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COMPLETELY POSITIVE MAPS ON AMALGAMATED PRODUCT C^*-ALGEBRAS

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Let F_N be the free group on N generators $a_1,\ldots,a_N (N \in \mathsf{N} \cup \{\infty\}, N \geqq 2)$; for $g = a_{i_1}^{k_1}\ldots a_{i_n}^{k_n}$, $i_1 \neq \ldots \neq i_n$, $k_i \in \mathsf{Z}$, $k_i \neq 0$, denote its length by $|g| = |k_1| + \ldots + |k_n|$. Haagerup [10] proved that for any $r \in [0,1]$, the function $H_r(g) = r^{|g|}$ is positive definite on F_N.

In fact, the functions $\phi_z : \mathsf{Z} \to \mathsf{C}$, $\phi_z(k) = z^{[k]}$, $k \in \mathsf{Z}$, $z \in \mathsf{C}$, $|z| \leqq 1$, where $z^{[k]} = \begin{cases} z^k & \text{, for } k \in \mathsf{Z}_+; \\ \bar{z}^{-k} & \text{, for } k \in \mathsf{Z}_- \end{cases}$ are known to be positive definite [13] and $H_r = \phi_r * \phi_r$ on $\mathsf{F}_2 = \mathsf{Z} * \mathsf{Z}$, in the sense that $H_r(a_{i_1}^{k_1}\ldots a_{i_n}^{k_n}) = \phi_r(a_{i_1}^{k_1})\ldots\phi_r(a_{i_n}^{k_n})$ for any reduced word $a_{i_1}^{k_1}\ldots a_{i_n}^{k_n}$.

In this way, de Michele and Figà-Talamanca [9], Bożejko [5, 6] and Picardello [12] extended Haagerup's result. In [6] Bożejko proved that the free product of the unital positive definite functions $u_i : G_i \to \mathscr{L}(\mathscr{H})$ is still positive definite on the free product group $\underset{i}{*} G_i = G$ and a similar result for the free product of H-bivariant functions on the amalgamated product $\underset{H}{*} G_i$. In [4] we defined an analogue of this construction for amalgamated free product C^*-algebras, showing a class of completely positive maps on these C^*-algebras.

Whenever G_i are discrete groups, the positive definite functions $u_i : G_i \to \mathsf{C}$ yield states ϕ_i on $C^*(G_i)$. In [2] and [17] the state which corresponds to $u = \underset{i}{*} u_i$ is constructed, the free product of GNS representations π_{ϕ_i} is defined and one gets $\underset{i}{*}\pi_{\phi_i} = \pi_\phi$. Consequently, there is a canonical way for constructing the Naimark dilation of the positive definite function $u : \underset{i}{*} G_i \to \mathsf{C}$.

The aim of this note is to construct the Stinespring dilatation for the completely positive maps $\underset{i}{*}\Phi_i : \underset{B}{*} A_i \to \mathscr{L}(\mathscr{H})$ considered in [4] (here $\underset{B}{*} A_i$ denotes the full amalgamated product of the unital C^*-algebras A_i over a common C^*-subalgebra B with respect to a family of projections of norm one $E_i : A_i \to B$). On this way one can easily write the Naimark dilation for the operator valued map $\underset{H}{*} u_i : \underset{H}{*} G_i \to \mathscr{L}(\mathscr{H})$ from [6] and [12]. This is explicitly done for $G_i = \mathsf{Z}$, $H = \{0\}$, $T_i \in \mathscr{L}(\mathscr{H})$ contractions and $u_i(k) = T_i^{[k]}$, where

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$$T^{[k]} = \begin{cases} T^k, & \text{for } k \in \mathbf{Z}_+; \\ T^{*-k}, & \text{for } k \in \mathbf{Z}_- \end{cases}.$$

1.1 Let A be a unital C^*-algebra, $\mathscr{H}$ be a Hilbert space and $\Phi: A \to \mathscr{L}(\mathscr{H})$ a unital completely positive map. The Stinespring dilation $(\pi, \mathscr{K})$ of Φ, consisting of a Hilbert space $\mathscr{K}$ which includes $\mathscr{H}$ and of a unital $*$-representation $\rho: A \to \mathscr{L}(\mathscr{K})$ such that

(i) $\Phi(a) = P^{\mathscr{K}}_{\mathscr{H}} \rho(a)|_{\mathscr{H}}$ for $a \in A$

(ii) $\mathscr{K} = \overline{\text{span}}\, \rho(A)\mathscr{H}$

is unique up to unitary equivalence.

Denote by $\mathscr{K}^0 = \mathscr{K} \ominus \mathscr{H}$ the orthogonal complement of $\mathscr{H}$ into $\mathscr{K}$ and remark that for $a \in A$, $h, h' \in \mathscr{H}$ one has

$$\langle \rho(a)h - \Phi(a)h, h' \rangle = \langle P^{\mathscr{K}}_{\mathscr{H}} \rho(a)h, h' \rangle - \langle \Phi(a)h, h' \rangle = 0.$$

On the other hand, let $k \in \mathscr{K}^0 \ominus \text{span}\{\rho(a)h - \Phi(a)h;\ a \in A, h \in \mathscr{H}\}$. Then $\langle k, \Phi(a)h \rangle = 0$, hence $\langle k, \rho(a)h \rangle = 0$ for $a \in A$, $h \in \mathscr{H}$. By (ii) it follows that $k = 0$ and consequently

$$\mathscr{K}^0 = \overline{\text{span}}\, \{\rho(a)h - \Phi(a)h;\ a \in A, h \in \mathscr{H}\}.$$

1.2 LEMMA *Let A be a unital C^*-algebra, B be a unital C^*-subalgebra of A with a projection of norm one $E: A \to B$, $\chi: B \to \mathscr{L}(\mathscr{H})$ be a unital $*$-representation on the Hilbert space $\mathscr{H}$ and $\Phi: A \to \mathscr{L}(\mathscr{H})$ be a B-linear* (i.e. $\Phi(ab) = \Phi(a)\chi(b)$ *for $a \in A$, $b \in B$) completely positive map. Let $(\rho, \mathscr{K})$ be the Stinespring representation associated with Φ. Then $\mathscr{H}$ (and consequently $\mathscr{K}^0$) is $\rho(B)$-invariant and $\rho(b)h = \chi(b)h$ for $b \in B$, $h \in \mathscr{H}$.*

PROOF. Since span $\{\rho(a)h - \Phi(a)h; a \in A, h \in \mathscr{H}\}$ is dense in $\mathscr{K}^0$, it is enough to remark that

$$\begin{aligned} \langle \rho(b)h, \rho(a)h' - \Phi(a)h' \rangle &= \langle \rho(a^*b)h, h' \rangle - \langle \rho(b)h, \Phi(a)h' \rangle \\ &= \langle \Phi(a^*b)h, h' \rangle - \langle \Phi(b)h, \Phi(a)h' \rangle = 0, \end{aligned}$$

for all $a \in A$, $b \in B$, $h, h' \in \mathscr{H}$.

1.3 Let B be a unital C^*-algebra, $\mathscr{H}$ be a right Hilbert B-module, $\mathscr{K}$ be a Hilbert space and $\chi: B \to \mathscr{L}(\mathscr{K})$ a $*$-representation. Denote by $\mathscr{H} \otimes_\chi \mathscr{K}$ the completion of the vector space $\mathscr{H} \odot \mathscr{K}$ (the algebraic tensor product as vector spaces) with respect to the scalar product

$$\langle h_1 \otimes k_1, h_2 \otimes k_2 \rangle = \langle \chi(\langle h_2, h_1 \rangle_B) k_1, k_2 \rangle_{\mathscr{K}}, \quad h_1, k_2 \in \mathscr{H}, k_1, k_2 \in \mathscr{K}.$$

In this way $\mathscr{H} \otimes_\chi \mathscr{K}$ becomes a Hilbert space and the map $\theta: \mathscr{L}(\mathscr{H}) \to \mathscr{L}(\mathscr{H} \otimes_\chi \mathscr{K})$ given by $\theta(T)(h \otimes k) = Th \otimes k$ for $T \in \mathscr{L}(\mathscr{H})$, $h \in \mathscr{H}, k \in \mathscr{K}$ is a $*$-representation.

Given any *-representation $\sigma: A \to \mathscr{L}(\mathscr{H})$, one considers the representation $\sigma \otimes I: A \to \mathscr{L}(\mathscr{H} \otimes_\chi \mathscr{K})$, $\sigma \otimes I = \theta\sigma$.

Remark that for $\mathscr{H} = B$ and $\chi: B \to \mathscr{L}(\mathscr{K})$ unital *-representation, there is a natural identification between the Hilbert spaces $B \otimes_\chi \mathscr{K}$ and $\mathscr{K}$ given by the unitary $W(\sum_i b_i \otimes k_i) = \sum_i \chi(b_i)k_i$.

Consequently, under the assumptions of 1.2, the Hilbert space $\mathscr{K}^0$ becomes a left B-module. For $b \in B$, $k \in \mathscr{K}^0$ we shall denote simply bk instead of $\chi(b)k$.

1.4 At this moment we are concerned with Voiculescu's construction of C^*-reduced amalgamated free products ([17]), that we recall briefly.

Given $(A_i)_{i \in I}$ unital C^*-algebras with a common unital C^*-subalgebra B and projections of norm one $E_i: A_i \to B$, one denotes by $\mathscr{H}_i$ the separation and completion of A_i with respect to $\|a\|_{E_i} = \|E_i(a^*a)\|^{1/2}$. Denote also $\xi_i = 1_B \in \mathscr{H}_i$ and $A_i^0 = \operatorname{Ker} E_i$.

The B-valued inner product $\langle x, y \rangle_B = E_i(y^*x)$ on A_i yields an inner product on $\mathscr{H}_i$ which becomes a Hilbert B-module. The B-bimodule direct sum $A_i = B \oplus A_i^0$ gives rise to the orthogonal direct sum of Hilbert B-modules $\mathscr{H}_i = B \oplus \mathscr{H}_i^0$.

The left multiplication on A_i yields a unital GNS type *-morphism $\pi_i: A_i \to \mathscr{L}(\mathscr{H}_i)$ with $E_i(a) = \langle \pi_i(a)\xi_i, \xi_i \rangle_B$ for $a \in A_i$. Clearly $\pi_i(A_i^0)\xi_i$ is dense in $\mathscr{H}_i^0$.

One defines the free product of the pointed Hilbert B-modules $(\mathscr{H}_i, \xi_i)_{i \in I}$ by $(\mathscr{H}_0, \xi)$, where

$$\mathscr{H}_0 = B \oplus \oplus_{n \geqq 1, i_1 \neq \ldots \neq i_n} \mathscr{H}_{i_1}^0 \underset{B}{\otimes} \ldots \underset{B}{\otimes} \mathscr{H}_{i_n}^0 = B \oplus \mathscr{H}^0;\ \xi = I_B \oplus 0 \in \mathscr{H}_0.$$

Consider also the Hilbert B-modules

$$\mathscr{H}_l(i) = B \oplus \oplus_{n \geqq 1, i \neq i_1 \neq \ldots \neq i_n} \mathscr{H}_{i_1}^0 \underset{B}{\otimes} \ldots \underset{B}{\otimes} \mathscr{H}_{i_n}^0 = B \oplus \mathscr{H}_l^0(i);$$

$$\mathscr{H}_r(i) = B \oplus \oplus_{n \geqq 1, i_1 \neq \ldots \neq i_n \neq i} \mathscr{H}_{i_1}^0 \underset{B}{\otimes} \ldots \underset{B}{\otimes} \mathscr{H}_{i_n}^0 = B \oplus \mathscr{H}_r^0(i)$$

and the unitaries $V_i: \mathscr{H}_0 \to \mathscr{H}_i \underset{B}{\otimes} \mathscr{H}_l(i)$ defined by

$$V_i(h) = \begin{cases} \xi_i \oplus \xi & \text{, for } h = \xi; \\ h_1 \otimes (h_2 \otimes \ldots \otimes h_n) & \text{, for } h = h_1 \otimes \ldots \otimes h_n, i_1 = i, n \geqq 2; \\ h_1 \otimes \xi & \text{, for } h = h_1, i_1 = i; \\ \xi_i \otimes (h_1 \otimes \ldots \otimes h_n) & \text{, for } h = h_1, i_1 \neq i; \end{cases}$$

where $h_k \in \mathscr{H}_{i_k}^0$, $k = \overline{1,n}$, $i_1 \neq \ldots \neq i_n$.

Define the *-morphisms σ_i: $A_i \to \mathscr{L}(\mathscr{H}_0)$ by $\sigma_i = \lambda_i \pi_i$, where λ_i: $\mathscr{L}(\mathscr{H}_i) \to \mathscr{L}(\mathscr{H}_0)$, $\lambda_i(T) = V_i^{-1}(T \otimes I) V_i$. Then the reduced free product with amalgamation of $(A_i, E_i)_{i \in I}$ is the C^*-algebra A generated by $\cup_{i \in I} \sigma_i(A_i)$ in $\mathscr{L}(\mathscr{H}_0)$, B identifies with a *-subalgebra of A and $E(a) = \langle a\xi, \xi\rangle_B$ is a conditional expectation of A onto B.

LEMMA. i) *The direct B-submodule* $\mathscr{H}_r(j)$ *of* $\mathscr{H}_0$ *is* σ_i*-invariant for* $i \neq j$.

ii) *The direct B-submodule* $\mathscr{H}_r^0(i)$ *of* $\mathscr{H}_0$ *is* σ_i*-invariant.*

PROOF. i) Since $\sigma_i(b)$ acts by left multiplication by b on B and on $\mathscr{H}_{i_1}^0 \underset{B}{\otimes} \dots \underset{B}{\otimes} \mathscr{H}_{i_n}^0$ and $\sigma_i(a)\xi = \pi_i(a)\xi_i$ for $a \in A_i^0$ it is enough to check that $\sigma_i(a)h \in \mathscr{H}_r(j)$ for any $a \in A_i^0, h \in \mathscr{H}_r(j)$, this being obtained by the following relations:

$$\sigma_i(a)h_1 = \langle \pi_i(a)h_1, \xi_i\rangle_B \xi + h_1' \in B \oplus \mathscr{H}_i^0 \text{ for } i_1 = i;$$

$$\sigma_i(a)(h_1 \otimes \dots \otimes h_n) = \langle \pi_i(a)h_1, \xi_i\rangle_B h_2 \otimes \dots \otimes h_n + h_1' \otimes h_2 \otimes \dots \otimes h_n$$
$$\text{for } i_1 = i, n \geqq 2 \text{ (in both cases } h_1' \in \mathscr{H}_i^0);$$

$$\sigma_i(a)(h_1 \otimes \dots \otimes h_n) = \pi_i(a)\xi_i \otimes h_1 \otimes \dots \otimes h_n \text{ for } i_1 \neq i, n \geqq 1.$$

ii) The above last two formulas show us that

$$\sigma_i(A_i^0)(\mathscr{H}_{i_1}^0 \underset{B}{\otimes} \dots \underset{B}{\otimes} \mathscr{H}_{i_n}^0) \subset \mathscr{H}_r^0(i) \text{ for } i_1 \neq i, n \geqq 1.$$

1.5 Let A_i, B and E_i as in 1.4 and look at the algebraic free product $A = \underset{B}{\circledast} A_i$ with amalgamation over B, which is a B-ring. The B-bimodule decompositions $A_i = B \oplus A_i^0$ yield the following B-bimodule decomposition ([7]):

$$\underset{B}{\circledast} A_i = B \oplus \oplus_{i \neq \dots \neq i_n; n \geqq 1} A_{i_1}^0 \underset{B}{\otimes} \dots \underset{B}{\otimes} A_{i_n}^0.$$

There is a natural *-operation which turns A into a complex *-algebra. Moreover, since each A_i is spanned by the unitary group $\mathscr{U}(A_i)$, the unital *-algebra $A = \underset{B}{\circledast} A_i$ is spanned by $\mathscr{U}(A)$. It is a well-known remark that such a *-algebra satisfies the Combes axiom i.e. for each $x \in A$, there is an $\lambda(x) > 0$ such that $x^*x \leqq \lambda(x)$.

It is a routine exercise to check that the first statement in 1.1 is still true whenever one replaces A by a unital *-algebra satisfying the Combes axiom.

The full amalgamated product of $(A_i, E_i)_{i \in I}$, denoted $\underset{B}{*} A_i$ is the completion and separation of $\underset{B}{\circledast} A_i$ in the C^*-seminorm

$$\|a\| = \sup\{\|\pi(a)\|; \pi\text{*-representation of } \underset{B}{\circledast} A_i\}.$$

It is not difficult to prove that assuming each E_i faithful, $\| \ \|$ is in fact a C^*-norm and the A_i's identify canonically to some unital *-subalgebras of $\underset{B}{*} A_i$.

Let $\chi: B \to \mathscr{L}(\mathscr{H})$ be a unital *-representation and $\Phi_i: A_i \to \mathscr{L}(\mathscr{H})$ be B-linear completely positive maps. Let $(\rho_i, \mathscr{K}_i)$ be the Stinespring dilation of Φ_i. By 1.1 one gets $\mathscr{K}_i^0 = \mathscr{K}_i \ominus \mathscr{H} = \overline{\text{span}}\,(\rho_i - \Phi_i)(A)\mathscr{H}$ and $\rho_i(B)\mathscr{K}_i^0 \subset \mathscr{K}_i^0$. Denote $\rho_i^0 = \rho_i|_{\mathscr{K}_i^0}: B \to \mathscr{L}(\mathscr{K}_i^0)$ and consider the following Hilbert space:

$$\mathscr{K} = \mathscr{H} \oplus \bigoplus_i \mathscr{K}_i^0 \oplus \bigoplus_i \mathscr{H}_r^0(i) \otimes_{\rho_i^0} \mathscr{K}_i^0 =$$

$$= \mathscr{K}_i \oplus \mathscr{H}_r^0(i) \otimes_{\rho_i^0} \mathscr{K}_i^0 \oplus \oplus_{j \neq i} \mathscr{K}_j^0 \oplus \oplus_{j \neq i} \mathscr{H}_r^0(j) \otimes_{\rho_j^0} \mathscr{K}_j^0 =$$

$$= \mathscr{K}_i \oplus \mathscr{H}_r^0(i) \otimes_{\rho_i^0} \mathscr{K}_i^0 \oplus \oplus_{j \neq i} \mathscr{H}_r(j) \otimes_{\rho_j^0} \mathscr{K}_j^0$$

and the *-representations $\tilde{\rho}_i : A_i \to \mathscr{L}(\mathscr{K})$, $\tilde{\rho}_i(a) = \rho_i(a) \oplus \sigma_i(a)|_{\mathscr{H}_r^0(i)} \otimes 1_{\mathscr{K}_i^0} \oplus \oplus_{j \neq i} \sigma_{ij}(a)$, where

$$\sigma_{ij}(a) = (W_j \oplus I_j)(\sigma_i(a)|_{\mathscr{H}_r(j)} \otimes 1_{\mathscr{K}_j^0})(W_j^* \oplus I_j),$$

$$W_j : B \otimes_{\rho_j^0} \mathscr{K}_j^0 \to \mathscr{K}_j^0,\ W_j(\sum_r b_r \otimes k_r) = \sum_r \rho_j(b_r) k_r \text{ are unitaries and}$$
$$I_j = I_{\mathscr{H}_r^0(j) \otimes_{\rho_j^0} \mathscr{K}_j^0}.$$

Denoting by $\hat{a}$ the image of $a \in A_i$ in $\mathscr{H}_i$ one obtains:

$$\sigma_{ij}(a)k = (W_j \oplus I)(\sigma_i(a)\xi \otimes k) = (W_j \oplus I)(\hat{a} \otimes k) =$$
$$= (W_j \oplus I)(\hat{E}_B(a) \otimes k + (\hat{a} - \hat{E}_B(a)) \otimes k) =$$
$$= \rho_j(E_B(a))k + (\hat{a} - \hat{E}_B(a)) \otimes k, \quad \text{for } a \in A_i,\ k \in \mathscr{K}_j^0,$$

hence $\sigma_{ij}(b)k = \rho_j(b)k = bk$ for $b \in B$, $k \in \mathscr{K}_j^0$ and consequently

$$\tilde{\rho}_r(b)k = bk = \tilde{\rho}_s(b)k \text{ for } b \in B,\ k \in \mathscr{K}_j^0,\ r, s \in I.$$

In fact, remark that $\mathscr{K}$ is a left B-module, the left multiplication by $b \in B$ being:

$$bh = \chi(b)h \text{ for } h \in \mathscr{H};\ bk = \rho_j(b)k \text{ for } k \in \mathscr{K}_j^0;$$

$$b(h_1 \otimes \ldots \otimes h_n \otimes k) = \chi(b)h_1 \otimes h_2 \otimes \ldots \otimes h_n \otimes k, \text{ for } h_s \in \mathscr{H}_{i_s}^0,\ s = \overline{1,n},\ k \in \mathscr{K}_j^0,\ i_1 \neq \ldots \neq i_n \neq j$$

and

$$\tilde{\rho}_i(b)\xi = b\xi \text{ for } b \in B,\ \xi \in \mathscr{K},\ i \in I.$$

Consequently the *-representations $\tilde{\rho}_i$ agree on B and one defines the *-representation $\rho = *_i \tilde{\rho}_i : \circledast_B A_i \to \mathscr{L}(\mathscr{K})$.

1.6 THEOREM. *$(\rho, \mathscr{K})$ is the Stinespring dilation of the map $\Phi : \circledast_B A_i \to \mathscr{L}(\mathscr{H})$ defined by:*

$$\Phi(b) = \chi(b), \textit{for } b \in B;$$

$$\Phi(a_1 \ldots a_n) = \Phi_{i_1}(a_1) \ldots \Phi_{i_n}(a_n), \textit{for } a_j \in A_{i_j}^0,\ j = \overline{1,n},\ i_1 \neq \ldots \neq i_n.$$

Consequently Φ is completely positive and extends to a B-linear completely positive map $\ast_i \Phi_i : \ast_B A_i \to \mathscr{L}(\mathscr{H})$.

PROOF. By 1.5 $\rho(b)h = \chi(b)h$ for $b \in B$, $h \in H$ hence $\langle \rho(b)h, h' \rangle = \Phi(b)$, $b \in B$, $h, h' \in \mathscr{H}$. Pick now $h, h' \in \mathscr{H}$, $a_j \in A^0_{i_j}$, $j = \overline{1,n}$, $i_1 \neq \ldots \neq i_n$ and check that $\langle \tilde{\rho}_{i_1}(a_1) \ldots \tilde{\rho}_{i_n}(a_n)h, 'h \rangle = \Phi_{i_1}(a_1) \ldots \Phi_{i_n}(a_n)$. Remark first that

$$\tilde{\rho}_{i_n}(a_n)h = \rho_{i_n}(a_n)h = \Phi_{i_n}(a_n)h + (\rho_{i_n}(a_n)h - \Phi_{i_n}(a_n)h) = \Phi_{i_n}(a_n)h + k_n, \text{ with } k_n \in \mathscr{K}^0_{i_n},$$

$$\tilde{\rho}_{i_{n-1}}(a_{n-1})\tilde{\rho}_{i_n}(a_n)h = \tilde{\rho}_{i_{n-1}}(a_{n-1})\Phi_{i_n}(a_n)h + \tilde{\rho}_{i_{n-1}}(a_{n-1})k_n =$$

$$= \Phi_{i_{n-1}}(a_{n-1})\Phi_{i_n}(a_n)h + (\rho_{i_{n-1}}(a_{n-1}) - \Phi_{i_{n-1}}(a_{n-1}))\Phi_{i_n}(a_n)h + \sigma_{i_{n-1}}(a_{n-1}) \otimes k_n \in \mathscr{H} \oplus \mathscr{K}^0_{i_{n-1}} \oplus \mathscr{H}_{i_{n-1}} \otimes_{\rho^0_{i_{n-1}}} \mathscr{K}^0_{i_n}$$

Assume that $\tilde{\rho}_{i_{k+1}}(a_{k+1}) \ldots \tilde{\rho}_{i_n}(a_n)h = \Phi_{i_{k+1}}(a_{i_{k+1}}) \ldots \Phi_{i_n}(a_n)h \oplus \eta_k$, with $\eta_k \in \mathscr{K}^0_{i_{k+1}} \oplus \bigoplus_{r=k+1}^{n-1} (\mathscr{H}^0_{i_{k+1}} \underset{B}{\otimes} \cdots \underset{B}{\otimes} \mathscr{H}^0_{i_r}) \otimes_{\rho^0_{i_r}} \mathscr{K}^0_{i_r}$.
Since

$$\tilde{\rho}_{i_k}(a_{i_k})(\Phi_{i_{k+1}}(a_{k+1}) \ldots \Phi_{i_n}(a_n)h) = \Phi_{i_k}(a_k) \ldots \Phi_{i_n}(a_n)h \oplus (\rho_{i_k}(a_k) - \Phi_{i_k}(a_k))\Phi_{i_{k+1}}(a_{k+1}) \ldots \Phi_{i_n}(a_n))h \in \mathscr{H} \oplus \mathscr{K}^0_{i_k};$$

$$\tilde{\rho}_{i_k}(a_k)(\mathscr{K}^0_{i_{k+1}}) \subset \mathscr{H}^0_{i_k} \oplus_{\rho^0_{i_{k+1}}} \mathscr{K}^0_{i_{k+1}};$$

$$\tilde{\rho}_{i_k}(a_k)((\mathscr{H}^0_{i_{k+1}} \underset{B}{\otimes} \cdots \underset{B}{\otimes} \mathscr{H}^0_{i_r}) \otimes_{\rho^0_{i_{r+1}}} \mathscr{K}^0_{i_{r+1}}) \subset \mathscr{H}^0_{i_k} \underset{B}{\otimes} \cdots \underset{B}{\otimes} \mathscr{H}^0_{i_r} \otimes_{\rho^0_{i_{r+1}}} \mathscr{K}^0_{i_{r+1}},$$

it follows that $\tilde{\rho}_{i_1}(a_1) \ldots \tilde{\rho}_{i_n}(a_n)h = \Phi_{i_1}(a_1) \ldots \Phi_{i_n}(a_n)h + \eta_0$, with

$$\eta_0 \in \mathscr{K}^0_{i_1} \oplus \mathscr{H}^0_{i_1} \otimes_{\rho^0_{i_2}} \mathscr{K}^0_{i_2} \oplus \ldots \oplus (\mathscr{H}^0_{i_1} \underset{B}{\otimes} \cdots \underset{B}{\otimes} \mathscr{H}^0_{i_{n-1}}) \otimes_{\rho^0_{i_n}} \mathscr{K}^0_{i_n}$$

and consequently $\langle \rho(a_1 \ldots a_n)h, h' \rangle = \langle \Phi_{i_1}(a_1) \ldots \Phi_{i_n}(a_n)h, h' \rangle + \langle \eta_0, h' \rangle = \langle \Phi(a_1 \ldots a_n)h, h' \rangle$.
Finally, 1.1 and the definitions yield:

$$\rho(B)\mathscr{H} = \chi(B)\mathscr{H} = \mathscr{H};$$

$$\overline{\text{span}}\,(\rho - \Phi)(A_i)\mathscr{H} = \overline{\text{span}}\,(\rho_i - \Phi_i)(A_i)\mathscr{H} = \mathscr{K}^0_i \text{ for } i \in I;$$

$$\overline{\text{span}}\,\rho(A^0_{i_1} \underset{B}{\otimes} \cdots \underset{B}{\otimes} A^0_{i_n})\mathscr{K}^0_{i_{n+1}} = (\mathscr{H}^0_{i_1} \underset{B}{\otimes} \cdots \underset{B}{\otimes} \mathscr{H}^0_{i_n}) \otimes_{\rho^0_{i_{n+1}}} \mathscr{K}^0_{i_{n+1}}$$
$$\text{for } i_1 \neq \ldots \neq i_n \neq i_{n+1}$$

hence $\mathscr{K} = \overline{\text{span}}\,\pi(A)\mathscr{H}$.

COROLLARY 1. *If $(G_i)_{i \in I}$ are discrete groups, $H \subset \cap_i G_i$ a common subgroup and $\phi_i : G_i \to \mathscr{L}(\mathscr{H})$ are H-bivariant positive functions, then the function $\phi : \ast_H G_i \to \mathscr{L}(\mathscr{H})$ defined by*

$$\phi(hg_1 \dots g_n) = \phi_{i_1}(g_1) \dots \phi_{i_n}(g_n) \text{ for } g_j \in G_{i_j},\ i_1 \neq \dots \neq i_n$$

is positive definite.

PROOF. Since G_i are discrete, $C^*(H)$ is a C^*-subalgebra of $C^*(G_i)$ and there exist conditional expectations $E_i: C^*(G_i) \to C^*(H)$. It is not difficult to see that $C^*(G_1 \underset{H}{*} G_2) \cong C^*(G_1) *_{C^*(H)} C^*(G_2)$ and each ϕ_i corresponds to its unital extension $\Phi_i: C^*(G_i) \to \mathscr{L}(\mathscr{H})$, which is $C^*(H)$-linear.

COROLLARY 2. *Let A_i be unital C^*-algebras, $S_i \subset A_i$ be unital subspaces and $L_i: S_i :\to \mathscr{L}(\mathscr{H})$ be unital completely contractive maps. Then the L_i extend to a completely contractive map on the free product C^*-algebra $\underset{\mathbf{C}}{*} A_i$.*

PROOF. By Arveson's extension theorem [1], each L_i extends to a unital completely map $\Phi_i: A_i \to \mathscr{L}(\mathscr{H})$ and taking $B = \mathbf{C}$ and each E_i a state on A_i, the map $\underset{i}{*} \Phi_i$ is unital and completely positive on $\underset{\mathbf{C}}{*} A_i$.

Blecher and Paulsen defined in [3] the free product with amalgamation over $\mathbf{C}$ in the category consisting of unital operator algebras as objects and completely contractive morphisms as morphisms and pointed out the following corollary:

COROLLARY 3. *If A_1, A_2 and B are unital operator algebras and $\Phi_i: A_i \to B$ are unital completely contractive maps, then there is a common completely extension $\Phi: A_1 \underset{\mathbf{C}}{*} A_2 \to B$.*

The following is a noncommutative version of Prop. 4.23 in [16] (see also Th.10.8. in [11]).

COROLLARY 4. *Given $(A_i, E_i)_{i \in I}$ and $(B_i, F_i)_{i \in I}$ with $E_i: A_i \to B$, $F_i: B_i \to B$ faithful projections of norm one onto the unital C^*-subalgebra B and the B-linear completely positive maps $\phi_i: A_i \to B_i$, there is a common extension $\Phi: \underset{B}{*} A_i \to \underset{B}{*} B_i$ which is B-linear and completely positive.*

Let M be a finite von Neumann algebra with a faithful trace τ. Then M satisfies *Haagerup's approximation property* whenever there exists a net of unital completely positive maps $\Psi_i: M \to M, i \in I$ such that:

i) $\tau(\Psi_i(x^*x)) \leqq \tau(x^*x)$ for all $x \in M$;

ii) $\lim\limits_{i \in I} \|\Psi_i(x) - x\|_2 = 0$, for all $x \in M$;

iii) Each Ψ_i induces a compact bounded operator $T_{\Psi_i}: L^2(M, \tau) \to L^2(M, \tau)$.

Note that conditions i) shows that $\|\Psi_i(x)\|_2^2 = \tau(\Psi_i(x)^* \Psi_i(x)) \leqq \tau(\Psi_i(x^*x) \neq \|x\|_2^2$ i.e. that T_{Ψ_i} is a contraction. Consequently $\operatorname{Ker}(I - T_{\Psi_i}) = \operatorname{Ker}(I - T_{\Psi_i}^*)$ and since $T_{\Psi_i}\hat{1} = \hat{1}$ it follows that $\mathbf{C}\hat{1}$ is a reducible subspace for T_{Ψ_i}. Denote $T_{\Psi_i}^0 = T_{\Psi_i}|_{\mathbf{C}\hat{1}}$ and let $\Psi_{i,\varepsilon}(x) = (1 + \varepsilon)^{-1}(\Psi_i(x) + \varepsilon\tau(x))$ for $\varepsilon \geqq 0$.

It is easily seen that $\Psi_{i,\varepsilon}$ are unital, $\tau(\Psi_{i,\varepsilon}(x^*x)) \leqq \tau(x^*x)$ and

$\lim_{(i,\varepsilon)\in I_*} \|\Psi_{i,\varepsilon}(x) - x\|_2 = 0$ for all $x \in M$, where $I_* = I \times \mathrm{R}_+$ with the order $(i_1, \varepsilon_1) \leqq (i_2, \varepsilon_2)$ if $i_1 \leqq i_2$ and $\varepsilon_1 \geqq \varepsilon_2$. Moreover, $T^0_{\Psi_{i,\varepsilon}} = (1+\varepsilon)^{-1} T^0_{\Psi_i}$, hence $\|T^0_{\Psi_{i,\varepsilon}}\| < 1$. This remark shows that in fact we can always assume that $\|T^0_{\Psi_i}\| < 1$.

Let M be a finite von Neumann algebra with a faithful trace τ, acting standardly by left multiplication on $\mathscr{H}_\tau = L^2(M,\tau)$. For any $x \in M$ denote by x_τ its appropriate vector in $\mathscr{H}_\tau$. The vector 1_τ is a cyclic and separating trace vector from M. Denote by $\omega_{\xi,\eta}$, ξ, $\eta \in \mathscr{H}_\tau$ the vector form induced by ξ and η and $\omega_\xi = \omega_{\xi,\xi}$. Let $M_0 \subset M$ be a unital weakly dense *-subalgebra of M and let $\Phi_0 : M_0 \to M_0$ be a unital linear map such that $\omega_{1_\tau}\Phi_o = \omega_{1_\tau}$ and $\Phi_0(x)^*\Phi_0(x) \leqq \Phi_0(x^*x)$, $x \in M_0$. Then Φ_0 induces a contraction $T_{\Phi_0} \in \mathscr{L}(\mathscr{H}_\tau)$, $T_{\Phi_0}(x_\tau) = \Phi(x)_\tau$, $x \in M$ and for $a, x \in M_0$ we get

$$\omega_{a_\tau}(\Phi_0(x)) = \langle \Phi_0(x)\cdot a_\tau, a_\tau\rangle = \langle a^*\Phi_0(x)a\cdot 1_\tau, 1_\tau\rangle = \langle aa^*\Phi_0(x)\cdot 1_\tau, 1_\tau\rangle =$$
$$= \langle \Phi_0(x)\cdot 1_\tau, (aa^*)_\tau\rangle = \langle T_{\Phi_0}(x_\tau), (aa^*)_\tau\rangle = \langle x_\tau, T^*_{\Phi_0}((aa^*)_\tau)\rangle =$$
$$= \langle x\cdot 1_\tau, T^*_{\Phi_0}((aa^*)_\tau)\rangle = \omega_{1_\tau, T^*_{\Phi_0}((aa^*)_\tau)}(x),$$

hence $\omega_{a_\tau}\Phi_0$ coincides with a vector form on M_0. Since $\omega_{a_\tau}\Phi_0(x^*x) \geqq 0, x \in M_0$, it follows that $\omega_{a_\tau}\Phi_0$ extends to a state on the norm closure of M_0 and on this C^*-algebra we have $\|\omega_{a_\tau}\Phi_0\| = \omega_{a_\tau}\Phi_0(1) = \|a\|^2_{2,\tau}$, hence

$$\|\Phi_0(x)a\|_2^2 = \langle \Phi_0(x)^*\Phi_0(x)\cdot a_\tau, a_\tau\rangle \leqq \langle \Phi_0(x^*x)\cdot a_\tau, a_\tau\rangle =$$
$$= \omega_{a_\tau}\Phi_0(x^*x) \leqq \|a\|^2_{2,\tau}\|x\|^2,\ a,\ x \in M_0.$$

Therefore Φ_0 extends to a contractive map $\Phi : \overline{M_0}^{\|\|} \to \overline{M_0}^{\|\|}$ with the same properties as Φ_0 and then to a strongly continuous map $\Phi : M \to M$ due to the inequalities $\tau(\Phi(x)^*\Phi(x)) \leqq \tau(x^*x)$, $\|\Phi(x)\| \leqq \|x\|$, $x \in M_0$, the Kaplansky density theorem and the faithfulness of τ.

Haagerup proved [10] that the II_1-factor associated to the free group on two generator satisfies this property. Actually one obtains the following corollary:

COROLLARY 5. *Let (M_1, τ_1) and (M_2, τ_2) be finite von Neumann algebras (τ_1 and τ_2 are faithful traces) with Haagerup property. Then any von Neumann subalgebra of $(M,\tau) = (M_1,\tau_1) \underset{C}{*} (M_2,\tau_2)$ has the Haagerup approximation property.*

PROOF. Let $(\Phi_i)_{i\in I}$ and $(\Psi_j)_{j\in J}$ be as in the definition, relatively to $(\pi_{\tau_1}(M_1)'', \omega_{1_{\tau_2}})$ and $(\pi_{\tau_2}(M_2)'', \omega_{1_{\tau_2}})$. We can assume that $\|T^0_{\Phi_i}\| < 1$, $\|T^0_{\Psi_j}\| < 1$ for all $i \in I, j \in J$ and using the product net that $I = J$. By the previous remark, each $\Phi_i * \Psi_i$ extends to a normal τ-preserving completely positive map on the finite von Neumann algebra M in its standard representation on $L^2(M,\tau)$, where τ denotes the free trace $\tau_1 * \tau_2$. Since $\mathrm{C}1_\tau$ is reducible subspace for both T_{Φ_i} and T_{Ψ_i}, it is easy to check that $(T_{\Phi_i}, 1_{\tau_1}) * (T_{\Psi_i}, 1_{\tau_2}) = (T_{\Phi_i * \Psi_i}, 1_\tau)$ (see [17]). Since

$\|T^0_{\Phi_i}\| < 1$ and $\|T^0_{\Psi_i}\| < 1$, it follows that $T_{\Phi_i * \Psi_i}$ is compact. Finally, its easy to check that $\|(\Phi_i * \Psi_i)(x) - x\|_{2,\tau} \to 0$ for all $x \in M$.

The statement follows using the remark that if N is a von Neumann algebra of M and M has the Haagerup approximation property given by the net of completely positive maps $\Phi_i: M \to M$ with respect to the trace τ on M, then $\tilde{\Phi}_i: N \to N, \tilde{\Phi}_i = E\Phi_i|_N$ approximate in a convenient way the identity of N (E denotes the τ-preserving conditional expectation from M onto N).

By [8, Th. 3] (there is also a simple argument in [13, Th. 4.3.1]) one gets:

COROLLARY 6. *Let M be a factor of type* II_1 *with property* T *of Connes. Then M is not a subfactor of the Neumann algebra* $(M,\tau) = (M_1,\tau_1) \underset{C}{*} (M_2,\tau_2)$, *with* (M_1,τ_1) *and* (M_2,τ_2) *of Haagerup type.*

1.7 Finally, we shall look at the particular example when $A_i = C(\mathrm{T}) = C^*(\mathrm{Z})$, $B = \mathrm{C}$ and $E_i = \tau$, the canonical trace on the full group C^*-algebra A_i. Let $(T_i)_i$ be a family of contractions on the Hilbert space $\mathscr{H}$. Denote by ι_n the function $\iota_n(z) = z^n, z \in \mathrm{T}, n \in \mathrm{Z}$ and $T^{[k]} = \begin{cases} T^k, & \text{for } k \in \mathrm{Z}_+ \\ T^{*-k}, & \text{for } k \in \mathrm{Z}_- \end{cases}$. Let $\Phi_i: C(\mathrm{T}) \to \mathscr{L}(\mathscr{H})$ be the completely positive map determined by $\Phi_i(\iota_n) = T_i^{[n]}$, $n \in \mathrm{Z}$. Denote also by $a_1, \ldots, a_N (N \in \mathrm{N} \cup \{\infty\})$ the generators of the free group on N generators F_N.

In this case we are interested to find the Naimark dilation of the positive definite function $\phi: \mathrm{F}_N \to \mathscr{L}(\mathscr{H})$,

$$\phi(g) = \begin{cases} \phi_{i_1}(a_{i_1}^{k_1}) \ldots \phi_{i_n}(a_{i_n}^{k_n}) & , \text{ for } g = a_{i_1}^{k_1} \ldots a_{i_n}^{k_n}, i_1 \neq \ldots \neq i_n, k_j \in \mathrm{Z} \setminus \{0\}; \\ I_{\mathscr{H}} & , \text{ for } g = e \end{cases}$$

where $\phi_i: \mathrm{Z} \to \mathscr{L}(\mathscr{H})$, $\phi_i(k) = T_i^{[k]}$ are positive defined functions ([15]).

By the classical theorem of Szökefalvi-Nagy [15] it is known that in fact the Naimark dilation $(\pi_i, \mathscr{K}_i)$ of ϕ_i is given by $\mathscr{K}_i = \mathscr{H} \oplus \mathscr{K}_i^0$ with $\mathscr{K}_i^0 = l^2(\mathrm{Z}_-) \otimes \mathscr{D}_{T_i^*} \oplus l^2(\mathrm{Z}_+) \otimes \mathscr{D}_{T_i}$ and $U_i = \pi_i(1) \in \mathscr{U}(\mathscr{K}_i)$ are defined by:

$$U_i v = \begin{cases} T_i h \oplus \xi_0 \otimes D_{T_i} h & , \text{ for } v = h \in \mathscr{H}; \\ \xi_{k+1} \otimes D_{T_i} h & , \text{ for } v = \xi_k \otimes D_{T_i} h, k \geqq 0, h \in \mathscr{H}; \\ \xi_{k+1} \otimes D_{T_i^*} h & , \text{ for } v = \xi_k \otimes D_{T_i^*} h, k \leqq -1, h \in \mathscr{H}; \\ D_{T_i^*}^2 h \oplus \xi_0 \otimes (-D_{T_i} T_i^* h) & , \text{ for } v = \xi_0 \otimes D_{T_i^*} h, h \in \mathscr{H} \end{cases}$$

where $(\xi_k)_{k \geq 0}$ (respectively $(\xi_k)_{k \leq 0}$) denotes the canonical orthonormal basis in $l^2(\mathrm{Z}_+)$ (respectively in $l^2(\mathrm{Z}_-)$), $D_{T_i} = (I - T_i^* T_i)^{1/2}$, $D_{T_i^*} = (I - T_i T_i^*)^{1/2}$, $\mathscr{D}_{T_i^*} = \overline{D_{T_i}\mathscr{H}}$, $\mathscr{D}_{T_i^*} = \overline{D_{T_i^*}\mathscr{H}}$.

Actually Theorem 1.6 yields a Hilbert space $\mathscr{K} \supset \mathscr{H}$ and the unitaries $\tilde{U}_i \in \mathscr{L}(\mathscr{K})$ such that

$$T_{i_1}^{[k_1]} \dots T_{i_n}^{[k_n]} = P_{\mathscr{H}}^{\mathscr{K}} \tilde{U}_{i_1}^{[k_1]} \dots \tilde{U}_{i_n}^{[k_n]}|_{\mathscr{H}}, \text{for } i_1 \neq \dots \neq i_n, n \geqq 1, k_j \in \mathbf{Z}, k_j \neq 0;$$
$$\mathscr{K} = \overline{\text{span}}\,(\mathscr{H} \cup \{U_{i_1}^{k_1} \dots U_{i_n}^{k_n} \mathscr{H}; i_1 \neq \dots \neq i_n, n \geqq 1, k_j \in \mathbf{Z}, k_j \neq 0\}).$$

It is easily seen that this dilation is unique up to unitary equivalence and in fact it can be spatially described as follows. Denote

$\mathscr{K}_i^0 = \mathscr{K}_i \ominus \mathscr{H} = l^2(\mathbf{Z}_-) \otimes \mathscr{D}_{T_i^*} \oplus l^2(\mathbf{Z}_+) \otimes \mathscr{D}_{T_i}$;
$\mathsf{F}_{N,i} = \{w = a_{i_1}^{k_1} \dots a_{i_n}^{k_n}$; w reduced word in F_N, $n \geqq 1$, $i_n \neq i\}$;
$l^2(\mathsf{F}_{N,i}) = \{f \in l^2(\mathsf{F}_N); \operatorname{supp} f \subset \mathsf{F}_{N,i}\}$;
$(\xi_w)_{w \in \mathsf{F}_N}$ the orthonormal basis of $l^2(\mathsf{F}_N)$ given by $\xi_w(w') = \delta_{ww'}$, $w, w' \in \mathsf{F}_N$.

Then $\mathscr{K} = \mathscr{H} \oplus \bigoplus_i \mathscr{K}_i^0 \oplus \bigoplus_i l^2(\mathsf{F}_{N,i}) \otimes \mathscr{K}_i^0$ and the unitaries $\tilde{U}_i \in \mathscr{L}(\mathscr{K})$ are defined by

$$\tilde{U}_i v = \begin{cases} U_i h = T_i h \oplus \xi_0 \otimes D_{T_i} h & , \text{ for } v = h \in \mathscr{H}; \\ U_i \eta_i & , \text{ for } v = \eta_i \in \mathscr{K}_i^0; \\ \xi_{a_i, a_j^k} \otimes D_{T_j} h & , \text{ for } v = \xi_k \otimes D_{T_j} h, j \neq i, k \geqq 0, h \in \mathscr{H}; \\ \xi_{a_i, a_j^k} \otimes D_{T_j^*} h & , \text{ for } v = \xi_k \otimes D_{T_j^*} h, j \neq i, k \geqq 0, h \in \mathscr{H}; \\ \eta & , \text{ for } v = \xi_{a_i^{-1}} \otimes \eta, \eta \in \mathscr{K}_j^0. \end{cases}$$

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