\delta_1 \ldots \delta_n$, *the basis of the dual space to the maximal torus of* $\mathfrak{g}_0$*) of their irreducible components are given in Table* 1.

TABLE 1

m	n	$H^{1,2}_{\mathfrak{g}_0}$	$H^{2,2}_{\mathfrak{g}_0}$
2	2	$2\varepsilon_1 - \varepsilon_2 + \delta_1 - 2\delta_2$ $\varepsilon_1 - \delta_2$	–
2	3	$2\varepsilon_1 - \varepsilon_2 + \delta_1 - 2\delta_3$	–
2	$\geqq 4$	$2\varepsilon_1 - \varepsilon_2 + \delta_1 - 2\delta_n$	$\varepsilon_1 + \varepsilon_2 + \delta_1 - \delta_{n-2} - \delta_{n-1} - \delta_n$
3	2	$2\varepsilon_1 - \varepsilon_3 + \delta_1 - 2\delta_2$	–
$\geqq 4$	2	$2\varepsilon_1 - \varepsilon_m + \delta_1 - 2\delta_2$	$\varepsilon_1 + \varepsilon_2 + \varepsilon_3 - \varepsilon_m - \delta_1 - \delta_2$
$\geqq 3$	$\geqq 3$	$2\varepsilon_1 - \varepsilon_m + \delta_1 - 2\delta_n$ $\varepsilon_1 + \varepsilon_2 - \varepsilon_m + \delta_1 - \delta_{n-1} - \delta_n$ $\varepsilon_1 - \delta_n$ (if $m = n$)	–

2.3.3. THEOREM, CASE 2.1.b). The nonzero SFs are of orders 1 and 2. The $\mathfrak{g}_0$-module $H^{2,2}_{\mathfrak{g}_0}$ splits into the direct sum of irreducible components whose highest weights are given in Table 2. Table 2 also contains the highest weights (with respect to the bases $\varepsilon_1, \dots, \varepsilon_{m+q}$ and $\delta_1 \dots \delta_{n-q}$ of the dual spaces to the maximal tori of $\mathfrak{gl}(m|q)$ and $\mathfrak{sl}(n-q)$, respectively) of irreducible components of $H^{1,2}_{\mathfrak{g}_0}$ for the cases when $H^{1,2}_{\mathfrak{g}_0}$ does split into the direct sum of irreducible $\mathfrak{g}_0$-modules.

TABLE 2

m	q	$n-q$	$H^{1,2}_{\mathfrak{g}_0}$	$H^{2,2}_{\mathfrak{g}_0}$
$\geqq 2$	$\geqq 1$	$\geqq 3$	$m \mp q \pm 1$ $2\varepsilon_1 - \varepsilon_{m+q} + \delta_1 - 2\delta_{n-q}$ $\varepsilon_1 + \varepsilon_2 - \varepsilon_{m+q} + \delta_1 - \delta_{n-q-1} - \delta_{n-q}$ $\varepsilon_1 - \delta_{n-q}$ (if $m = n$)	–
$\geqq 3$	$\geqq 1$	2	$m \mp q - 1$ $2\varepsilon_1 - \varepsilon_{m+q} + \delta_1 - 2\delta_2$ $\varepsilon_1 - \delta_2$ (if $m = n$)	$\varepsilon_1 + \varepsilon_2 + \varepsilon_3 -$ $-\varepsilon_{m+q} - \delta_1 - \delta_2$
2	$\geqq 2$	2	$q \mp 3$ $2\varepsilon_1 - \varepsilon_{q+2} + \delta_1 - 2\delta_2$	$\varepsilon_1 + \varepsilon_2 + \varepsilon_3 -$ $-\varepsilon_{q+2} - \delta_1 - \delta_2$
2	1	2	$2\varepsilon_1 - \varepsilon_3 + \delta_1 - 2\delta_2$	$\varepsilon_1 + \varepsilon_3 - \delta_1 - \delta_2$
1	$\geqq 1$	$\geqq 3$	$q \mp 2$ $2\varepsilon_1 - \varepsilon_{q+1} + \delta_1 - 2\delta_{n-q}$ $\varepsilon_1 + \varepsilon_2 - \varepsilon_{q+1} + \delta_1 - \delta_{n-q-1} - \delta_{n-q}$	–

m	q	$n-q$	$H^{1,2}_{\mathfrak{g}_0}$	$H^{2,2}_{\mathfrak{g}_0}$
1	$\geqq 1$	2	$q \neq 2$ $2\varepsilon_1 - \varepsilon_{q+1} + \delta_1 - 2\delta_2$	$\varepsilon_1 + 2\varepsilon_2 - \varepsilon_{q+1} -$ $-\delta_1 - \delta_2 (q \neq 1)$ $2\varepsilon_2 - \delta_1 - \delta_2 (q = 1)$
0	2	2	–	$3\varepsilon_1 - \varepsilon_2 - \delta_1 - \delta_2$ $\varepsilon_1 + \varepsilon_2 + \delta_1 - 3\delta_2$
0	2	$\geqq 3$	$2\varepsilon_1 - \varepsilon_2 + \delta_1 - \delta_{n-3} - \delta_{n-2}$	$\varepsilon_1 + \varepsilon_2 + \delta_1 - 3\delta_{n-2}$
0	$\geqq 3$	2	$\varepsilon_1 + \varepsilon_2 - \varepsilon_q + \delta_1 - 2\delta_2$	$3\varepsilon_1 - \varepsilon_q - \delta_1 - \delta_2$
0	$\geqq 3$	$\geqq 3$	$2\varepsilon_1 - \varepsilon_q + \delta_1 - \delta_{n-q-1} - \delta_{n-q}$ $\varepsilon_1 + \varepsilon_2 - \varepsilon_q + \delta_1 - 2\delta_{n-q}$	–

Exceptional cases are as follows:
if $m = q - 1$, $m > 1$, $n - q \geqq 3$, then

(1) $$H^{1,2}_{\mathfrak{g}_0} = V_{\varepsilon_1+\varepsilon_2-\varepsilon_{m+q}+\delta_1-\delta_{n-q-1}-\delta_{n-q}} \oplus X,$$

where
X is given by the non-split exact sequence of $\mathfrak{g}_0$-modules

(2) $$0 \to V_{2\varepsilon_1-\varepsilon_{m+q}+\delta_1-2\delta n-q} \to X \to \Pi(V_{\varepsilon_1+\delta_1-2\delta_{n-q}}) \to 0;$$

if $m = q - 1$, $m > 1$, $n - q = 2$, then

(3) $$H^{1,2}_{\mathfrak{g}_0} = X,$$

where X is given by (2);
if $m = 1$, $q = 2$, $n - q \geqq 3$, then

(4) $$H^{1,2}_{\mathfrak{g}_0} = \Pi(V_{\varepsilon_1+\varepsilon_2-\varepsilon_3+\delta_1-\delta_{n-q-1}-\delta_{n-q}}) \oplus X,$$

where X is given by the non-split exact sequence of $\mathfrak{g}_0$-modules

(5) $$0 \to V_{2\varepsilon_1-\varepsilon_3+\delta_1-2\delta_{n-q}} \oplus \Pi(V_{-\varepsilon_1+\varepsilon_2+\varepsilon_3+\delta_1-2\delta_{n-q}}) \to X \to \Pi(V_{\varepsilon_1+\delta_1-2\delta_{n-q}}) \to 0;$$

if $m = 1$, $q = 2$, $n - q = 2$, then

(6) $$H^{1,2}_{\mathfrak{g}_0} = X,$$

where X is given by (5);
if $m = q + 1$, $n - q \geqq 3$, then

(7) $$H^{1,2}_{\mathfrak{g}_0} = V_{2\varepsilon_1-\varepsilon_{m+q}+\delta_1-2\delta_{n-q}} \oplus X,$$

where X is given by the non-split exact sequence of $\mathfrak{g}_0$-modules

$$0 \to V_{\varepsilon_1+\varepsilon_2-\varepsilon_{m+q}+\delta_1-\delta_{n-q-1}-\delta_{n-q}} \to X \to \Pi(V_{\varepsilon_1+\delta_1-\delta_{n-q-1}-\delta_{n-q}}) \to 0 \quad (q \geqq 2),$$

$$(8) \qquad 0 \to V_{\varepsilon_1-\varepsilon_2+\varepsilon_3+\delta_1-\delta_{n-2}-\delta_{n-1}} \oplus V_{\varepsilon_1+\varepsilon_2-\varepsilon_3+\delta_1-\delta_{n-2}-\delta_{n-1}} \to X$$
$$\to \Pi(V_{\varepsilon_1+\delta_1-\delta_{n-2}-\delta_{n-1}}) \to 0 \quad (q = 1).$$

2.3.4. In this generic case the weights are given with respect to the bases $\varepsilon_1, \dots, \varepsilon_{m-p+q}$ and $\delta_1, \dots, \delta_{p+n-q}$ of the dual spaces to the maximal tori of $\mathfrak{gl}(m-p|q)$ and $\mathfrak{sl}(p|n-q)$, respectively.

THEOREM, CASE 2.1.a). The nonzero SFs are of orders 1 and 2. The $\mathfrak{g}_0$-module $H^{2,2}_{\mathfrak{g}_0}$ splits into the direct sum of irreducible components whose highest weights are given in Table 3. Table 3 also contains the highest of irreducible components of $H^{1,2}_{\mathfrak{g}_0}$ for the cases when $H^{1,2}_{\mathfrak{g}_0}$ does split into the direct sum of irreducible $\mathfrak{g}_0$-modules.

TABLE 3

$m-p$	q	p	$n-q$	$H^{1,2}_{\mathfrak{g}_0}$	$H^{2,2}_{\mathfrak{g}_0}$
0	2	2	0	$2\varepsilon_1-\varepsilon_2+\delta_1-2\delta_2$ $\varepsilon_1-\delta_2$	–
0	2	3	0	$2\varepsilon_1-\varepsilon_2+\delta_1-2\delta_3$	–
0	3	2	0	$2\varepsilon_1-\varepsilon_3+\delta_1-2\delta_2$	–
0	2	$\geqq 4$	0	$2\varepsilon_1-\varepsilon_2+\delta_1-2\delta_p$	$\varepsilon_1+\varepsilon_2+\delta_1-$ $-\delta_{p-2}-\delta_{p-1}-\delta_p$
0	$\geqq 4$	2	0	$2\varepsilon_1-\varepsilon_q+\delta_1-2\delta_2$	$\varepsilon_1+\varepsilon_2+\varepsilon_3-$ $-\varepsilon_q-\delta_1-\delta_2$
0	$\geqq 3$	$\geqq 3$	0	$2\varepsilon_1-\varepsilon_q+\delta_1-2\delta_p$ $\varepsilon_1+\varepsilon_2-\varepsilon_q+\delta_1-\delta_{p-1}-\delta_p$ $\varepsilon_1-\delta_p (m=n)$	–
0	2	$\geqq 1$	$\geqq 1$	$n \mp p+q+1$ $2\varepsilon_1-\varepsilon_2+\delta_1-\delta_{p+n-3}-\delta_{p+n-2}$ $\varepsilon_1-\delta_{p+n-2} (m=n)$	$\varepsilon_1+\varepsilon_2+$ $+\delta_1-3\delta_{p+n-2}$
$\geqq 1$	$\geqq 1$	2	0	$m \mp p+q+1$ $\varepsilon_1+\varepsilon_2-\varepsilon_{m-p+q}+\delta_1-2\delta_2$ $\varepsilon_1-\delta_2 (m=n)$	$3\varepsilon_1-\varepsilon_{m-p+q}-$ $-\delta_1-\delta_2$
0	$\geqq 3$	$\geqq 1$	$\geqq 1$	$n \mp p+q\pm 1$ $\varepsilon_1+\varepsilon_2-\varepsilon_q+\delta_1-2\delta_{p+n-q}$ $2\varepsilon_1-\varepsilon_q+\delta_1-\delta_{p+n-q-1}-\delta_{p+n-q}$ $\varepsilon_1-\delta_{p+n-q} (m=n)$	–

$m-p$	q	p	$n-q$	$H^{1,2}_{\mathfrak{g}_0}$	$H^{2,2}_{\mathfrak{g}_0}$
$\geqq 1$	$\geqq 1$	$\geqq 3$	0	$m \neq p+q \pm 1$ $2\varepsilon_1 - \varepsilon_{m-p+q} + \delta_1 - \delta_{p-1} - \delta_p$ $\varepsilon_1 + \varepsilon_2 - \varepsilon_{m-p+q} + \delta_1 - 2\delta_p$ $\varepsilon_1 - \delta_p (m = n)$	–
$\geqq 1$	$\geqq 1$	$\geqq 1$	$\geqq 1$	$m, n \neq p+q \pm 1$ $2\varepsilon_1 - \varepsilon_{m-p+q} + \delta_1 - 2\delta_{p+n-q}$ $\varepsilon_1 + \varepsilon_2 - \varepsilon_{m-p+q} + \delta_1 - \delta_{p+n-q-1} - \delta_{p+n-q}$ $\varepsilon_1 - \delta_{p+n-q} (m = n \geqq 3)$	–

Exceptional cases are $m = p + q \pm 1$ and $n = p + q \pm 1$. More precisely: if $m = p + q + 1, n \neq p + q \pm 1, q$, then

$$H^{1,2}_{\mathfrak{g}_0} = V_{2\varepsilon_1 - \varepsilon_{m-p+q} + \delta_1 - 2\delta_{p+n-q}} \oplus Y, \tag{1}$$

where Y is given by the non-split exact sequence of $\mathfrak{g}_0$-modules

$$0 \to V_{\varepsilon_1+\varepsilon_2-\varepsilon_3+\delta_1-\delta_{p+n-2}-\delta_{p+n-1}} \oplus V_{\varepsilon_1-\varepsilon_2+\varepsilon_3+\delta_1-\delta_{p+n-2}-\delta_{p+n-1}} \to Y$$
$$\to V_{\varepsilon_1+\delta_1-\delta_{p+n-2}-\delta_{p+n-1}} \to 0 \quad (\text{if } q = 1),$$

$$0 \to V_{\varepsilon_1+\varepsilon_2-\varepsilon_{m-p+q}+\delta_1-\delta_{p+n-q-1}-\delta_{p+n-q}} \to Y \tag{2}$$
$$\to V_{\varepsilon_1+\delta_1-\delta_{p+n-q-1}-\delta_{p+n-q}} \to 0 \quad (\text{if } q \geqq 2);$$

if $m = p + q + 1, n = p + q - 1$, then

$$H^{1,2}_{\mathfrak{g}_0} = X \oplus Y, \tag{3}$$

where X is given by the non-split exact sequence of $\mathfrak{g}_0$-modules

$$0 \to V_{2\varepsilon_1-\varepsilon_{m-2+q}+\delta_1-2\delta_3} \oplus V_{2\varepsilon_1-\varepsilon_{m-2+q}-\delta_1-\delta_2+\delta_3} \to$$
$$X \to V_{2\varepsilon_1-\varepsilon_{m-2+q}-\delta_3} \to 0 \quad (\text{if } p = 2), \tag{4}$$
$$0 \to V_{2\varepsilon_1-\varepsilon_{m-p+q}+\delta_1-2\delta_{p+n-q}} \to X$$
$$\to V_{2\varepsilon_1-\varepsilon_{m-p+q}-\delta_{p+n-q}} \to 0 \quad (\text{if } p \geqq 3),$$

and Y is given by the non-split exact sequence of $\mathfrak{g}_0$-modules

$$0 \to V_{\varepsilon_1+\varepsilon_2-\varepsilon_3+\delta_1-\delta_{p+n-2}-\delta_{p+n-1}} \oplus V_{\varepsilon_1-\varepsilon_2+\varepsilon_3+\delta_1-\delta_{p+n-2}-\delta_{p+n-1}} \to$$
$$Y \to V_{\varepsilon_1+\delta_1-\delta_{p+n-2}-\delta_{p+n-1}} \to 0 \quad (\text{if } q = 1), \tag{5}$$
$$0 \to V_{\varepsilon_1+\varepsilon_2-\varepsilon_{m-p+q}+\delta_1-\delta_{p+n-q-1}-\delta_{p+n-q}} \to Y$$
$$\to V_{\varepsilon_1+\delta_1-\delta_{p+n-q-1}-\delta_{p+n-q}} \to 0 \quad (\text{if } q \geqq 2);$$

if $m = p + q + 1$, $n = q$, then

$$H^{1,2}_{\mathfrak{g}_0} = V_{2\varepsilon_1 - \varepsilon_{m-p+q} + \delta_1 - \delta_{p-1} - \delta_p} \oplus Y \quad (\text{if } p \geqq 3),$$
$$\text{or } H^{1,2}_{\mathfrak{g}_0} = Y \quad (\text{if } p = 2), \tag{6}$$

where Y is given by the non-split exact sequence of $\mathfrak{g}_0$-modules

$$0 \to V_{\varepsilon_1+\varepsilon_2-\varepsilon_3+\delta_1-2\delta_p} \oplus V_{\varepsilon_1-\varepsilon_2+\varepsilon_3+\delta_1-2\delta_p} \to Y \to V_{\varepsilon_1+\delta_1-2\delta_p} \to 0 \quad (\text{if } q = 1),$$
$$0 \to V_{\varepsilon_1+\varepsilon_2-\varepsilon_{m-p+q}+\delta_1-2\delta_p} \to Y \to V_{\varepsilon_1+\delta_1-2\delta_p} \to 0 \quad (\text{if } q \geqq 2); \tag{7}$$

if $n = p + q + 1$, $m \neq p + q \pm 1, p$, then

$$H^{1,2}_{\mathfrak{g}_0} = V_{2\varepsilon_1-\varepsilon_{m-p+q}+\delta_1-2\delta_{p+n-q}} \oplus Y, \tag{8}$$

where

$$0 \to V_{\varepsilon_1+\varepsilon_2-\varepsilon_{m-1+q}+\delta_1-\delta_2-\delta_3} \oplus V_{\varepsilon_1+\varepsilon_2-\varepsilon_{m-1+q}-\delta_1+\delta_2-\delta_3} \to$$
$$Y \to V_{\varepsilon_1+\varepsilon_2-\varepsilon_{m-1+q}-\delta_3} \to 0 \quad (\text{if } p = 1),$$
$$0 \to V_{\varepsilon_1+\varepsilon_2-\varepsilon_{m-p+q}+\delta_1-\delta_{p+n-q-1}-\delta_{p+n-q}} \to Y$$
$$\to V_{\varepsilon_1+\varepsilon_2-\varepsilon_{m-p+q}-\delta_{p+n-q}} \to 0 \quad (\text{if } p \geqq 2); \tag{9}$$

if $n = p + q + 1$, $m = p + q - 1$, then

$$H^{1,2}_{\mathfrak{g}_0} = X \oplus Y, \tag{10}$$

where

$$0 \to V_{2\varepsilon_1-\varepsilon_3+\delta_1-2\delta_{p+n-2}} \oplus V_{-\varepsilon_1+\varepsilon_2+\varepsilon_3+\delta_1-2\delta_{p+n-2}} \to X$$
$$\to V_{\varepsilon_1+\delta_1-2\delta_{p+n-2}} \to 0 \quad (\text{if } q = 2),$$
$$0 \to V_{2\varepsilon_1-\varepsilon_{m-p+q}+\delta_1-2\delta_{p+n-q}} \to X \to V_{\varepsilon_1+\delta_1-2\delta_{p+n-q}} \to 0 \quad (\text{if } q \geqq 3), \tag{11}$$

and Y is given by the non-split exact sequence of $\mathfrak{g}_0$-modules

$$0 \to V_{\varepsilon_1+\varepsilon_2-\varepsilon_{m-1+q}+\delta_1-\delta_2-\delta_3} \oplus V_{\varepsilon_1+\varepsilon_2-\varepsilon_{m-1+q}-\delta_1+\delta_2-\delta_3}$$
$$\to Y \to V_{\varepsilon_1+\varepsilon_2-\varepsilon_{m-1+q}-\delta_3} \to 0 \quad (\text{if } p = 1),$$
$$0 \to V_{\varepsilon_1+\varepsilon_2-\varepsilon_{m-p+q}+\delta_1-\delta_{p+n-q-1}-\delta_{p+n-q}} \to Y$$
$$\to V_{\varepsilon_1+\varepsilon_2-\varepsilon_{m-p+q}-\delta_{p+n-q}} \to 0 \quad (\text{if } p \geqq 2); \tag{12}$$

if $n = p + q + 1$, $m = p$, then

$$H^{1,2}_{\mathfrak{g}_0} = V_{\varepsilon_1+\varepsilon_2-\varepsilon_q+\delta_1-2\delta_{p+n-q}} \oplus Y \quad (\text{if } q \geqq 3), \text{ or } H^{1,2}_{\mathfrak{g}_0} = Y \quad (\text{if } q = 2), \tag{13}$$

where

$$0 \to V_{2\varepsilon_1-\varepsilon_q+\delta_1-\delta_2-\delta_3} \oplus V_{2\varepsilon_1-\varepsilon_q-\delta_1+\delta_2-\delta_3} \to Y \to V_{2\varepsilon_1-\varepsilon_q-\delta_3} \to 0 \text{ (if } p = 1),$$

(14)

$$0 \to V_{2\varepsilon_1-\varepsilon_q+\delta_1-\delta_{p+n-q-1}-\delta_{p+n-q}} \to Y \to V_{2\varepsilon_1-\varepsilon_q-\delta_{p+n-q}} \to 0 \quad (\text{if } p \geqq 2);$$

if $m = p + q - 1$, $n \neq p + q \pm 1, q$, then

(15)
$$H^{1,2}_{\mathfrak{g}_0} = X \oplus V_{\varepsilon_1+\varepsilon_2-\varepsilon_{m-p+q}+\delta_1-\delta_{p+n-q-1}-\delta_{p+n-q}},$$

where X is given by the non-split exact sequence of $\mathfrak{g}_0$-modules

$$0 \to V_{2\varepsilon_1-\varepsilon_3+\delta_1-2\delta_{p+n-2}} \oplus V_{-\varepsilon_1+\varepsilon_2+\varepsilon_3+\delta_1-2\delta_{p+n-2}} \to$$

(16)
$$X \to V_{\varepsilon_1+\delta_1-2\delta_{p+n-2}} \to 0 \quad (\text{if } q = 2),$$

$$0 \to V_{2\varepsilon_1-\varepsilon_{m-p+q}+\delta_1-2\delta_{p+n-q}} \to X \to V_{\varepsilon_1+\delta_1-2\delta_{p+n-q}} \to 0 \quad (\text{if } q \geqq 3);$$

if $m = p + q - 1$, $n = q$, then

(17)
$$H^{1,2}_{\mathfrak{g}_0} = X \oplus V_{\varepsilon_1+\varepsilon_2-\varepsilon_{m-p+q}+\delta_1-2\delta_p} \quad (p \geqq 3),$$

where X is given by the non-split exact sequence of $\mathfrak{g}_0$-modules

$$0 \to V_{2\varepsilon_1-\varepsilon_3+\delta_1-\delta_{p-1}-\delta_p} \oplus V_{-\varepsilon_1+\varepsilon_2+\varepsilon_3+\delta_1-\delta_{p-1}-\delta_p} \to$$

(18)
$$X \to V_{\varepsilon_1+\delta_1-\delta_{p-1}-\delta_p} \to 0 \quad (\text{if } q = 2);$$

$$0 \to V_{2\varepsilon_1-\varepsilon_{m-p+q}+\delta_1-\delta_{p-1}-\delta_p} \to X \to V_{\varepsilon_1+\delta_1-\delta_{p-1}-\delta_p} \to 0 \quad (\text{if } q \geqq 3);$$

if $n = p + q - 1$, $m \neq p + q \pm 1, p$, then

(19)
$$H^{1,2}_{\mathfrak{g}_0} = X \oplus V_{\varepsilon_1+\varepsilon_2-\varepsilon_{m-p+q}+\delta_1-\delta_{p+n-q-1}-\delta_{p+n-q}},$$

where X is given by the non-split exact sequence of $\mathfrak{g}_0$-modules

$$0 \to V_{2\varepsilon_1-\varepsilon_{m-2+q}+\delta_1-2\delta_3} \oplus V_{2\varepsilon_1-\varepsilon_{m-2+q}-\delta_1-\delta_2+\delta_3} \to$$

(20)
$$X \to V_{2\varepsilon_1-\varepsilon_{m-2+q}-\delta_3} \to 0 \quad (\text{if } p = 2),$$

$$0 \to V_{2\varepsilon_1-\varepsilon_{m-p+q}+\delta_1-2\delta_{p+n-q}} \to X \to V_{2\varepsilon_1-\varepsilon_{m-p+q}-\delta_{p+n-q}} \to 0 \quad (\text{if } p \geqq 3);$$

if $n = p + q - 1$, $m = p$, then

(21)
$$H^{1,2}_{\mathfrak{g}_0} = X \oplus V_{2\varepsilon_1-\varepsilon_q+\delta_1-\delta_{p+n-q-1}-\delta_{p+n-q}} \quad (q \geqq 3),$$

where X is given by the non-split exact sequence of $\mathfrak{g}_0$-modules

$$0 \to V_{\varepsilon_1+\varepsilon_2-\varepsilon_q+\delta_1-2\delta_3} \oplus V_{\varepsilon_1+\varepsilon_2-\varepsilon_q-\delta_1-\delta_2+\delta_3} \to X \to V_{\varepsilon_1+\varepsilon_2-\varepsilon_q-\delta_3} \to 0 \quad (\text{if } p = 2),$$

(22)
$$0 \to V_{\varepsilon_1+\varepsilon_2-\varepsilon_q+\delta_1-2\delta_{p+n-q}} \to X \to V_{\varepsilon_1+\varepsilon_2-\varepsilon_q-\delta_{p+n-q}} \to 0 \quad (\text{if } p \geqq 3);$$

if $m = n = p + q + 1$, then

(23)
$$H^{1,2}_{\mathfrak{g}_0} = X \oplus V_{2\varepsilon_1-\varepsilon_{m-p+q}+\delta_1-2\delta_{p+n-q}} \oplus V_{\varepsilon_1-\delta_{p+n-q}},$$

where X is given by the non-split exact sequence of $\mathfrak{g}_0$-modules

(24) $$0 \to Y \to X \to V_{\varepsilon_1 - \delta_{p+n-q}} \to 0,$$

and Y is given by the non-split exact sequence of $\mathfrak{g}_0$-modules

(25) $$0 \to V_{\varepsilon_1+\varepsilon_2-\varepsilon_{m-p+q}+\delta_1-\delta_{p+n-q-1}-\delta_{p+n-q}} \to Y \to V_{\varepsilon_1+\delta_1-\delta_{p+n-q-1}-\delta_{p+n-q}} \oplus V_{\varepsilon_1+\varepsilon_2-\varepsilon_{m-p+q}-\delta_{p+n-q}} \to 0 \quad (\text{if } p \geqq 2,\ q \geqq 2),$$

$$0 \to V_{\varepsilon_1+\varepsilon_2-\varepsilon_{m-1+q}+\delta_1-\delta_2-\delta_3} \oplus V_{\varepsilon_1+\varepsilon_2-\varepsilon_{m-1+q}-\delta_1+\delta_2-\delta_3} \to Y \to V_{\varepsilon_1+\varepsilon_2-\varepsilon_{m-1+q}-\delta_3} \oplus V_{\varepsilon_1+\delta_1-\delta_2-\delta_3} \oplus V_{\varepsilon_1-\delta_1+\delta_2-\delta_3} \to 0 \quad (\text{if } p = 1,\ q \geqq 2),$$

$$0 \to V_{\varepsilon_1+\varepsilon_2-\varepsilon_3+\delta_1-\delta_{p+n-2}-\delta_{p+n-1}} \oplus V_{\varepsilon_1-\varepsilon_2+\varepsilon_3+\delta_1-\delta_{p+n-2}-\delta_{p+n-1}} \to Y \to V_{\varepsilon_1+\varepsilon_2-\varepsilon_3-\delta_{p+n-1}} \oplus V_{\varepsilon_1+\delta_1-\delta_{p+n-2}-\delta_{p+n-1}} \oplus V_{\varepsilon_1-\varepsilon_2+\varepsilon_3-\delta_{p+n-1}} \to 0 \quad (\text{if } p \geqq 2,\ q = 1),$$

$$0 \to V_{\varepsilon_1+\varepsilon_2-\varepsilon_3+\delta_1-\delta_2-\delta_3} \oplus V_{\varepsilon_1-\varepsilon_2+\varepsilon_3+\delta_1-\delta_2-\delta_3} \oplus V_{\varepsilon_1+\varepsilon_2-\varepsilon_3-\delta_1+\delta_2-\delta_3} \oplus V_{\varepsilon_1-\varepsilon_2+\varepsilon_3-\delta_1+\delta_2-\delta_3} \to Y \to V_{\varepsilon_1+\varepsilon_2-\varepsilon_3-\delta_3} \oplus V_{\varepsilon_1-\varepsilon_2+\varepsilon_3-\delta_3} \oplus V_{\varepsilon_1+\delta_1-\delta_2-\delta_3} \oplus V_{\varepsilon_1-\delta_1+\delta_2-\delta_3} \to 0 \quad (\text{if } p = 1,\ q = 1);$$

if $m = n = p + q - 1$, then

(26) $$H^{1,2}_{\mathfrak{g}_0} = X \oplus V_{\varepsilon_1+\varepsilon_2-\varepsilon_{m-p+q}+\delta_1-\delta_{p+n-q-1}-\delta_{p+n-q}} \oplus V_{\varepsilon_1-\delta_{p+n-q}},$$

where X is given by the non-split exact sequence of $\mathfrak{g}_0$-modules

(27) $$0 \to Y \to X \to V_{\varepsilon_1-\delta_{p+n-q}} \to 0,$$

and Y is given by the non-split exact sequence of $\mathfrak{g}_0$-modules

$$0 \to V_{2\varepsilon_1-\varepsilon_{m-p+q}+\delta_1-2\delta_{p+n-q}} \to Y \to V_{2\varepsilon_1-\varepsilon_{m-p+q}-\delta_{p+n-q}} \oplus V_{\varepsilon_1+\delta_1-2\delta_{p+n-q}} \to 0 \quad (\text{if } p \geqq 3,\ q \geqq 3),$$

$$0 \to V_{2\varepsilon_1-\varepsilon_{m-2+q}+\delta_1-2\delta_3} \oplus V_{2\varepsilon_1-\varepsilon_{m-2+q}-\delta_1-\delta_2+\delta_3} \to Y \to V_{2\varepsilon_1-\varepsilon_{m-2+q}-\delta_3} \oplus V_{\varepsilon_1+\delta_1-2\delta_3} \oplus V_{\varepsilon_1-\delta_1-\delta_2+\delta_3} \to 0 \quad (\text{if } p = 2,\ q \geqq 3),$$

(28) $$0 \to V_{2\varepsilon_1-\varepsilon_3+\delta_1-2\delta_{p+n-2}} \oplus V_{-\varepsilon_1+\varepsilon_2+\varepsilon_3+\delta_1-2\delta_{p+n-2}} \to Y \to V_{\varepsilon_1+\delta_1-2\delta_{p+n-2}} \oplus V_{2\varepsilon_1-\varepsilon_3-\delta_{p+n-2}} \oplus V_{-\varepsilon_1+\varepsilon_2+\varepsilon_3-\delta_{p+n-2}} \to 0 \quad (\text{if } p \geqq 3,\ q = 2),$$

$$0 \to V_{2\varepsilon_1-\varepsilon_3+\delta_1-2\delta_3} \oplus V_{-\varepsilon_1+\varepsilon_2+\varepsilon_3+\delta_1-2\delta_3} \oplus V_{2\varepsilon_1-\varepsilon_3-\delta_1-\delta_2+\delta_3} \oplus V_{-\varepsilon_1+\varepsilon_2+\varepsilon_3-\delta_1-\delta_2+\delta_3} \to Y \to V_{2\varepsilon_1-\varepsilon_3-\delta_3} \oplus V_{-\varepsilon_1+\varepsilon_2+\varepsilon_3-\delta_3} \oplus V_{\varepsilon_1+\delta_1-2\delta_3} \oplus V_{\varepsilon_1-\delta_1-\delta_2+\delta_3} \to 0 \quad (\text{if } p = 2,\ q = 2).$$

REMARK. The irreducible $\mathfrak{g}_0$-modules in the above listed non-split exact sequences are given regardless of their parity, which can be easily recovered from the corresponding highest weights.

3. Proofs.

Proof of Main Theorem will be given for all the key subcases of the SL case; the Q case is similar.

LEMMA ([St]). *Consider the first and the second differentials around the* $(k, 2)$nd *term of the Spencer complex:*

$$\mathfrak{g}_{k-1} \otimes \mathfrak{g}_{-1}^* \xrightarrow{\partial_{\mathfrak{g}_0}^{k+1,1}} \mathfrak{g}_{k-2} \otimes E^2(\mathfrak{g}_{-1}^*) \xrightarrow{\partial_{\mathfrak{g}_0}^{k,2}} \mathfrak{g}_{k-3} \otimes E^3(\mathfrak{g}_{-1}^*).$$

Then $\ker \partial_{\mathfrak{g}_0}^{k+1,1} = \mathfrak{g}_k$.

Case 2.3.2. *Order* 1. We have to find the second cohomology of the complex

$$\mathfrak{g}_0 \otimes \mathfrak{g}_{-1}^* \xrightarrow{\partial_{\mathfrak{g}_0}^{2,1}} \mathfrak{g}_{-1} \otimes S^2(\mathfrak{g}_{-1}^*) \xrightarrow{\partial_{\mathfrak{g}_0}^{1,2}} 0. \tag{1}$$

Notice that since $\mathfrak{g}_{-1}$ is odd, the operator E turns to S. By Lemma $\ker \partial_{\mathfrak{g}_0}^{2,1} = \mathfrak{g}_1$. So all we have to do is compare the weights of irreducible constituents of the first and the second terms of the complex (1).

Let V be the standard (identity) $\mathfrak{gl}(m)$-module, and W be the standard $\mathfrak{gl}(n)$-module. Let $\varepsilon_1, \dots, \varepsilon_m$ be a basis of V and $\tilde{e}_1, \dots, \tilde{e}_m$ be the dual basis of V^* normalized so that

$$\tilde{e}_i(e_j) = \delta_{ij}.$$

Let $f_1, \dots, f_n$ and $\tilde{f}_1, \dots, \tilde{f}_n$ be similar bases of W and W^*. Recall that the weights ε_i and δ_i are defined on the diagonal matrices $\operatorname{diag}(x_1, \dots, x_m)$ and $\operatorname{diag}(y_1, \dots, y_n)$, respectively, by the formulas

$$\varepsilon_i(\operatorname{diag}(x_1, \dots, x_m)) = x_i,$$

$$\delta_i(\operatorname{diag}(y_1, \dots, y_n)) = y_i.$$

Then $V = R(\varepsilon_1)$, $W = R(\delta_1)$, $V^* = R(-\varepsilon_m)$, $W^* = R(-\delta_n)$, and the highest weight vectors constituting $\mathfrak{g}_{-1} \otimes S^2(\mathfrak{g}_{-1}^*) = (V^* \otimes W) \otimes S^2(V \otimes W^*)$ and their weights are:

parameters	highest weight vectors	weights
$m \geqq 2$	$(\tilde{e}_m \otimes f_1) \otimes (e_1 \otimes \tilde{f}_n)(e_1 \otimes \tilde{f}_n)$	$2\varepsilon_1 - \varepsilon_m + \delta_1 - 2\delta_n$
$n \geqq 2$	$\sum_{1 \leqq i \leqq n}(\tilde{e}_i \otimes f_1) \otimes (e_i \otimes \tilde{f}_n)(e_1 \otimes \tilde{f}_n)$	$\varepsilon_1 + \delta_1 - 2\delta_n$
	$\sum_{1 \leqq i \leqq n}(\tilde{e}_m \otimes f_i) \otimes (e_1 \otimes \tilde{f}_i)(e_1 \otimes \tilde{f}_n)$	$2\varepsilon_1 - \varepsilon_m - \delta_n$
	$(\tilde{e}_m \otimes f_1) \otimes ((e_1 \otimes \tilde{f}_{n-1})(e_2 \otimes \tilde{f}_n) -$ $- (e_1 \otimes \tilde{f}_n)(e_2 \otimes \tilde{f}_{n-1}))$	$\varepsilon_1 + \varepsilon_2 - \varepsilon_m +$ $+ \delta_1 - \delta_{n-1} - \delta_n$
	$\sum_{j=1,n}\sum_{i=1,m}(\tilde{e}_i \otimes f_j) \otimes (e_i \otimes \tilde{f}_j)(e_1 \otimes \tilde{f}_n)$	$\varepsilon_1 - \delta_n$
either $m \neq 2$ or $n \neq 2$	$\sum_{j=1,n}\sum_{i=1,m}(\tilde{e}_i \otimes f_j) \otimes (e_i \otimes \tilde{f}_n)(e_1 \otimes \tilde{f}_j)$	$\varepsilon_1 - \delta_n$
$m \geqq 3$	$\sum_{i=1,m}(\tilde{e}_i \otimes f_1) \otimes ((e_i \otimes \tilde{f}_{n-1})(e_1 \otimes \tilde{f}_n) -$ $- (e_i \otimes \tilde{f}_n)(e_1 \otimes \tilde{f}_{n-1}))$	$\varepsilon_1 + \delta_1 - \delta_{n-1} - \delta_n$
$n \geqq 3$	$\sum_{i=1,n}(\tilde{e}_m \otimes f_i) \otimes ((e_1 \otimes \tilde{f}_i)(e_2 \otimes \tilde{f}_n) -$ $- (e_1 \otimes \tilde{f}_n)(e_2 \otimes \tilde{f}_i))$	$\varepsilon_1 + \varepsilon_2 - \varepsilon_m - \delta_n$

The highest weight vectors constituting $\mathfrak{g}_0 \otimes \mathfrak{g}^*_{-1} = (\mathfrak{c}(\mathfrak{sl}(m) \oplus \mathfrak{sl}(n))) \otimes (V \otimes W^*) = (((V \otimes V^*)/\langle 1_m \rangle) \oplus ((W \otimes W^*)/\langle 1_n \rangle) \oplus \mathbb{C}z) \otimes (V \otimes W^*)$, where the center z is $z = n\sum_{i=1,m}(e_i \otimes \tilde{e}_i) + m\sum_{i=1,n}(f_i \otimes \tilde{f}_i)$, and their weights are:

for $\mathfrak{sl}(m) \otimes (V \otimes W^*)$

parameters	highest weight vectors	weights
$m \geqq 2$	$(e_1 \otimes \tilde{e}_m) \otimes (e_1 \otimes \tilde{f}_n)$	$2\varepsilon_1 - \varepsilon_m - \delta_n$
$n \geqq 2$	$((m-1)e_1 \otimes \tilde{e}_1 - \sum_{i=2,m} e_i \otimes \tilde{e}_i) \otimes (e_1 \otimes \tilde{f}_n) +$ $+ m\sum_{i=2,m}(e_1 \otimes \tilde{e}_i) \otimes (e_i \otimes \tilde{f}_n)$	$\varepsilon_1 - \delta_n$
$m \geqq 3$	$(e_1 \otimes \tilde{e}_m) \otimes (e_2 \otimes \tilde{f}_n) -$ $- (e_2 \otimes \tilde{e}_m) \otimes (e_1 \otimes \tilde{f}_n)$	$\varepsilon_1 + \varepsilon_2 - \varepsilon_m - \delta_n$

for $\mathfrak{sl}(n) \otimes (V \otimes W^*)$

parameters	highest weight vectors	weights
$m \geqq 2$	$(f_1 \otimes \tilde{f}_n) \otimes (e_1 \otimes \tilde{f}_n)$	$\varepsilon_1 + \delta_1 - 2\delta_n$
$n \geqq 2$	$((n-1)f_n \otimes \tilde{f}_n - \sum_{i=1,n-1} f_i \otimes \tilde{f}_i) \otimes (e_1 \otimes \tilde{f}_n) +$ $+ n\sum_{i=1,n-1}(f_i \otimes \tilde{f}_n) \otimes (e_1 \otimes \tilde{f}_i)$	$\varepsilon_1 - \delta_n$

parameters	highest weight vectors	weights
$n \geqq 3$	$(f_1 \otimes \tilde{f}_n) \otimes (e_1 \otimes \tilde{f}_{n-1}) -$ $- (f_1 \otimes \tilde{f}_{n-1}) \otimes (e_1 \otimes \tilde{f}_n)$	$\varepsilon_1 + \delta_1 - \delta_{n-1} - \delta_n$

for $\mathbb{C}z \otimes (V \otimes W^*)$

parameters	highest weight vectors	weights
$m \geqq 2$ $n \geqq 2$	$(\sum_{i=1,m}(e_i \otimes \tilde{e}_i) + (m/n)\sum_{i=1,n}(f_i \otimes \tilde{f}_i)) \otimes (e_1 \otimes \tilde{f}_n)$	$\varepsilon_1 - \delta_n$

Since for $m \neq n$ $\mathfrak{g}_1$ is an irreducible $\mathfrak{g}_0$-module with highest weight $\varepsilon_1 - \delta_n$ and for $m = n$ $\mathfrak{g}_1$ consists of two irreducible components whose highest weights are both equal to $\varepsilon_1 - \delta_n$, we are done.

Order 2. We have to find the second cohomology of the complex

$$\mathfrak{g}_1 \otimes \mathfrak{g}^*_{-1} \xrightarrow{\partial^{3,1}_{\mathfrak{g}_0}} \mathfrak{g}_0 \otimes S^2(\mathfrak{g}^*_{-1}) \xrightarrow{\partial^{2,2}_{\mathfrak{g}_0}} \mathfrak{g}_{-1} \otimes S^3(\mathfrak{g}^*_{-1}). \tag{2}$$

In what follows we will need the vectors v_i and u_i, $i = 1,\ldots,8$, and c_1, c_2 introduced in the following three tables.

The highest weight vectors constituting $\mathfrak{g}_0 \otimes S^2(\mathfrak{g}^*_{-1})$ and their weights are:

for $\mathfrak{sl}(m) \otimes S^2(V \otimes W^*)$

parameters	highest weight vectors	weights
$m \geqq 2$	$v_1 = (e_1 \otimes \tilde{e}_m) \otimes (e_1 \otimes \tilde{f}_n)(e_1 \otimes \tilde{f}_n)$	$3\varepsilon_1 - \varepsilon_m - 2\delta_n$
	$v_2 = m\sum_{i=1,m}(e_1 \otimes \tilde{e}_i) \otimes (e_i \otimes \tilde{f}_n)(e_1 \otimes \tilde{f}_n) -$ $- \sum_{i=1,m}(e_i \otimes \tilde{e}_i) \otimes (e_1 \otimes \tilde{f}_n)(e_1 \otimes \tilde{f}_n)$	$2\varepsilon_1 - 2\delta_n$
	$v_3 = \sum_{i=1,m}((e_1 \otimes \tilde{e}_i) \otimes (e_i \otimes \tilde{f}_n)(e_2 \otimes \tilde{f}_n) -$ $- (e_2 \otimes \tilde{e}_i) \otimes (e_i \otimes \tilde{f}_n)(e_1 \otimes \tilde{f}_n))$	$\varepsilon_1 + \varepsilon_2 - 2\delta_n$
	$v_4 = (e_1 \otimes \tilde{e}_m) \otimes (e_1 \otimes \tilde{f}_{n-1})(e_2 \otimes \tilde{f}_n) -$ $- (e_1 \otimes \tilde{e}_m) \otimes (e_2 \otimes \tilde{f}_{n-1})(e_1 \otimes \tilde{f}_n)$	$2\varepsilon_1 + \varepsilon_2 - \varepsilon_m -$ $-\delta_{n-1} - \delta_n$

parameters	highest weight vectors	weights
$m \geqq 3$	$v_5 = (e_1 \otimes \tilde{e}_m) \otimes (e_2 \otimes \tilde{f}_n)(e_1 \otimes \tilde{f}_n) - (e_2 \otimes \tilde{e}_m) \otimes (e_1 \otimes \tilde{f}_n)(e_1 \otimes \tilde{f}_n)$	$2\varepsilon_1 + \varepsilon_2 - \varepsilon_m - 2\delta_n$
	$v_6 = \sum_{i=1,m}(e_1 \otimes \tilde{e}_i) \otimes ((e_i \otimes \tilde{f}_{n-1})(e_1 \otimes \tilde{f}_n) - (e_i \otimes \tilde{f}_n)(e_1 \otimes \tilde{f}_{n-1}))$	$2\varepsilon_1 - \delta_{n-1} - \delta_n$
	$v_7 = 2\sum_{i=1,m}(e_i \otimes \tilde{e}_i) \otimes ((e_1 \otimes \tilde{f}_{n-1})(e_2 \otimes \tilde{f}_n) - (e_1 \otimes \tilde{f}_n)(e_2 \otimes \tilde{f}_{n-1})) + m\sum_{i=1,m}(e_1 \otimes \tilde{e}_i) \otimes ((e_2 \otimes \tilde{f}_{n-1})(e_i \otimes \tilde{f}_n) - (e_2 \otimes \tilde{f}_n)(e_i \otimes \tilde{f}_{n-1})) + m\sum_{i=1,m}(e_2 \otimes \tilde{e}_i) \otimes ((e_1 \otimes \tilde{f}_n)(e_i \otimes \tilde{f}_{n-1}) - (e_1 \otimes \tilde{f}_{n-1})(e_i \otimes \tilde{f}_n))$	$\varepsilon_1 + \varepsilon_2 - \delta_{n-1} - \delta_n$
$m \geqq 4$	$v_8 = (e_1 \otimes \tilde{e}_m) \otimes (e_2 \otimes \tilde{f}_{n-1})(e_3 \otimes \tilde{f}_n) - (e_1 \otimes \tilde{e}_m) \otimes (e_2 \otimes \tilde{f}_n)(e_3 \otimes \tilde{f}_{n-1}) + (e_2 \otimes \tilde{e}_m) \otimes (e_3 \otimes \tilde{f}_{n-1})(e_1 \otimes \tilde{f}_n) - (e_2 \otimes \tilde{e}_m) \otimes (e_3 \otimes \tilde{f}_n)(e_1 \otimes \tilde{f}_{n-1}) + (e_3 \otimes \tilde{e}_m) \otimes (e_1 \otimes \tilde{f}_{n-1})(e_2 \otimes \tilde{f}_n) - (e_3 \otimes \tilde{e}_m) \otimes (e_1 \otimes \tilde{f}_n)(e_2 \otimes \tilde{f}_{n-1})$	$\varepsilon_1 + \varepsilon_2 + \varepsilon_3 - \varepsilon_m - \delta_{n-1} - \delta_n$

for $\mathfrak{sl}(n) \otimes S^2(V \otimes W^*)$

parameters	highest weight vectors	weights
$n \geqq 2$	$u_1 = (f_1 \otimes \tilde{f}_n) \otimes (e_1 \otimes \tilde{f}_n)(e_1 \otimes \tilde{f}_n)$	$2\varepsilon_1 + \delta_1 - 3\delta_n$
	$u_2 = n\sum_{i=1,n}(f_i \otimes \tilde{f}_n) \otimes (e_1 \otimes \tilde{f}_i)(e_1 \otimes \tilde{f}_n) - \sum_{i=1,n}(f_i \otimes \tilde{f}_i)(e_1 \otimes \tilde{f}_n)(e_1 \otimes \tilde{f}_n)$	$2\varepsilon_1 - 2\delta_n$
	$u_3 = \sum_{i=1,n}((f_i \otimes \tilde{f}_n) \otimes (e_1 \otimes \tilde{f}_{n-1})(e_1 \otimes \tilde{f}_i) - (f_i \otimes \tilde{f}_{n-1}) \otimes (e_1 \otimes \tilde{f}_n)(e_1 \otimes \tilde{f}_i))$	$2\varepsilon_1 - \delta_{n-1} - \delta_n$
	$u_4 = (f_1 \otimes \tilde{f}_n) \otimes (e_1 \otimes \tilde{f}_{n-1})(e_2 \otimes \tilde{f}_n) - (f_1 \otimes \tilde{f}_n) \otimes (e_1 \otimes \tilde{f}_n)(e_2 \otimes \tilde{f}_{n-1})$	$\varepsilon_1 + \varepsilon_2 + \delta_1 - \delta_{n-1} - 2\delta_n$

parameters	highest weight vectors	weights
$n \geqq 3$	$u_5 = (f_1 \otimes \tilde{f}_n) \otimes (e_1 \otimes \tilde{f}_{n-1})(e_1 \otimes \tilde{f}_n) -$ $- (f_1 \otimes \tilde{f}_{n-1}) \otimes (e_1 \otimes \tilde{f}_n)(e_1 \otimes \tilde{f}_n)$ $u_6 = \sum_{i=1,n}(f_i \otimes \tilde{f}_n) \otimes ((e_1 \otimes \tilde{f}_i)(e_2 \otimes \tilde{f}_n) -$ $- (e_2 \otimes \tilde{f}_i)(e_1 \otimes \tilde{f}_n))$ $u_7 = 2\sum_{i=1,n}(f_i \otimes \tilde{f}_i) \otimes ((e_2 \otimes \tilde{f}_{n-1})(e_1 \otimes \tilde{f}_n) -$ $- (e_1 \otimes \tilde{f}_{n-1})(e_2 \otimes \tilde{f}_n)) + n\sum_{i=1,n}(f_i \otimes \tilde{f}_n) \otimes$ $\otimes ((e_1 \otimes \tilde{f}_{n-1})(e_2 \otimes \tilde{f}_i) - (e_2 \otimes \tilde{f}_{n-1})(e_1 \otimes \tilde{f}_i)) +$ $+ n\sum_{i=1,n}(f_i \otimes \tilde{f}_{n-1}) \otimes ((e_1 \otimes \tilde{f}_i)(e_2 \otimes \tilde{f}_n) -$ $- (e_2 \otimes \tilde{f}_i)(e_1 \otimes \tilde{f}_n))$	$2\varepsilon_1 + \delta_1 -$ $- \delta_{n-1} - 2\delta_n$ $\varepsilon_1 + \varepsilon_2 - 2\delta_n$ $\varepsilon_1 + \varepsilon_2 -$ $- \delta_{n-1} - \delta_n$
$n \geqq 4$	$u_8 = (f_1 \otimes \tilde{f}_{n-2}) \otimes (e_1 \otimes \tilde{f}_{n-1})(e_2 \otimes \tilde{f}_n) -$ $- (f_1 \otimes \tilde{f}_{n-2}) \otimes (e_2 \otimes \tilde{f}_{n-1})(e_1 \otimes \tilde{f}_n) +$ $+ (f_1 \otimes \tilde{f}_{n-1}) \otimes (e_1 \otimes \tilde{f}_n)(e_2 \otimes \tilde{f}_{n-2}) -$ $- (f_1 \otimes \tilde{f}_{n-1}) \otimes (e_2 \otimes \tilde{f}_n)(e_1 \otimes \tilde{f}_{n-2}) +$ $+ (f_1 \otimes \tilde{f}_n) \otimes (e_1 \otimes \tilde{f}_{n-2})(e_2 \otimes \tilde{f}_{n-1}) -$ $- (f_1 \otimes \tilde{f}_n) \otimes (e_2 \otimes \tilde{f}_{n-2})(e_1 \otimes \tilde{f}_{n-1})$	$\varepsilon_1 + \varepsilon_2 + \delta_1 -$ $- \delta_{n-2} - \delta_{n-1} - \delta_n$

for $\mathbf{C}z \otimes S^2(V \otimes W^*)$

parameters	highest weight vectors	weights
$m \geqq 2$ $n \geqq 2$	$c_1 = (\sum_{i=1,m}(e_i \otimes \tilde{e}_i) + (m/n)\sum_{i=1,n}(f_i \otimes \tilde{f}_i)) \otimes$ $\otimes (e_1 \otimes \tilde{f}_n)(e_1 \otimes \tilde{f}_n)$ $c_2 = (\sum_{i=1,m}(e_i \otimes \tilde{e}_i) + (m/n)\sum_{i=1,n}(f_i \otimes \tilde{f}_i)) \otimes$ $\otimes ((e_1 \otimes \tilde{f}_n)(e_2 \otimes \tilde{f}_{n-1}) -$ $- (e_1 \otimes \tilde{f}_{n-1})(e_2 \otimes \tilde{f}_n))$	$2\varepsilon_1 - 2\delta_n$ $\varepsilon_1 + \varepsilon_2 -$ $-\delta_{n-1} - \delta_n$

We leave it to the reader to verify that $\ker \partial^{2,2}_{\mathfrak{g}_0}$ consists of the following $\mathfrak{g}_0$-modules:

parameters	highest weight vectors	weights
$m = 2, n \geqq 3$	$v_2 + (2/n)u_2 + c_1$	$2\varepsilon_1 - 2\delta_n$
	$v_4 + u_3$	$2\varepsilon_1 - \delta_{n-1} - \delta_n$
	$v_3 + u_6$	$\varepsilon_1 + \varepsilon_2 - 2\delta_n$
	$u_7 - nc_2$	$\varepsilon_1 + \varepsilon_2 - \delta_{n-1} - \delta_n$

parameters	highest weight vectors	weights
$m = 2, n \geqq 4$	u_8	$\varepsilon_1 + \varepsilon_2 + \delta_1 - \delta_{n-2} - \delta_{n-1} - \delta_n$
$m = 3, n = 2$	$v_2 + (m/2)u_2 + c_1$ $v_6 - u_3$ $v_3 + u_4$ $v_7 + 2c_2$	$2\varepsilon_1 - 2\delta_n$ $2\varepsilon_1 - \delta_{n-1} - \delta_n$ $\varepsilon_1 + \varepsilon_2 - 2\delta_n$ $\varepsilon_1 + \varepsilon_2 - \delta_{n-1} - \delta_n$
$m \geqq 4, n = 2$	v_8	$\varepsilon_1 + \varepsilon_2 + \varepsilon_3 - \varepsilon_m - \delta_{n-1} - \delta_n$
$m \geqq 3, n \geqq 3$	$v_2 + (m/n)u_2 + c_1$ if $m \neq n$ $v_2 + u_2, c_1$ if $m = n$ $v_6 - u_3$ $v_3 + u_6$ $v_7 - (m/n)u_7 + 2c_2$ if $m \neq n$ $v_7 - u_7, c_2$ if $m = n$	$2\varepsilon_1 - 2\delta_n$ $2\varepsilon_1 - 2\delta_n$ $2\varepsilon_1 - \delta_{n-1} - \delta_n$ $\varepsilon_1 + \varepsilon_2 - 2\delta_n$ $\varepsilon_1 + \varepsilon_2 - \delta_{n-1} - \delta_n$ $\varepsilon_1 + \varepsilon_2 - \delta_{n-1} - \delta_n$
$m = n = 2$	$v_2 + u_2, c_1$ $v_4 + u_3$ $v_3 + u_4$ c_2	$2\varepsilon_1 - 2\delta_2$ $2\varepsilon_1 - \delta_1 - \delta_2$ $\varepsilon_1 + \varepsilon_2 - 2\delta_2$ $\varepsilon_1 + \varepsilon_2 - \delta_1 - \delta_2$

Since the $\mathfrak{g}_0$-module $\mathfrak{g}_1$ is irreducible for $m \neq n$ and consists of two irreducible components for $m = n$ whose highest weights are both equal to $\varepsilon_1 - \delta_n$, then the $\mathfrak{g}_0$-module $\mathfrak{g}_1 \otimes \mathfrak{g}^*_{-1}$ consists of the irreducible components with the following highest weights: $2\varepsilon_1 - 2\delta_n, 2\varepsilon_1 - \delta_{n-1} - \delta_n, \varepsilon_1 + \varepsilon_2 - 2\delta_n, \varepsilon_1 + \varepsilon_2 - \delta_{n-1} - \delta_n$, each with multiplicity 1 if $m \neq n$ and each with multiplicity 2 if $m = n$.

For $m \neq n$ we have $\mathfrak{g}_2 = 0$. For $m = n = 2$ the highest weights of the irreducible $\mathfrak{g}_0$-modules that constitute $\mathfrak{g}_2$ are $\varepsilon_1 + \varepsilon_2 - \delta_1 - \delta_2$, $2\varepsilon_1 - \delta_1 - \delta_2$, and $\varepsilon_1 + \varepsilon_2 - 2\delta_2$; Now apply the Lemma and this case of the Theorem is also done.

Order 3. We have to find the second cohomology of the complex

$$\mathfrak{g}_2 \otimes \mathfrak{g}^*_{-1} \xrightarrow{\partial^{4,1}_{\mathfrak{g}_0}} \mathfrak{g}_1 \otimes S^2(\mathfrak{g}^*_{-1}) \xrightarrow{\partial^{3,2}_{\mathfrak{g}_0}} \mathfrak{g}_0 \otimes S^3(\mathfrak{g}^*_{-1}). \tag{3}$$

Notice that, as $\mathfrak{g}_0$-module, $\mathfrak{g}_1$ is isomorphic to $\mathfrak{g}^*_{-1}$ for $m \neq n$ and to $\mathfrak{g}^*_{-1} \oplus \mathfrak{g}^*_{-1}$ otherwise. The highest weight vectors constituting $\mathfrak{g}^*_{-1} \otimes S^2(\mathfrak{g}^*_{-1})$ and their weights are given in the following table (s and t denote the cyclic permutations of $(1, 2, 3)$ and of $(n-2, n-1, n)$, respectively):

parameters	highest weight vectors	weights
$m \geqq 2$ $n \geqq 2$	$v'_1 = (e_1 \otimes \tilde f_n) \otimes (e_1 \otimes \tilde f_n)(e_1 \otimes \tilde f_n)$ $v'_2 = (e_1 \otimes \tilde f_n) \otimes (e_2 \otimes \tilde f_n)(e_1 \otimes \tilde f_n) -$ $- (e_2 \otimes \tilde f_n) \otimes (e_1 \otimes \tilde f_n)(e_1 \otimes \tilde f_n)$ $v'_3 = (e_1 \otimes \tilde f_{n-1}) \otimes (e_1 \otimes \tilde f_n)(e_1 \otimes \tilde f_n)$ $- (e_1 \otimes \tilde f_n) \otimes (e_1 \otimes \tilde f_{n-1})(e_1 \otimes \tilde f_n)$ $v'_4 = (e_1 \otimes \tilde f_n) \otimes ((e_2 \otimes \tilde f_{n-1})(e_1 \otimes \tilde f_n) -$ $- (e_1 \otimes \tilde f_{n-1})(e_2 \otimes \tilde f_n))$ $v'_5 = (e_1 \otimes \tilde f_n) \otimes (e_2 \otimes \tilde f_{n-1})(e_1 \otimes \tilde f_n) -$ $- (e_2 \otimes \tilde f_n) \otimes (e_1 \otimes \tilde f_{n-1})(e_1 \otimes \tilde f_n) +$ $+ (e_2 \otimes \tilde f_{n-1}) \otimes (e_1 \otimes \tilde f_n)(e_1 \otimes \tilde f_n) -$ $- (e_1 \otimes \tilde f_{n-1}) \otimes (e_1 \otimes \tilde f_n)(e_2 \otimes \tilde f_n)$	$3\varepsilon_1 - 3\delta_n$ $2\varepsilon_1 + \varepsilon_2 - 3\delta_n$ $3\varepsilon_1 - \delta_{n-1} - 2\delta_n$ $2\varepsilon_1 + \varepsilon_2 -$ $-\delta_{n-1} - 2\delta_n$ $2\varepsilon_1 + \varepsilon_2 -$ $-\delta_{n-1} - 2\delta_n$
$m \geqq 3$	$v'_6 = \sum_{i=0,2} (e_{s^i(1)} \otimes \tilde f_n) \otimes ((e_{s^i(2)} \otimes \tilde f_{n-1})(e_{s^i(3)} \otimes \tilde f_n) -$ $- (e_{s^i(2)} \otimes \tilde f_n)(e_{s^i(3)} \otimes \tilde f_{n-1}))$	$\varepsilon_1 + \varepsilon_2 + \varepsilon_3 -$ $-\delta_{n-1} - 2\delta_n$
$n \geqq 3$	$v'_7 = \sum_{i=0,2} (e_1 \otimes \tilde f_{t^i(n-2)}) \otimes ((e_2 \otimes \tilde f_{t^i(n-1)})(e_1 \otimes \tilde f_{t^i(n)}) -$ $- (e_1 \otimes \tilde f_{t^i(n-1)})(e_2 \otimes \tilde f_{t^i(n)}))$	$2\varepsilon_1 + \varepsilon_2 - \delta_{n-2}$ $-\delta_{n-1} - \delta_n$
$m \geqq 3$ $n \geqq 3$	$v'_8 = \sum_{i=0,2} (e_{s^i(1)} \otimes \tilde f_{n-2}) \otimes ((e_{s^i(2)} \otimes \tilde f_{n-1})(e_{s^i(3)} \otimes \tilde f_n) -$ $- (e_{s^i(2)} \otimes \tilde f_n)(e_{s^i(3)} \otimes \tilde f_{n-1})) + (e_{s^i(1)} \otimes \tilde f_{n-1}) \otimes$ $\otimes ((e_{s^i(2)} \otimes \tilde f_{n-2})(e_{s^i(3)} \otimes \tilde f_n) - (e_{s^i(2)} \otimes \tilde f_n)(e_{s^i(3)} \otimes \tilde f_{n-2})) +$ $+ (e_{s^i(1)} \otimes \tilde f_n) \otimes ((e_{s^i(2)} \otimes \tilde f_{n-2})(e_{s^i(3)} \otimes \tilde f_{n-1}) -$ $- (e_{s^i(2)} \otimes \tilde f_{n-1})(e_{s^i(3)} \otimes \tilde f_{n-2}))$	$\varepsilon_1 + \varepsilon_2 + \varepsilon_3 -$ $- \delta_{n-2} - \delta_{n-1} - \delta_n$

If $m \neq n$, then $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_* = \mathfrak{sl}(m|n) = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, where the highest weight of the irreducible $\mathfrak{g}_0$-module $\mathfrak{g}_1$ is $\varepsilon_1 - \delta_n$ and the highest vector is

$$\sum_{i=1,m} (e_1 \otimes \tilde e_i) \otimes (e_i \otimes \tilde f_n) + \sum_{i=1,n} (f_i \otimes \tilde f_n) \otimes (e_1 \otimes \tilde f_i).$$

If $m = n$, then $(\mathfrak{g}_{-1}, \hat{\mathfrak{g}}_0)_* = \mathfrak{psl}(n|n) = \mathfrak{g}_{-1} \oplus \hat{\mathfrak{g}}_0 \oplus \hat{\mathfrak{g}}_1$ (if $n > 2$) and $(\mathfrak{g}_{-1}, \hat{\mathfrak{g}}_0)_* = \mathfrak{h}(0|4) = \mathfrak{g}_{-1} \oplus \hat{\mathfrak{g}}_0 \oplus \hat{\mathfrak{g}}_1 \oplus \mathbf{C}$ (if $n = 2$), where the highest weight of the irreducible $\mathfrak{g}_0$-module $\hat{\mathfrak{g}}_1$ is $\varepsilon_1 - \delta_n$ and the highest vector is

$$n\left(\sum_{i=1,n} (e_1 \otimes \tilde e_i) \otimes (e_i \otimes \tilde f_n) + \sum_{i=1,n} (f_i \otimes \tilde f_n) \otimes (e_1 \otimes \tilde f_i) \right) -$$

$$\left(\sum_{i=1,n} e_i \otimes \tilde e_i + \sum_{i=1,n} f_i \otimes \tilde f_i \right) \otimes (e_1 \otimes \tilde f_n),$$

whereas $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_* = (\mathfrak{g}_{-1}, \hat{\mathfrak{g}}_0)_* \ltimes S^*(\mathfrak{g}_{-1}^*) = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \dots$, where $\mathfrak{g}_1 =$

$\hat{\mathfrak{g}}_1 \oplus \tilde{\mathfrak{g}}_1$ and $\tilde{\mathfrak{g}}_1$ is an irreducible $\mathfrak{g}_0$-module with the same highest weight $\varepsilon_1 - \delta_n$ and the highest weight vector is

$$\Big(\sum_{i=1,n} e_i \otimes \tilde{e}_i + \sum_{i=1,n} f_i \otimes \tilde{f}_i\Big) \otimes (e_1 \otimes \tilde{f}_n).$$

Now, let v'_1 and u'_i, $i = 1, \ldots, 8$, introduced above, be the highest weight vectors of the modules $\hat{\mathfrak{g}}_1 \otimes S^2(\mathfrak{g}^*_{-1})$ and $\tilde{\mathfrak{g}}_1 \otimes S^2(\mathfrak{g}^*_{-1})$, respectively. We leave it as an exercise to the reader to verify that $\ker \partial^{3,2}_{\mathfrak{g}_0}$ consists of the following $\mathfrak{g}_0$-modules:

parameters	highest weight vectors	weights
$m \geqq 2, n \geqq 2, m \neq n$	–	–
$m = n = 2$	u'_2 u'_3 $v'_5, u'_5 - 2u'_4$	$2\varepsilon_1 + \varepsilon_2 - 3\delta_2$ $3\varepsilon_1 - \delta_1 - 2\delta_2$ $2\varepsilon_1 + \varepsilon_2 - \delta_1 - 2\delta_2$
$m = n \geqq 3$	u'_2 u'_3 $u'_5 - 2u'_4$ u'_6 u'_7	$2\varepsilon_1 + \varepsilon_2 - 3\delta_n$ $3\varepsilon_1 - \delta_{n-1} - 2\delta_n$ $2\varepsilon_1 + \varepsilon_2 - \delta_{n-1} - 2\delta_n$ $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 - \delta_{n-1} - 2\delta_n$ $2\varepsilon_1 + \varepsilon_2 - \delta_{n-2} - \delta_{n-1} - \delta_n$

Therefore, it is clear that SFs of order 3 vanish if $m \neq n$.

If $m = n$, then the highest weights of the irreducible components of the $\mathfrak{g}_0$-module $\mathfrak{g}_2$ are as follows:

$$2\varepsilon_1 - \delta_1 - \delta_2, \varepsilon_1 + \varepsilon_2 - 2\delta_2, \varepsilon_1 + \varepsilon_2 - \delta_1 - \delta_2 \text{ for } n = 2;$$

$$\varepsilon_1 + \varepsilon_2 - 2\delta_n, 2\varepsilon_1 - \delta_{n-1} - \delta_n \text{ for } n \geqq 3.$$

The highest weights of the irreducible components of the $\mathfrak{g}_0$-module $\mathfrak{g}_2 \otimes \mathfrak{g}^*_{-1}$ are as follows:

$3\varepsilon_1 - \delta_1 - 2\delta_2, 2\varepsilon_1 + \varepsilon_2 - 3\delta_2, 2\varepsilon_1 + \varepsilon_2 - \delta_1 - 2\delta_2$ (of multiplicity 3) for $n = 2$;

$2\varepsilon_1 + \varepsilon_2 - 3\delta_n, 3\varepsilon_1 - \delta_{n-1} - 2\delta_n, 2\varepsilon_1 + \varepsilon_2 - \delta_{n-1} - 2\delta_n$ (of multiplicity 2),

$\varepsilon_1 + \varepsilon_2 + \varepsilon_3 - 3\delta_n, 3\varepsilon_1 - \delta_{n-2} - \delta_{n-1} - \delta_n, \varepsilon_1 + \varepsilon_2 + \varepsilon_3 - \delta_{n-1} - 2\delta_n,$

$2\varepsilon_1 + \varepsilon_2 - \delta_{n-2} - \delta_{n-1} - \delta_n$ for $n > 2$.

The highest weights of the irreducible components of the $\mathfrak{g}_0$-module $\mathfrak{g}_3$ are as follows:

$2\varepsilon_1 + \varepsilon_2 - \delta_1 - 2\delta_2$ for $n = 2$;

$\varepsilon_1 + \varepsilon_2 + \varepsilon_3 - 3\delta_n, 3\varepsilon_1 - \delta_{n-2} - \delta_{n-1} - \delta_n, 2\varepsilon_1 + \varepsilon_2 + \delta_{n-1} - 2\delta_n$ for $n > 2$.

By the Lemma, $\operatorname{im} \partial^{4,1}_{\mathfrak{g}_0}$ and $\mathfrak{g}_2 \otimes \mathfrak{g}^*_{-1}/\mathfrak{g}_3$ are isomorphic as $\mathfrak{g}_0$-modules and, therefore, SFs of order 3 vanish for $m = n > 1$ as well.

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