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LIFTINGS OF MÖBIUS GROUPS TO MATRIX GROUPS

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Abstract.

It is well known that any discrete subgroup G of PSL(2, C) can be lifted to a subgroup of SL(2, C) if and only if G does not have two-torsion ([1], [4]).

In this note we show the existence of liftings of certain discrete subgroups of the group $M(U)$ of orientation preserving and reversing Möbius transformations mapping the upper half-plane U onto itself. The orientation preserving Möbius transformations mapping the upper half-plane onto itself form the connected subgroup PSL(2, R) of $M(U)$. The lifting of subgroups of $M(U)$ to SL(2, C) has two complications: first of all, the group $M(U)$ is not connected, and, secondly, there is no natural mapping from PSL(2, C) to $M(U)$. In this note we consider these problems and show that all subgroups G of $M(U)$ for which U/G is a compact Klein surface can be lifted to SL(2, C). This result is closely related with the general considerations of M. Culler ([1]) and is based on the arguments presented in [8].

We also study the uniqueness of liftings of subgroups of PSL(2, C) to SL(2, C). Assume that $G \subset$ PSL(2, C) can be lfted to a subgroup of SL(2, C). Consider

$$\hat{G} = \cap_{\tilde{G}} \tilde{G}$$

where $\tilde{G}$ goes through all liftings of G to subgroups of SL(2, C). Let $G^{*} \subset G$ be the projection of $\hat{G}$. The group G^{*} is non-trivial because it contains, e.g., all squares of elements of G. We show that – under rather general conditions – G^{*} is actually generated by squares and commutators of elements of G.

1. Preliminaries.

Recall that SL(2, C) is the group of complex 2×2-matrices with determinant 1. The quotient $\mathrm{PSL}(2, \mathrm{C}) = \mathrm{SL}(2, \mathrm{C})/\{\pm I\}$ is the group of Möbius transformations of the extended comples plane $\hat{\mathrm{C}}$.

Consider the exact sequence

$$0 \to \mathrm{Z}_2 \to \mathrm{SL}(2, \mathrm{C}) \xrightarrow{\pi} \mathrm{PSL}(2, \mathrm{C}) \to 0. \tag{1}$$

Here π: SL(2, C) $\to$ PSL(2, C) is the natural projection. Let $G \subset$ PSL(2, C) be a subgroup. A homomorphism φ: $G \to$ SL(2, C) is a *lifting* of G if $\pi \circ \varphi$ is the identity mapping of G. It follows that a lifting φ is an isomorphism of G onto $\varphi(G)$

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(cf. Example 4). For a lifting $\varphi\colon G \to \mathrm{SL}(2,\mathbf{C})$, we also call the image $\varphi(G) \subset \mathrm{SL}(2,\mathbf{C})$ of G, a *lifting of* G.

If G contains the Möbius transformation $g(z) = -1/z$, then G has evidently no liftings because the matrices $\tilde{g}_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\tilde{g}_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ are of order 4 while g is of order two. This observation can be generalized as follows:

PROPOSITION 1. *Let $G \subset \mathrm{PSL}(2,\mathbf{C})$ be a group and let S be a generating set of G. Suppose that $\varphi\colon S \to \mathrm{SL}(2,\mathbf{C})$ is a mapping satisfying $\pi(\varphi(s)) = s$ for all $s \in S$. Then φ can be extended to a lifting of G if and only if the following holds: If $s_1, \ldots, s_n \in S$ such that $s_n^{\varepsilon_n} \ldots s_2^{\varepsilon_2} s_1^{\varepsilon_1} = \mathrm{id}$, $\varepsilon_j = \pm 1, j = 1, \ldots, n$, then*

$$\varphi(s_n)^{\varepsilon_n} \ldots \varphi(s_2)^{\varepsilon_2}\varphi(s_1)^{\varepsilon_1} = I. \tag{2}$$

EXAMPLE 1. Let $k > 1$. For any positive integer n define $g_n(z) = k^{1/n}z$. Consider the group F generated by all elements g_n, $n = 1, 2, \ldots$. It is obvious that a lifting of F is generated by the matrices

$$\begin{pmatrix} k^{\frac{1}{2n}} & 0 \\ 0 & k^{-\frac{1}{2n}} \end{pmatrix}.$$

On the other hand, every generator g_n is a square of some other generator, namely g_{2n}, i.e., $g_n(z) = g_{2n}(g_{2n}(z))$. This implies that for any lifting of F to $\mathrm{SL}(2,\mathbf{C})$, all matrices corresponding to generators of F have positive trace. We conclude that the group F has only one lifting to a subgroup of $\mathrm{SL}(2,\mathbf{C})$.

This example indicates that the exact sequence (1) is not simple as it looks like. It is well known that a Fuchsian subgroup G of $\mathrm{PSL}(2,\mathbf{C})$, which does not contain elliptic elements, can always be lifted to a subgroup of $\mathrm{SL}(2,\mathbf{C})$ (see [4] and references given there). Furthermore, if G acts in the upper half-plane and U/G is a genus g Riemann surface, any choice of matrices corresponding to standard generators of G generates a lifting of G. This fact has been discovered and rediscovered several times independently by many authors during this century. In particular, G has 2^{2g} liftings to subgroups of $\mathrm{SL}(2,\mathbf{C})$.

The usual proofs of this fact rely on the geometry of the Riemann surface U/G. Some proofs are based on the fact that the canonical bundle of U/G has a square root, some others on the fact the Euler characteristic of an orientable surface is an even number.

2. Free groups.

Let $G \subset \mathrm{PSL}(2,\mathbf{C})$ be a group, and suppose that S is a generating set of G. By definition, G is a *free* group and G is *generated freely* by S if any mapping from

S into an arbitrary group X can be extended to a homomorphism from G into X. Since the projection π: SL(2, C) → PSL(2, C) is a homomorphism, the following is obvious:

PROPOSITION 2. *Suppose that G is generated freely by S. Then every mapping $\varphi: S \to \mathrm{SL}(2, \mathrm{C})$ satisfying $\pi(\varphi(s)) = s$ for all $s \in S$ can be extended to a lifting of G.*

COROLLARY 1. *If G is generated freely by n elements, then G has 2^n different liftings.*

3. Möbius transformations fixing the upper half-plane.

The group PSL(2, R) consist of all Möbius transformations

$$g(z) = \frac{az + b}{cz + d}, \quad ad - bc = 1,$$

which map upper half-plane U onto itself. Hence PSL(2, R) contains besides the identity hyperbolic, parabolic and elliptic transformations only.

Every hyperbolic g is conjugate to a unique standard form m_k: $z \mapsto kz$, $k > 1$. Denote $k = k(g)$.

Let g and h be hyperbolic elements of PSL(2, R) sharing no fixed points. The cross-ratio $t = (r(g), r(h), a(h), a(g))$ of the repelling and attracting fixed points of g and h, as well as the multipliers $k_1 = k(g)$ and $k_2 = k(h)$ are invariant under conjugation by elements of PSL(2, C).

The matrix product $\tilde{c} = \tilde{h}\tilde{g}^{-1}\tilde{h}^{-1}\tilde{g}$ does not depend on the choice of the liftings $\tilde{g}$ and $\tilde{h}$. Hence the commutator

$$c = [g, h] = hg^{-1}h^{-1}g$$

has a well-defined trace $\operatorname{tr} c = tr\, \tilde{c}$.

PROPOSITION 3. $\operatorname{tr} c = 2 - t(1 - t)(\sqrt{k_1} - 1/\sqrt{k_1})^2(\sqrt{k_2} - 1/\sqrt{k_2})^2$.

For given values k_1 and k_2 of $k(g)$ and $k(h)$, respectively, consider the function

$$F(t) = 2 - t(1 - t)(\sqrt{k_1} - 1/\sqrt{k_1})^2(\sqrt{k_2} - 1/\sqrt{k_2})^2.$$

Then $F(0) = F(1) = 2$, $F(t) = -2$ if and only if

$$t(1 - t) = \frac{4}{(\sqrt{k_1} - 1/\sqrt{k_1})^2(\sqrt{k_2} - 1/\sqrt{k_2})^2},$$

and $F(t) = 0$ if and only if

$$t(1 - t) = \frac{2}{(\sqrt{k_1} - 1/\sqrt{k_1})^2(\sqrt{k_2} - 1/\sqrt{k_2})^2}. \tag{3}$$

EXAMPLE 2. Consider hyperbolic transformations $g, h \in PSL(2, R)$ with intersecting axes. The pair (g, h) can be chosen such that (3) is satisfied. In this case c is an elliptic transformation of order two and the group $\langle g, h\rangle$ has no liftings (cf. Proposition 1).

The axes of g and h intersect if and only if $0 < t < 1$. If, in this case, $\alpha(g, h)$ denotes the acute angle between the axes of g and h, then

$$\sin \alpha(g, h) = 2\sqrt{t(1-t)}.$$

We have proved the following result recently obtained also by Gilman ([2, Theorem 1 and Corollary]):

PROPOSITION 4. *The following conditions are equivalent:*
(i) *The axes of g and h intersect and the commutator $c = [g, h]$ is hyperbolic.*
(ii) *The axes of g and h intersect and*

$$\sin \alpha(g, h) > \frac{4}{(\sqrt{k_1} - 1/\sqrt{k_1})(\sqrt{k_2} - 1/\sqrt{k_2})}.$$

(iii) $\operatorname{tr} c < -2$.

The next proposition is almost a restatement of Theorem 8 in [5]. The proof, however, is based on different arguments which emphasize the Schottky group structure of $\langle g, h\rangle$.

PROPOSITION 5. *Suppose that the hyperbolic transformations g and h have intersecting axes. If the commutator $c = [g, h]$ is hyperbolic, then $\langle g, h\rangle$ is a discrete Schottky group generated freely by g and h.*

PROOF. Symmetry in the action of the commutator will be highlighted if we normalize such that g and h fix the unit disk and choose the origin as the intersection point of the axes of g and h. Denote

$$c_1 = gcg^{-1} = ghg^{-1}h^{-1},$$
$$c_2 = h^{-1}c_1h = h^{-1}ghg^{-1},$$
$$c_3 = g^{-1}c_2g = g^{-1}h^{-1}gh.$$

Then $hc_3h^{-1} = c$. The cyclic order of the axes of g, h, c, c_1, c_2 and c_3 is given by Figure 1 (cf. [9, Figure 1.7]). Let K_1 and K_2 be orthogonal circles of the unit circle such that K_1 is orthogonal to the axes of c and c_3 and K_2 is orthogonal to the axes of c_2 and c_3. Then $K_3 = g(K_1)$ is orthogonal to the axes of c_1 and c_2 and $K_4 = h(K_2)$ is orthogonal to the axes of c and c_1. Since the circles K_1, K_2, K_3 and K_4 are external to one another, it follows that $\langle g, h\rangle$ is a Schottky group and hence it is generated freely by g and h.

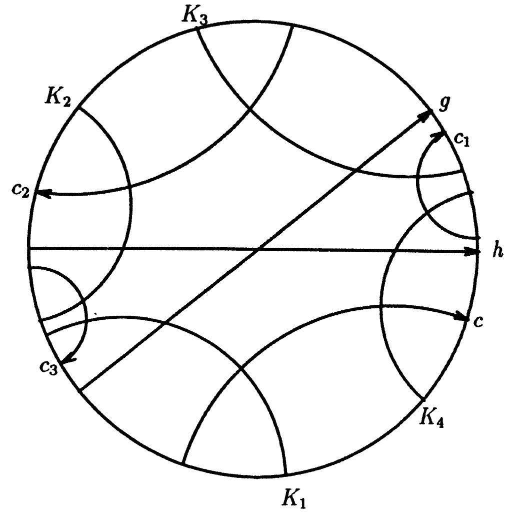

Figure 1.

Combining Propositions 2 and 5 we obtain the following result:

THEOREM 2. *Let $g, h \in \mathrm{PSL}(2, \mathrm{R})$ be hyperbolic transformations with intersecting axes. If the commutator $c = [g, h]$ is hyperbolic, then every mapping $\varphi: \{g, h\} \to \mathrm{SL}(2, \mathrm{R})$, $\pi(\varphi(g)) = g$, $\pi(\varphi(h)) = h$, has a continuation to a lifting of the group $\langle g, h\rangle$.*

The following example shows that the existence of liftings of Fuchsian groups representing compact Riemann surfaces is by no means obvious.

EXAMPLE 3. Consider two pairs (g_1, h_1) and (g_2, h_2) of hyperbolic transformations fixing U. Choosing $k_1 = k(g_1)$ and $k_2 = k(h_1)$ such that

$$(\sqrt{k_1} - 1/\sqrt{k_1})(\sqrt{k_2} - 1/\sqrt{k_2}) > 4.$$

Then, by Propositions 3 and 4, it is possible to choose $t_1 = (r(g_1), r(h_1), a(h_1), a(g_1))$ such that $c_1 = [g_1, h_1]$ is hyperbolic and $\operatorname{tr} c_1 < -2$. Finally, conjugate such that $a(c_1) = \infty$, $r(c_1) = 0$ and $r(g_1) = 1$ (Figure 2).

On the other hand, it is possible to choose $k(g_2)$, $k(h_2)$ and

$$t_2 = (r(g_2), r(h_2), a(h_2), a(g_2)) > 1$$

such that $\operatorname{tr} c_2 = -\operatorname{tr} c_1$ for $c_2 = [g_2, h_2]$. Conjugate such that $a(c_2) = 0$, $r(c_2) = \infty$ and $a(g_2) = -1$ (Figure 3). Then $c_2 = c_1^{-1}$.

Consider the group $G = \langle g_1, h_1, g_2, h_2\rangle$. The generators satisfy relation

$$[g_2, h_2][g_1, h_1] = \mathrm{id}.$$

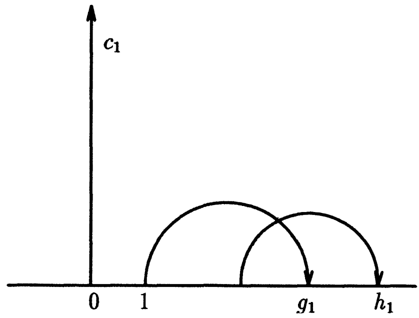

Figure 2.

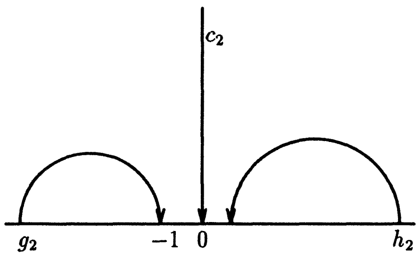

Figure 3.

However, we have

$$[\tilde{g}_2, \tilde{h}_2][\tilde{g}_1, \tilde{h}_1] = -I$$

for all choices of the liftings $\tilde{g}_j = \varphi(g_j)$ and $\varphi(h_j), j = 1, 2$. Hence the group G has no liftings (cf. Proposition 1).

4. Elements with unique liftings.

Let $G \subset \mathrm{PSL}(2, \mathbf{C})$ be a group with a nonempty set $\mathscr{F}$ of liftings φ: $G \to \mathrm{SL}(2, \mathbf{C})$. The group

$$G^{\#} = \pi\left(\bigcap_{\varphi \in \mathscr{F}} \varphi(G)\right)$$

consists of all elements g of G which have a unique lifting, i.e., $\varphi(g)$ does not depend on the lifting $\varphi \in \mathscr{F}$.

A choice of a lifting of G associates a unique matrix to any element of G. It is, on the other hand, clear that, for any choice of a lifting of G, the matrices associated to the commutators and squares of elements of G are independent of the lifting.

PROPOSITION 6. *$G^{\#}$ is a normal subgroup of G containing the squares g^2 and the commutators $[g, h]$ of all elements of G.*

The group G can be *unlimitedly lifted* if there exists a generating set S of G such that every mapping $\varphi\colon S \to SL(2, \mathrm{C})$ satisfying $\pi(\varphi(s)) = s$ for all $s \in S$ can be extended to a lifting of G. All free subgroups of $\mathrm{PSL}(2, \mathrm{C})$ can be unlimitedly lifted by Proposition 2. On the other hand, if G acts discontinuously and fixed-point-freely in U and U/G is compact, then G can be unlimitedly lifted.

THEOREM 3. *If G can be unlimitedly lifted, then $G^{\#}$ is generated by the squares and commutators of the elements of G.*

PROOF. Let S be a generating set of G with the unlimited lifting property.

Any element $g \in G$ has a representation

$$g = \chi\sigma_m^{t_m} \dots \sigma_1^{t_1},$$

where χ is in the commutator subgroup of G, $\sigma_1, \dots, \sigma_m$ are distinct elements of S and $t_1, \dots, t_m$ are integers.

By Proposition 6, the lifting $\tilde{\chi}$ of χ does not depend on the choice of the liftings $\tilde{s}$ of the generators $s \in S$. Denote

$$\tilde{g} = \tilde{\chi}\tilde{\sigma}_m^{t_m} \dots \tilde{\sigma}_1^{t_1}.$$

Suppose that $t_{j_0} \in \{t_1, \dots, t_m\}$ is odd. Then we can change $\tilde{g}$ to $-\tilde{g}$ by changing $\tilde{\sigma}_{j_0}$ to $-\tilde{\sigma}_{j_0}$ but keeping the liftings of all other generators unchanged. Hence $g \notin G^{\#}$.

If $G \subset \mathrm{PSL}(2, \mathrm{R})$ acts discontinuously and does not have fixed-points in U, then U/G is a Riemann surface. Since G is either a free group or U/G is compact, the group G can be unlimitedly lifted. Then $G^{\#} \neq G$ which implies that G contains elements which are not products of commutators and squares of elements of G. In particular, the fundamental group of a compact Riemann surface contains such elements. On the other hand, the group G considered in Example 1 satisfies $G^{\#} = G$. Since this group is generated by the squares of its elements, the converse of Theorem 3 does not hold.

5. Liftings of orientation reversing Möbius transformations.

Let $M(\hat{\mathrm{C}})$ denote the group of all orientation preserving or orientation reversing Möbius transformations $g\colon \hat{\mathrm{C}} \to \hat{\mathrm{C}}$. For every matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathrm{C})$ there exists two transformations in $M(\hat{\mathrm{C}})$ with the same coefficients a, b, c and d, namely the orientation preserving transformation

$$g(z) = \frac{az + b}{cz + d}, \quad ad - bc = 1, \tag{4}$$

and the orientation reversing transformation

$$g(z) = \frac{a\bar{z} + b}{c\bar{z} + d}, \quad ad - bc = 1. \tag{5}$$

Although we have no natural projection $SL(2, C) \to M(\hat{C})$, the matrices $\pm\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ are called *liftings* of the transformations (4) and (5).

Let g be defined by (5) and let h be any transformation with the liftings $\pm\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$. Since the composed transformation gh has

$$\pm\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \bar{\alpha} & \bar{\beta} \\ \bar{\gamma} & \bar{\delta} \end{pmatrix}$$

as liftings, the matrix product

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

is a lifting of gh if and only if either $(\alpha, \beta, \gamma, \delta) \in R^4$ or $(i\alpha, i\beta, i\gamma, i\delta) \in R^4$.

Let $G \subset M(\hat{C})$ be a group. A homomorphism φ: $G \to SL(2, C)$ is a *lifting* of G if φ is injective and $\varphi(g)$ is a lifting of g for every $g \in G$. Since we have no homomorphic projection $SL(2, C) \to M(\hat{C})$, Proposition 2 does not hold for all free groups $G \subset M(\hat{C})$.

Example 4. Let $g(z) = \bar{z}$. Then the matrices I and $-I$ are liftings of g. Define $\varphi(g) = -I$ and $\varphi(\mathrm{id}) = I$. Then φ: $\langle g \rangle \to SL(2, C)$ is a lifting of $\langle g \rangle$, and the group $\langle g \rangle$ has no other liftings. On the other hand, we can define a homomorphism ψ: $\langle g \rangle \to SL(2, C)$ by setting $\psi(g) = \psi(\mathrm{id}) = I$. Hence the injectiveness of a lifting φ: $G \to SL(2, C)$ does not any more follow from the facts that φ is homomorphic and $\varphi(g)$ is a lifting of g for all $g \in G$ (cf. § 1). It is interesting to note that $\langle g \rangle$ is a discrete group containing a two-torsion (cf. [1]).

Let $s(z) = z + i$ and $h = sgs^{-1}$. Then the matrices $\pm\begin{pmatrix} 1 & 2i \\ 0 & 1 \end{pmatrix}$ are liftings of h. Since $\left(\pm\begin{pmatrix} 1 & 2i \\ 0 & 1 \end{pmatrix}\right)^2 = \begin{pmatrix} 1 & 4i \\ 0 & 1 \end{pmatrix}$ but $h^2 = \mathrm{id}$, the group $\langle h \rangle$ has no liftings although the groups $\langle g \rangle$ and $\langle h \rangle$ are conjugate subgroups of $M(\hat{C})$.

The transformations (4) and (5) with the same coefficients a, b, c and d agree on the extended real axis $\hat{R} = R \cup \{\infty\}$. They map $\hat{R}$ onto itself if and only if either $(a, b, c, d) \in R^4$ or $(ia, ib, ic, id) \in R^4$.

Theorem 4. *Suppose that $g \in M(\hat{C})$ is orientation reversing. If the group $\langle g \rangle$ has liftings, then $g(\hat{R}) = \hat{R}$.*

Proof. Suppose that $\varphi\colon \langle g\rangle \to \mathrm{SL}(2,\mathbf{C})$ is a lifting, and let $\varphi(g) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Since $\varphi(g)^2 = \varphi(g^2)$, we have

$$\begin{pmatrix} a & b \\ a & b \end{pmatrix}\begin{pmatrix} a & b \\ a & b \end{pmatrix} = \pm\begin{pmatrix} a & b \\ a & b \end{pmatrix}\begin{pmatrix} \bar a & \bar b \\ \bar c & \bar d \end{pmatrix}.$$

Hence $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \pm\begin{pmatrix} \bar a & \bar b \\ \bar c & \bar d \end{pmatrix}$ and the assertion follows.

Corollary 5. *Suppose that $G \subset M(\hat{\mathbf{C}})$ is a group containing orientation reversing elements. If G has a lifting $\varphi\colon G \to \mathrm{SL}(2,\mathbf{C})$, then all elements of G map the extended real axis $\hat{\mathbf{R}}$ onto itself.*

The converse of Theorem 4 does not hold. For instance, the group $\langle g\rangle$ generated by $g(\bar z) = 1/\hat z$ has no liftings although $g(\hat{\mathbf{R}}) = \hat{\mathbf{R}}$.

6. Liftings of NEC groups representing Klein surfaces.

An important application of the lifting theorem concerning Fuchsian subgroups of $\mathrm{PSL}(2,\mathbf{R})$ is the representation of Teichmüller spaces of smooth complex projective curves as a component of an affine real algebraic variety ([3]). This result can be generalized to smooth projective real algebraic curves by considering the complexifications of real algebraic curves ([7]). These complexifications are simply complex algebraic curves *with orientation reversing symmetries.*

Real algebraic curves can be viewed as non-classical Klein surfaces. Excluding certain elementary cases such Klein surfaces can always be expressed in the form U/G where G is a discrete subgroup of the group $M(U)$ (see abstract).

Observe that the topological group $M(U)$ has two components: one containing all orientation preserving Möbius transformations and one containing all orientation reversing ones.

Let

$$\tilde M(U) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbf{C}) \,\middle|\, a, b, c, d \in \mathbf{R} \text{ or } ia, ib, ic, id \in \mathbf{R}\right\}.$$

Then $\tilde M(U)$ is a subgroup of $\mathrm{SL}(2,\mathbf{C})$ having two components. Define the projection

$$\pi\colon \tilde M(U) \to M(U)$$

by setting

$$\pi\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{cases} z \mapsto \dfrac{az+b}{cz+d} & \text{if } a, b, c, d \text{ are real,} \\ z \mapsto \dfrac{a\bar{z}+b}{c\bar{z}+d} & \text{if } a, b, c, d \text{ are purely imaginary.} \end{cases}$$

With this definition of the projection π, the exact sequence (1) generalizes to the exact sequence

$$0 \to \mathbf{Z}_2 \to \tilde{M}(U) \xrightarrow{\pi} M(U) \to 0. \tag{6}$$

Then π: $\tilde{M}(U) \to M(U)$ is a surjective homomorphism whose kernel is $\{I, -I\}$. It follows that Propositions 1, 2 and 6 can be applied to all groups $G \subset M(U)$.

PROPOSITION 7. *Suppose that $g \in M(U)$ is orientation reversing. If $\tilde{g} \in \tilde{M}(U)$ is a lifting of g, then* $\operatorname{tr} \tilde{g}^2 \leqq -2$.

PROOF. If $\tilde{g} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $ad - bc = 1$, then a, b, c and d are purely imaginary. Hence

$$\operatorname{tr} \tilde{g}^2 = a^2 + d^2 + 2bc = (a+d)^2 - 2 \leqq -2.$$

Assume that Σ is a non-orientable compact C^∞-surface whose Euler-characteristic is negative. If Σ is equipped with a metric d of constant curvature -1, then $X = (\Sigma, d)$ is a Klein surface. Such surfaces can always be represented as $X = U/G$ where G is a non-Euclidean crystallographic group (a NEC group), i.e., a properly discontinuous subgroup of $M(U)$. *Since* $X = U/G$ is compact, the orientation preserving elements of G are hyperbolic transformations whereas orientation reversing ones are *glide reflections*. Every glide reflection $s \in M(U)$ admits a representation $s = \sigma \circ g$ where $g \in M(U)$ is hyperbolic and σ is the reflection in the axis of g. Hence $s^2 = g^2$ is hyperbolic (in this case).

Let G_+ be the subgroup of G containing the orientation preserving elements of G. Then G_+ is a Fuchsian group of the first kind, $X^0 = U/G_+$ is a compact Riemann surface and the projection $X^0 \to X$ is an unramified double covering map.

If the genus p of X^0 is even, then G has a standard set S of $p+1$ generators $g_1, h_1, \ldots, g_{p/2}, h_{p/2}, s$ which generate G with one defining relation

$$s^2[g_{p/2}, h_{p/2}] \cdots [g_1, h_1] = \mathrm{id}.$$

Here s is a glide reflection and g_i and h_j are hyperbolic transformations with intersecting axes, $j = 1, \ldots, p/2$, but there are no other intersections between the axes of the elements of S. The geodesics on X^0 corresponding to the transformations $c_1 = [g_1, h_1], \ldots, c_{p/2} = [g_{p/2}, h_{p/2}]$, $c_{p/2+1} = s^2, c_2c_1, c_3c_2c_1, \ldots, c_{p/2} \cdots c_3c_2c_1$ are simple and pairwise disjoint.

If the genus p of X^0 is odd, the standard set S consist of $p-1$ hyperbolic generators $g_1, h_1, \ldots, g_{(p-1)/2}, h_{(p-1)/2}$ and two glide reflections s_1 and s_2. The defining relation is now

$$s_2^2 s_1^2 [g_{(p-1)/2}, h_{(p-1)/2}] \cdots [g_1, h_1] = \text{id},$$

and the axes of the generators and the geodesics on X^0 corresponding to the transformations $c_1 = [g_1, h_1], \ldots, c_{(p-1)/2} = [g_{(p-1)/2}, h_{(p-1)/2}]$, $c_{(p+1)/2} = s_1^2$, $c_{(p+3)/2} = s_2^2$, $c_2 c_1$, $c_3 c_2 c_1$, $c_{(p+1)/2} \cdots c_3 c_2 c_1$ have similar properties as in the previous case (cf. [6, §4]).

THEOREM 6. *Suppose that $G \subset M(U)$ is a* NEC *group representing a compact Klein surface $X = U/G$. Let S be a standard set of generators of G. Then every mapping $\varphi\colon S \to \mathrm{SL}(2, \mathrm{C})$ satisfying $\pi(\sigma(s)) = s$ for all $s \in S$ has an extension to a lifting of G.*

PROOF. Suppose that the genus p of X^0 is even. If p is odd, the assertion can be shown similarly.

Choose liftings $\tilde{g}_j = \varphi(g_j)$, $\tilde{h}_j = \varphi(h_j)$, $\tilde{s} = \varphi(s)$, and denote $\tilde{c}_j = [\tilde{g}_j, \tilde{h}_j]$, $j = 1, \ldots, p/2$, $\tilde{c}_{p/2+1} = \tilde{s}^2$. Since s^2 is hyperbolic, $\operatorname{tr} \tilde{c}_{p/2+1} < -2$ by Proposition 7. Similarly, $\operatorname{tr} \tilde{c}_j < -2$, $j = 1, \ldots, p/2$, by Proposition 4. The assertion follows now similarly as in the proof of Theorem 4.1 in [6] by considering the group G_+ and the Riemann surface $X^0 = U/G_+$.

This is a generalization of the classical result and allows one to represent the Teichmüller space of the non-classical Klein surface U/G as a component of an affine real algebraic variety (cf. [3] and [7]).

Suppose that $G \subset M(U)$ is a NEC group representing a compact Klein surface $X = U/G$. Then, by the proof of Theorem 6, G can be unlimitedly lifted. The Kleinian group.

$$\bar{G} = G_+ \cup \{z \mapsto g(\bar{z}) \mid g \in G \backslash G_+\}$$

has the same liftings and the same limit set $\hat{\mathbb{R}}$ as G. Since G acts on U, the group $\bar{G}$ has no invariant components. Hence there exist non-elementary finitely generated Kleinian groups which are not function groups but which can be unlimitedly lifted. A similar example has been constructed by M. Culler ([1]).

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