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EXTENSION OF BILIPSCHITZ MAPS OF COMPACT POLYHEDRA

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Introduction.

Let X and Y be metric spaces with distance written as $|a - b|$. A map $f: X \to Y$ is called *bilipschitz*, abbreviated BL, if there is $L \geqq 1$ such that

$$|x - y|/L \leqq |fx - fy| \leqq L|x - y|$$

for all $x, y \in X$. In this situation we also say that f is L-BL. A 1-BL map $f: X \to Y$ will be called an *isometry*.

A set $A \subset X$ has the *bilipschitz extension property*, abbreviated BLEP, in (X, Y) if there is $L_0 = L_0(A, X, Y) > 1$ such that if $1 \leqq L \leqq L_0$, then every L-BL map $f: A \to Y$ has an L_1-BL extension $g: X \to Y$, where $L_1 = L_1(L, A, X, Y) \to 1$ as $L \to 1$. This definition is from [V, p. 239].

In this paper we prove that a compact polyhedron $X \subset \mathsf{R}^n$ has the BLEP in (R^n, Y) whenever Y is a linear subspace of the Hilbert space l_2 with $\dim Y \geqq n$. In the special case where X is a finite union of n-simplexes in R^n, this follows from [V, 6.2]. This result and regular neighborhoods allow us to reduce the theorem to proving that the BLEP in (R^n, Y) is preserved under an elementary simplicial collapse $K' \searrow K$ in R^n; the definition of collapsing will be recalled at the beginning of the proof of Theorem 1.2. This reduction is accomplished in Section 1.

Supposing that K' collapses to K through a p-simplex Δ we prove in Sections 2–5 that the BLEP of $|K'|$ indeed implies the BLEP of $|K|$. Our method resembles that used in [TV] to prove that R^p has the BLEP in $(\mathsf{R}^n, \mathsf{R}^n)$ for $1 \leqq p \leqq n - 1$.

Sections 2 and 3 contain some auxiliary constructions. We first consider a normalized situation where Δ is a standard p-simplex; in the general case we use an auxiliary affine map which carries the standard simplex onto Δ. We obtain a Whitney type decomposition $\mathscr{A}$ of a set $A = \cup\mathscr{A}$ containing $\Delta\setminus|K|$. The elements of $\mathscr{A}$ are p-cubes. To each $Q \in \mathscr{A}$ we associate a set $E_Q \subset |K|$ near Q and of roughly the same size as Q.

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In Section 4 we begin the task of extending a given L-BL map $f\colon |K| \to Y$ to $|K'| = |K| \cup \Delta$. If $Q \in \mathscr{A}$, we let T_Q and T_Q^* be the affine subspaces of R^n spanned by E_Q and $E_Q^* = E_Q \cup \Delta$, respectively. We first approximate f in each E_Q by an isometry $h_Q\colon T_Q \to Y$. Then we extend these isometries h_Q to isometries $h_Q^*\colon T_Q^* \to Y$ in such a way that if $Q, R \in \mathscr{A}$ intersect, then h_Q^* and h_R^* do not differ much in $Q \cup R$. This is a crucial and laborious technical step in our proof.

In Section 5 we obtain a Whitney triangulation $\mathscr{T}$ of A by triangulating each cube $Q \in \mathscr{A}$ in a suitable way. For each vertex v of $\mathscr{T}$ we then choose a cube $Q(v) \in \mathscr{A}$ with $v \in Q(v)$. The desired extension g of f is obtained by setting $g(v) = h_{Q(v)}^*(v)$ for the vertices v of $\mathscr{T}$ and extending affinely to the simplexes of $\mathscr{T}$. We prove that if $L = 1 + \varepsilon$ with ε small enough, then all the steps described above are possible and that the map $g\colon |K'| \to Y$ is L_1-BL with $L_1 = L_1(L, K', Y) \to 1$ as $L \to 1$.

In Section 6 we apply our main theorem to show that also certain unbounded polyhedra have the BLEP in $(\mathsf{R}^n, \mathsf{R}^n)$.

We thank the referee for pointing out and filling a gap in the proof of 4.18 and for other suggestions and corrections.

NOTATION. Our notation on PL topology is fairly standard. Given a set K of simplexes, we write $|K| = \cup K$. If $\tau \in K$, we let $\operatorname{st}(\tau, K)$ denote the set of the simplexes of K containing τ. The sets of the vertices of K and τ are written as K^0 and τ^0, respectively.

If $1 \leqq p \leqq n - 1$, we identify R^p with the subset $\{x : x_{p+1} = \dots = x_n = 0\}$ of R^n. The distance between two sets A, B in a metric space is written as $d(A, B)$ with the agreement that $d(A, B) = \infty$ if A or B is empty. The diameter of A is $d(A)$ with $d(\emptyset) = 0$. We let $T(A)$ denote the affine subspace of R^n spanned by a set $A \subset \mathsf{R}^n$. For $r > 0$ we set

$$\bar{B}^n(A, r) = \{x \in \mathsf{R}^n : d(x, A) \leqq r\}.$$

In the Hilbert space l_2 of all square summable sequences of real numbers, we let $x \cdot y$ denote the inner product of x and y, and $|x| = (x \cdot x)^{1/2}$ is the norm of x. If f and g are two functions defined in a set A and with values in l_2, we write

$$|f - g|_A = \sup_{x \in A} |fx - gx|.$$

We usually omit parentheses writing fx instead of $f(x)$. A map $f\colon A \to l_2$ with $A \subset \mathsf{R}^n$ is a *similarity* if there is $\lambda > 0$ such that

$$|fx - fy| = \lambda |x - y|$$

for all $x, y \in A$. Every similarity $f\colon A \to l_2$ is the restriction of a unique affine similarity $g\colon T(A) \to l_2$. The corresponding statement for isometries is also true.

We let R, Z and N denote the set of real numbers, integers and nonnegative integers, respectively.

1. The main result and a reduction.

This paper is devoted to proving the following result:

1.1. THEOREM. *Let $X \subset \mathsf{R}^n$ be a compact polyhedron and let Y be a linear subspace of the Hilbert space l_2 with* $\dim Y \geqq n$. *Then X has the* BLEP *in* (R^n, Y).

PROOF. Let N be a regular neighborhood of X in R^n. Since N is a finite union of n-simplexes, N is *thick* in R^n in the sense of [V, 6.1]. Hence N has the BLEP in (R^n, Y) by [V, 6.2].

A standard result of PL topology (cf. [Gl, p. 77]) gives a triangulation (N', X') of (N, X) such that N' collapses to X' through a finite sequence of elementary simplicial collapses

$$N' = N_0 \searrow N_1 \searrow \dots \searrow N_s = X'.$$

Since $|N'| = N$ has the BLEP in (R^n, Y) and since $|X'| = X$, we have reduced the theorem to the following result:

1.2. THEOREM. *Let K and K' be finite simplicial complexes in* R^n *and suppose that there is an elementary simplicial collapse $K' \searrow K$ through a p-simplex Δ, $1 \leqq p \leqq n$. Suppose also that $|K'|$ has the* BLEP *in* (R^n, Y). *Then $|K|$ has the* BLEP *in* (R^n, Y).

PROOF. Let $v_0, \dots, v_p$ be the vertices of Δ. Then Δ is their join $v_0 \dots v_p$, and the $(p-1)$-faces of Δ are the simplexes $\sigma_i = v_0 \dots \hat{v}_i \dots v_p$ with vertices $v_j, j \neq i$. The collapsing condition $K' \searrow K$ through Δ means that $K' \backslash K = \{\Delta, \sigma_i\}$ for some i. We may assume that $K' \backslash K = \{\Delta, \sigma_0\}$. Using an auxiliary isometry of R^n, we may also normalize the situation so that $v_0 = 0$ and $\Delta \subset \mathsf{R}^p$.

We divide the proof of Theorem 1.2 into four parts, which are presented in Sections 2–5.

If $p = 1$, it is possible that $\{0\}$ is a isolated simplex of K. This easy special case will be considered in 5.20. Until then, we assume that 0 is not isolated in $|K|$.

2. The decomposition.

We shall construct a Whitney type decomposition $\mathscr{A}$ of a suitable set containing $\Delta \backslash |K|$. Let $e_0 = 0$ and let $(e_1, \dots, e_p)$ be the standard basis of R^p. We first consider a special case, assuming that $v_i = e_i$ for $0 \leqq i \leqq p$ until the end of Section 3.

2.1. NOTATION. For each nonempty subset ν of $\{1, \dots, p\}$ we set

$$T_\nu = \{x \in \mathsf{R}^p : x_j = 0 \ \text{ for all } \ j \in \nu\}.$$

We let q_v denote the orthogonal projection $q_v\colon \mathsf{R}^p \to T_v$. As a special case we obtain the coordinate hyperplanes $T_i = T_{\{i\}} = T(\sigma_i)$ of R^p. We set $\mathsf{R}^p_0 = T_1 \cup \ldots \cup T_p$.

2.2. THE CUBE FAMILY $\mathscr{J}$. Setting $I = [0, 1]$ we have $\Delta \subset I^p \subset 2I^p = [0, 2]^p$. We define some auxiliary families of p-cubes of R^p. First, let $\mathcal{J}_{-1} = \{2I^p\}$. Proceeding inductively, we obtain $\mathcal{J}_{k+1}$ from $\mathcal{J}_k$ by bisecting the sides of each cube in $\mathcal{J}_k$. Let $\mathcal{J}$ be the union of all $\mathcal{J}_k$, $k \geqq -1$, $k \in \mathsf{Z}$.

For any cube $Q \subset \mathsf{R}^p$, we let λ_Q denote the side length and z_Q the center of Q. If $Q \in \mathcal{J}$, we let $k(Q)$ denote the unique integer with $Q \in \mathcal{J}_{k(Q)}$. Then $\lambda_Q = 2^{-k(Q)}$ for $Q \in \mathcal{J}$.

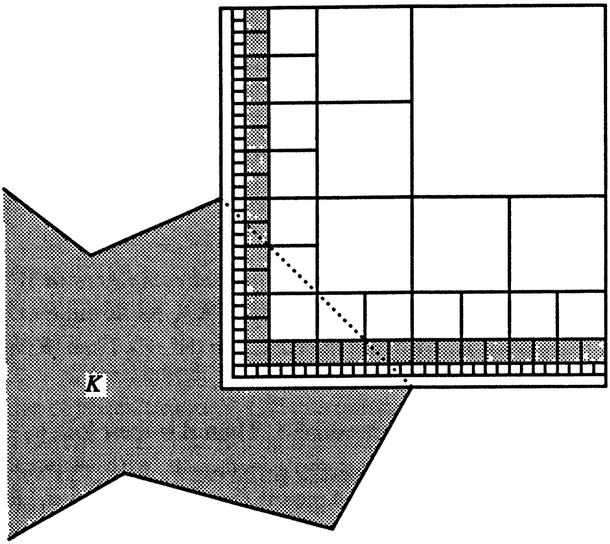

Figure 1. The decomposition $\mathscr{J}$.

More interesting is the subfamily $\mathscr{J} = \cup_{k=0}^{\infty} \mathscr{J}_k$ of $\mathcal{J}$ where

$$\mathscr{J}_k = \{Q \in \mathcal{J}_k : d(Q, \mathsf{R}^p_0) = \lambda_Q = 2^{-k}\}.$$

The cubes $Q \in \mathscr{J}$ give a decomposition of $2I^p \setminus \mathsf{R}^p_0$ into closed p-cubes with disjoint interiors. In Figure 1 we have $p = 2$, but the reader should remember that the corresponding three-dimensional picture is a better guide to a sufficient understanding of some important phenomena. The members of $\mathscr{J}_3$ are shaded. The simplex Δ is in the lower left corner, and σ_0 is the dotted line segment.

If $Q \subset \mathsf{R}^p$ is a p-cube and $t > 0$, we let $Q(t)$ be the p-cube of R^p with center z_Q, side length $t\lambda_Q$ and edges parallel to those of Q.

We state without proof some obvious properties of $\mathscr{J}$:

2.3. LEMMA. (1) *Let* $Q \in \mathscr{J}_k$, $R \in \mathscr{J}_{k+1}$ *and let* $Q \cap R \neq \emptyset$. *Then* $R(t) \subset Q(t)$ *for* $t \geqq 3$.

(2) *Let* $Q, R \in \mathscr{J}_k$ *and let* $Q \cap R \neq \emptyset$. *Then* $R(t) \subset Q(t+2)$ *for all* $t > 0$.

(3) *Let* $Q_1, \ldots, Q_s$ *be a sequence of cubes in* $\mathscr{J}$ *such that* $k(Q_{j+1}) = k(Q_j) + 1$ *and* $Q_{j+1} \cap Q_j \neq \emptyset$ *for all* $j \in \{1, \ldots, s-1\}$. *Then* $Q_s \subset Q_1(3)$.

2.4. NOTATION. We let $\# S$ denote the cardinality of a set S. If $Q \in \mathscr{J}$, we define

$$\nu_Q = \{j : d(Q, T_j) = 2^{-k(Q)}\}, \; l_Q = \# \nu_Q.$$

Then $1 \leqq l_Q \leqq p$. If $1 \leqq j \leqq p$ and $k \in \mathbf{N}$, we set

$$\mathscr{J}_k^j = \{Q \in \mathscr{J}_k : j \in \nu_Q\} = \{Q \in \mathscr{J}_k : d(Q, T_j) = 2^{-k}\}.$$

Then $\mathscr{J}_k = \mathscr{J}_k^1 \cup \ldots \cup \mathscr{J}_k^p$. In Figure 1, $\mathscr{J}_3^1$ is the vertical and $\mathscr{J}_3^2$ the horizontal row of shaded squares.

2.5. PREDECESSORS AND FOLLOWERS. Suppose that $R \in \mathscr{J}_k$ with $k \geqq 1$. Then there is a unique cube Q in $\mathscr{J}_{k-1}$ satisfying the conditions

$$\nu_R \subset \nu_Q, \; q_{\nu_R} R \subset q_{\nu_R} Q, \; Q \cap R \neq \emptyset. \tag{2.6}$$

We say that Q is the *predecessor* of R and R is a *follower* of Q, and we write $R \lhd Q$ and $Q \rhd R$. We let $\mathscr{F}(Q)$ denote the family of all followers of Q. Then $\mathscr{F}(Q)$ is the union of the mutually disjoint families

$$\mathscr{F}^\nu(Q) = \{R \in \mathscr{F}(Q) : \nu_R = \nu\},$$

$\emptyset \neq \nu \subset \nu_Q$. In particular, for $j \in \nu_Q$ we have the sets $\mathscr{F}^j(Q) = \mathscr{F}^{\{j\}}(Q)$, each containing 2^{p-1} cubes. We let P_Q^j be the unique cube of $\mathscr{F}^j(Q)$ closest to the origin. We call P_Q^j the *principal follower* of Q in the direction $j \in \nu_Q$. If $l_Q = 1$, Q has only one principal follower, written as P_Q. This is the case with most cubes in $\mathscr{J}$. For example, if $p = 2$, then only the corner cube of $\mathscr{J}_k$ has $l_Q = 2$; for all other cubes of $\mathscr{J}_k$ we have $l_Q = 1$.

2.7. THE FAMILY $\mathscr{A}$. The p-simplex $\Delta = e_0 \ldots e_p$ is a corner of the cube $2I^p$. We set $\varrho = \sigma_1 \cup \ldots \cup \sigma_p = \Delta \cap \mathbf{R}_0^p$. We are mainly interested in cubes $Q \in \mathscr{J}$ sufficiently near ϱ. We define a family $\mathscr{A}$ of such cubes:

$$\mathscr{A} = \bigcup_{k=0}^{\infty} \mathscr{A}_k, \; \mathscr{A}_k = \{Q \in \mathscr{J}_k : Q(5) \cap \varrho \neq \emptyset\}.$$

We also set

$$A = \cup \mathscr{A}, \quad \mathscr{A}_k^j = \mathscr{A} \cap \mathscr{J}_k^j,$$

where $1 \leqq j \leqq p, k \in \mathbf{N}$.

2.8. LEMMA. (1) $\Delta \setminus \varrho \subset A$.

(2) *If* $R \in \mathscr{A}_{k+1}$, $R \lhd Q \in \mathscr{J}_k$, $S \in \mathscr{J}_k$ *and* $Q \cap S \neq \emptyset$, *then* $S \in \mathscr{A}_k$.

(3) *If* $R \in \mathscr{A}_{k+1}$, $Q \in \mathscr{J}_k$ *and* $Q \cap R \neq \emptyset$, *then* $Q \in \mathscr{A}_k$.

(4) *If* $Q, R \in \mathscr{J}_k$, $Q \cap R \neq \emptyset$ *and* $R \lhd S$, *then* $Q \cap S \neq \emptyset$.

PROOF. The statement (1) is obvious, (3) follows from Lemma 2.3(1), and (4) is easy to verify. To prove (2), assume its situation and observe that it suffices to prove that $R(5) \cap \varrho \subset S(5)$. Let $x \in R(5) \cap \varrho$ and $i \in \{1, \dots, p\}$. If $i \in \nu_R$, then (2.6) gives $i \in \nu_Q$. Hence $0 \leqq x_i \leqq z_{Ri} + 5\lambda_R/2 = 2\lambda_Q$ and $z_{Qi} = 3\lambda_Q/2$. This implies

$$|x_i - z_{Si}| \leqq |x_i - z_{Qi}| + |z_{Qi} - z_{Si}| \leqq 3\lambda_Q/2 + \lambda_Q = 5\lambda_S/2.$$

If $i \notin \nu_R$, we get

$$|x_i - z_{Si}| \leqq |x_i - z_{Ri}| + |z_{Ri} - z_{Qi}| + |z_{Qi} - z_{Si}| \leqq 5\lambda_R/2 + \lambda_R/2 + \lambda_S = 5\lambda_S/2.$$

It follows that $x \in S(5)$, and thus $R(5) \cap \varrho \subset S(5)$.

2.9. THE RUBIK CUBES AND BOXES. Let C be a p-cube. By dividing all edges of C into three equal parts we get a subdivision of C into 3^p subcubes. The family Γ of these 3^p cubes is called a *Rubik p-cube*. Let $0 \leqq r \leqq p$, and let B be an r-dimensional face of C. The family

$$\Gamma_1 = \Gamma_1(\Gamma, B) = \{Q \in \Gamma : Q \cap B \neq \emptyset\} \tag{2.10}$$

will be called an *r-face* of Γ. Such cube families Γ_1 are also called *Rubik* (p, r)*-boxes*. The members of Γ_1 containing a vertex of the r-cube B are called the *vertex cubes* of Γ_1. We let Γ_1' denote the family of all 2^r vertex cubes of Γ_1. Given a Rubik (p,r)-box Γ_1, $0 \leqq r \leqq p - 1$, its representation in the form (2.10) is not unique, but Γ_1' is clearly independent of the representation. Some Rubik $(3, r)$-boxes are shown in Figure 2; the vertex cubes are shaded.

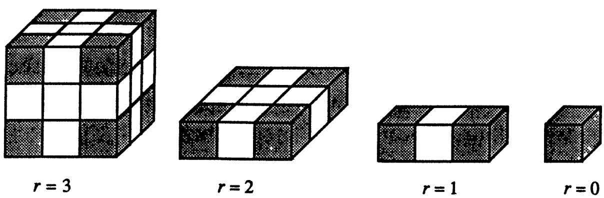

Figure 2. Rubik $(3, r)$-boxes

Let $j \in \{1, \dots, p\}$ and $k \in \mathbf{N}$. We consider the family

$$\mathscr{P}^j_{k+1} = \{P^j_Q : Q \in \mathscr{J}^j_k\}, \tag{2.11}$$

consisting of all principal followers of the cubes $Q \in \mathscr{J}^j_k$ in the direction j; for definitions, consult 2.4 and 2.5. Then $\nu_R = \{j\}$ for all $R \in \mathscr{P}^j_{k+1}$. The following statement is obviously true:

2.12. LEMMA. *Let $Q \in \mathscr{J}_{k+1}$. Suppose that $\nu_Q = \{j\}$ and that Q does not meet any of the hyperplanes $x_i = 2$, $1 \leqq i \leqq p$. Then Q belongs to a unique Rubik (p,r)-box Γ_Q with $\Gamma'_Q \subset \mathscr{P}^j_{k+1}$ and with $r \in \{0,\dots,p-1\}$ minimal. In fact, we have*

$$\Gamma'_Q = \{R \in \mathscr{P}^j_{k+1} : R \cap Q \neq \emptyset\}.$$

A somewhat less obvious result is:

2.13. LEMMA. *Suppose that $Q, R \in \mathscr{A}_k$, $k \geqq 2$, $\nu_Q = \nu_R = \{j\}$, and $Q \cap R \neq \emptyset$. Then there exists a Rubik $(p, p-1)$-box $\Gamma \subset \mathscr{J}^j_k$ containing Q and R such that the elements of Γ' are principal followers of some members of $\mathscr{A}_{k-1}$:*

$$\Gamma' = \{P^j_S : S \in \Gamma_0\}, \Gamma_0 \subset \mathscr{A}_{k-1}.$$

Moreover, $S_1 \cap S_2 \neq \emptyset$ for all $S_1, S_2 \in \Gamma_0$.

PROOF. Since $k \geqq 2$, the cubes Q and R do not meet any of the hyperplanes $x_i = 2$. Hence there clearly exists a Rubik $(p, p-1)$-box $\Gamma \subset \mathscr{J}^j_k$ containing Q and R such that $\Gamma' = \{P^j_S : S \in \Gamma_0\}$ for some $\Gamma_0 \subset \mathscr{J}_{k-1}$. Moreover, $S_1 \cap S_2 \neq \emptyset$ for all $S_1, S_2 \in \Gamma_0$, and the predecessors of Q and R belong to Γ_0. By Lemma 2.8(2) we have $\Gamma_0 \subset \mathscr{A}_{k-1}$.

3. Corners and estates.

In this section, we associate to every $Q \in \mathscr{A}$ an *estate* $E_Q \subset |K|$ in such a way that the numbers $d(E_Q)$, λ_Q and $d(Q, E_Q)$ are roughly equal. Moreover, if the affine subspace $T_Q = T(E_Q)$ spanned by E_Q is m_Q-dimensional, we want E_Q to contain the vertices of an m_Q-simplex σ_Q, which also is of the size λ_Q and not too flat.

3.1. CORNERS. We have already called the simplex $\Delta = e_0 \dots e_p$ a corner. More generally, we say that a set $\Theta \subset \mathsf{R}^p$ is a *corner* if there are $v \in \mathsf{R}^p$ and $\lambda > 0$ such that $\Theta = v + \lambda\Delta$. Here v and λ are uniquely determined by Θ. We say that v is the *basic vertex* and λ is the *size* of the corner Θ. Observe that a point $x \in \mathsf{R}^p$ is in Θ if and only if

$$\begin{aligned} &x_j \geqq v_j \text{ for all } j \in \{1,\dots,p\}, \\ &\sum_{j=1}^{p} (x_j - v_j) \leqq \lambda. \end{aligned} \tag{3.2}$$

Hence Θ is the intersection of the $p+1$ half spaces $x_j \geqq v_j$, $1 \leqq j \leqq p$, and $\sum_{j=1}^p x_j \leqq \sum_{j=1}^p v_j + \lambda$ of R^p. Conversely, the intersection of half spaces of the form $x_j \geqq v_j$ and $\sum_{j=1}^p x_j \leqq t$ is always a corner or a point or the empty set. From this we obtain:

3.3. LEMMA. *The intersection of two corners is either a corner or a point or empty.*

If Θ and Θ' are corners in R^p, there is a unique homeomorphism $f\colon \Theta \to \Theta'$ of the form $fx = \lambda x + a, \lambda > 0$. In the following lemma we set $f\sigma = \sigma'$ whenever σ is a face of Θ:

3.4. LEMMA. *Suppose that Θ and Θ' are corners in R^p such that $\Theta' \subset \Theta$, $\Theta' \neq \Theta$, and $\Theta' \cap \partial\Theta \neq \emptyset$. Then the following statements are true:*

(1) *If Θ' meets a $(p-1)$-face σ of Θ, then $\sigma' = \sigma \cap \Theta'$.*

(2) *There is a unique proper face τ of Θ such that $\Theta' \cap \partial\Theta = |\mathrm{st}(\tau', \partial\Theta')|$, where $\partial\Theta'$ is viewed as a complex in the natural way. In fact, τ is the intersection of all $(p-1)$-faces of Θ meeting Θ'.*

(3) $\tau' \subset \tau$.

(4) *If τ_1 is a face of Θ, then $\tau_1 \cap \Theta' \neq \emptyset \Leftrightarrow \tau \subset \tau_1$.*

PROOF. (1) The hyperplanes $T(\sigma)$ and $T(\sigma')$ are parallel. If $T(\sigma) \neq T(\sigma')$, then the condition $\Theta' \subset \Theta$ implies that $T(\sigma)$ and Θ' are on different sides of $T(\sigma')$. This is impossible, because $\sigma \cap \Theta' \neq \emptyset$. It follows that $T(\sigma) = T(\sigma')$, and hence $\sigma' = \sigma \cap \Theta'$.

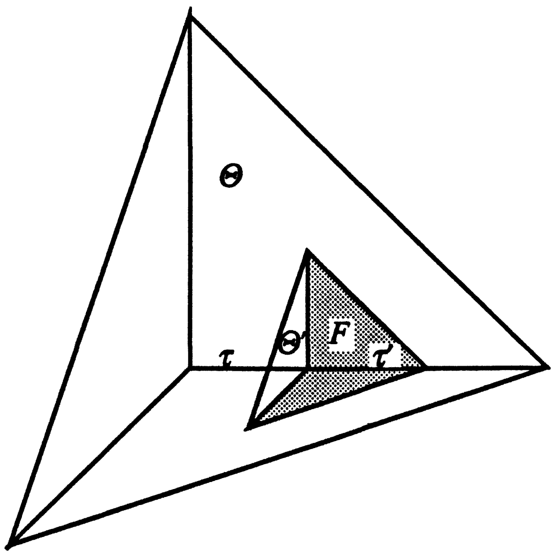

Figure 3

(2) Set $F = \Theta' \cap \partial\Theta$. Figure 3 illustrates a typical situation with $p = 3$, $\dim \tau = 1$; the set F is shaded. Let $a_0, \ldots, a_p$ be the vertices of Θ, and let σ_j be the $(p-1)$-face of Θ opposite to a_j, $0 \leq j \leq p$. Write

$$J_p = \{0, \ldots, p\}, J = \{j \in J_p : \sigma_j \cap \Theta' \neq \emptyset\}.$$

By (1) we have $F = \cup \{\sigma'_j : j \in J\}$. Since $\emptyset \neq F \neq \partial\Theta$, we have $\emptyset \neq J \neq J_p$. It is now easy to see that the simplex $\tau = \cap \{\sigma_j : j \in J\}$ with vertices $a_j, j \in J_p \setminus J$, is the unique face of Θ satisfying the condition

$$F = \bigcup_{j \in J} \sigma'_j = |\mathrm{st}(\tau', \partial\Theta')|.$$

(3) By the proof of (2), we have

$$\tau = \bigcap_{j \in J} \sigma_j, \tau' = \bigcap_{j \in J} \sigma'_j.$$

By (1) we have $\sigma'_j \subset \sigma_j$ for all $j \in J$. Hence $\tau' \subset \tau$.

(4) If $\tau \subset \tau_1$, then (3) implies that $\emptyset \neq \tau' \subset \tau_1 \cap \Theta'$. Conversely, suppose that $\tau_1 \cap \Theta' \neq \emptyset$. If $\tau_1 = \Theta$, then trivially $\tau \subset \tau_1$. If $\tau_1 \neq \Theta$, then τ_1 is the intersection of all $(p-1)$-faces σ of Θ containing it. For each such σ, the assertion (2) implies that $\tau \subset \sigma$, because $\emptyset \neq \tau_1 \cap \Theta' \subset \sigma \cap \Theta'$. Hence $\tau \subset \tau_1$.

3.5. Estates. Let us consider the situation of Theorem 1.2, where $K' = K \cup \{\Delta, \sigma_0\}$. We first assume that $\Delta = e_0 \ldots e_p$ is the standard corner of R^p.

Recall the cube family $\mathscr{A}$ from 2.7 and suppose that $Q \in \mathscr{A}$. We let Δ_Q denote the smallest corner containing the cube $Q(7)$. Then the size of Δ_Q is $7p\lambda_Q$, and the basic vertex of Δ_Q is

$$v_Q = z_Q - \frac{7\lambda_Q}{2} \sum_{j=1}^{p} e_j.$$

Hence Δ_Q is the set of all $x \in \mathrm{R}^p$ satisfying the inequalities

$$x_j \geqq z_{Qj} - 7\lambda_Q/2, \sum_{j=1}^{p} (x_j - z_{Qj}) \leqq 7p\lambda_Q/2.$$

Writing $\varrho = \Delta \cap \mathrm{R}^p_0$ as in 2.7 we set

(3.6) $$E^0_Q = \Delta_Q \cap \varrho, K_Q = \{\tau \in K : \tau \cap E^0_Q \neq \emptyset\}.$$

We define the *estate* E_Q of Q by the formula

(3.7) $$E_Q = \cup \{\bar{B}^n(\tau \cap E^0_Q, \lambda_Q) \cap \tau : \tau \in K_Q\}.$$

3.8. Remark. We list some observations about the estates E_Q:

1. Let $Q \in \mathscr{A}$. Since $Q(5) \cap \varrho \neq \emptyset$ and since $Q(7) \subset \Delta_Q$, the set $\Delta'_Q = \Delta_Q \cap \Delta$ is a corner by Lemma 3.3. Moreover, we have $\Delta'_Q \cap \varrho = E^0_Q \neq \emptyset$.

If $\Delta'_Q \neq \Delta$, we can apply Lemma 3.4 (2) with the substitution $\Theta' \mapsto \Delta'_Q, \Theta \mapsto \Delta$. We get a proper face τ of Δ such that

$$\Delta'_Q \cap \partial\Delta = |\mathrm{st}(\tau', \partial\Delta'_Q)|.$$

Here $\tau' \neq \sigma'_0$, and we can write

$$E_Q^0 = |\mathrm{st}(\tau', \varrho')|,$$

where ϱ' corresponds ϱ viewed as a complex in the natural way.

The case $\Delta'_Q = \Delta$ can only occur for a finite number of cubes $Q \in \mathscr{A}$. In this case we have

$$E_Q^0 = \varrho = |\mathrm{st}\,(0, \varrho)|.$$

2. The diameter $d(\Delta_Q)$ is $7p\lambda_Q\sqrt{2}$ if $p \geqq 2$ and $7\lambda_Q$ if $p = 1$. Hence we have

$$d(E_Q) \leqq (7p\sqrt{2} + 2)\lambda_Q$$

for all $Q \in \mathscr{A}$.

3.9. LEMMA. *Suppose that $Q, R \in \mathscr{A}$.*
(1) *If $Q \cap R \neq \emptyset$, then $E_Q^0 \cap E_R^0 \neq \emptyset$.*
(2) *If $Q \cap R \neq \emptyset$ and $k(R) = k(Q) + 1$, then $E_R^0 \subset E_Q^0$ and $E_R \subset E_Q$.*

PROOF. We first prove (2). Suppose that $Q \cap R \neq \emptyset$ and $k(R) = k(Q) + 1$. By 2.3(1), (3.6) and the definition of Δ_Q, we have $E_R^0 \subset E_Q^0$. Since $\lambda_R < \lambda_Q$, we get $E_R \subset E_Q$ from (3.6) and (3.7).

To prove (1), observe that by (2) we may assume that $k(R) = k(Q)$. Then $Q(5) \subset R(7)$ by 2.3(2), and hence $Q(5) \subset \Delta_Q \cap \Delta_R$. Since $Q \in \mathscr{A}$, we get $\emptyset \neq Q(5) \cap \varrho \subset E_Q^0 \cap E_R^0$.

For $Q \in \mathscr{A}$ we write

$$T_Q = T(E_Q), \quad m_Q = \dim T_Q. \tag{3.10}$$

The affine subspaces T_Q play an important role in the sequel. Since each K_Q contains at least one $\sigma_j = e_0 \ldots \hat{e}_j \ldots e_p$, we have $0 \in T_Q$ for all $Q \in \mathscr{A}$. Hence T_Q is always a linear subspace of R^n. The assumption that 0 is not isolated in $|K|$ implies that $m_Q \geqq 1$ even if $p = 1$.

The *flatness* $F(\alpha)$ of a k-simplex $\alpha = a_0 \ldots a_k \subset \mathsf{R}^n$, $k \geqq 1$, is defined by

$$F(\alpha) = \frac{d(\alpha)}{b(\alpha)}, \tag{3.11}$$

where $b(\alpha)$ is the smallest height of α. Explicitly,

$$b(\alpha) = \min_{0 \leqq j \leqq k} d(a_j, T(\alpha_j)), \quad \alpha_j = a_0 \ldots \hat{a}_j \ldots a_k.$$

The following lemma is the main goal of this section:

3.12. LEMMA. *For each $Q \in \mathscr{A}$ there is a simplex σ_Q such that*

(1) $\sigma_Q^0 \subset E_Q$,
(2) $T_Q = T(\sigma_Q)$,
(3) $c_1 \lambda_Q \leqq d(\sigma_Q) \leqq c_2 \lambda_Q$,
(4) $F(\sigma_Q) \leqq c_3$,

where c_1, c_2, c_3 are positive constants depending only on K'.

PROOF. Let $Q \in \mathscr{A}$ and set $\Delta'_Q = \Delta_Q \cap \Delta$ as in 3.8.1. Since $\Delta'_R = \Delta$ only for a finite number of cubes $R \in \mathscr{A}$, we may assume that $\Delta'_Q \neq \Delta$.

For each face τ of Δ with $\tau \subset \varrho$ we set $N(\tau) = |\mathrm{st}(\tau, K)|$. Since $0 \in N(\tau)$, we can choose a simplex $\sigma(\tau)$ with the properties

$$0 \in \sigma(\tau)^0 \subset N(\tau), \quad T(\sigma(\tau)) = T(N(\tau)). \tag{3.13}$$

It suffices to find a simplex σ_Q such that σ_Q satisfies the conditions (1)–(3) and is similar to some $\sigma(\tau)$.

As in 3.8.1 we can write $E_Q^0 = |\mathrm{st}\,(\tau', \varrho')|$. Here τ' is the face of Δ'_Q corresponding to the intersection τ of all $(p-1)$-faces of Δ meeting Δ'_Q; cf. 3.4(2). By 3.4(3) we have $\tau' \subset \tau$. We first prove that $K_Q = \mathrm{st}(\tau, K)$.

If $\sigma \in \mathrm{st}(\tau, K)$, then $\emptyset \neq \tau' \subset \sigma \cap E_Q^0$, and hence $\sigma \in K_Q$. Conversely, let $\sigma \in K_Q$. Then

$$\emptyset \neq \sigma \cap E_Q^0 = \sigma \cap \Delta'_Q = (\sigma \cap \Delta) \cap \Delta'_Q.$$

Since $\sigma \cap \Delta$ is a face of Δ, Lemma 3.4(4) implies that $\sigma \in \mathrm{st}(\tau, K)$. Hence $K_Q = \mathrm{st}\,(\tau, K)$.

There is $M = M(K') \geqq 1$ such that $d(N(\tau)) \leqq M$. Choose $v \in \tau'$ and consider the radial similarity $f\colon \mathsf{R}^n \to \mathsf{R}^n$ defined by

$$fx = \frac{\lambda_Q}{M}(x - v) + v.$$

Since $N(\tau)$ is starlike with respect to v and since $M \geqq 1 \geqq \lambda_Q$, we have $fN(\tau) \subset N(\tau) = |\mathrm{st}(\tau, K)| = |K_Q|$. Moreover, if $x \in N(\tau)$, then $|fx - v| = \lambda_Q|x - v|/M \leqq \lambda_Q$. Hence $fN(\tau) \subset |K_Q| \cap \bar{B}^n(v, \lambda_Q)$. Since $v \in \tau' \subset \sigma \cap E_Q^0$ for all $\sigma \in K_Q = \mathrm{st}\,(\tau, K)$, we have $fN(\tau) \subset E_Q$.

We show that $\sigma_Q = f\sigma(\tau)$ is the desired simplex. Since

$$\sigma_Q^0 = f\sigma(\tau)^0 \subset fN(\tau) \subset E_Q,$$

the condition (1) holds. Since $K_Q = \mathrm{st}(\tau, K)$, we have $E_Q \subset N(\tau)$. Hence

$$T_Q = T(E_Q) \subset T(N(\tau)) = T(\sigma(\tau)) = T(\sigma_Q) \subset T_Q,$$

which implies (2). Finally, we have $d(\sigma_Q) = \lambda_Q d(\sigma(\tau))/M$. Since K is finite, we get (3).

3.14. THE GENERAL CASE. In Sections 2 and 3 we have so far only considered the special case where $v_i = e_i$ for $i = 0, \dots, p$. Now we return to the situation described in Section 1. There we had $\Delta = v_0 \dots v_p, v_0 = 0$, $K' \setminus K = \{\Delta, \sigma_0\}$.

Let $\varkappa : \mathsf{R}^n \to \mathsf{R}^n$ be a linear isomorphism satisfying $\varkappa e_i = v_i$ for $i = 0, \dots, p$. We do not know yet whether $\varkappa^{-1}|K'|$ has the BLEP in (R^n, Y) or not; cf. [P, 3.7]. Despite of this we can apply the constructions of Sections 2 and 3 to the situation where $\varkappa^{-1}K'$ collapses to $\varkappa^{-1}K$ through $\varkappa^{-1}\Delta = e_0 \dots e_p$. Thus we have the cube families $\mathscr{J}, \mathscr{A}, \mathscr{F}(Q), \mathscr{P}^j_{k+1}$, the Rubik cubes and boxes Γ, the sets A, E^0_Q, K_Q, and the estates E_Q as before.

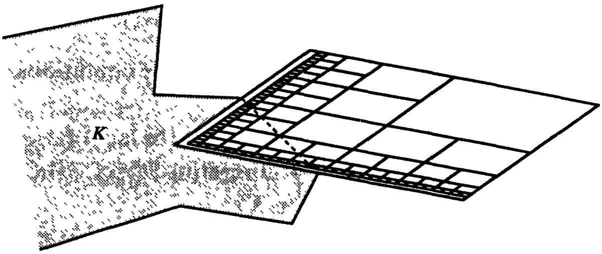

Figure 4. The decomposition $\varkappa\mathscr{J}$

Applying the linear isomorphism $\varkappa$ to these, we get the corresponding notions in the general situation. Thus we get the sets $\varkappa\varrho$, $\varkappa Q$, $\varkappa Q(t)$, $\varkappa A$, $\varkappa E^0_Q$, $\varkappa E_Q$, the families $\varkappa\mathscr{J} = \{\varkappa Q : Q \in \mathscr{J}\}$, $\varkappa\mathscr{A}$, $\varkappa\mathscr{F}(Q)$, $\varkappa\mathscr{P}^j_{k+1}$, $\varkappa\Gamma$ and $\varkappa K_Q$. We also get the linear supspaces $T_j = T(\varkappa^{-1}\sigma_j)$, T_Q, $\varkappa T_j$ and $\varkappa T_Q$ of dimensions $p - 1$ and m_Q. See Figure 4.

For $Q \in \mathscr{A}$ consider the simplex σ_Q given by Lemma 3.12. Choose $H \geqq 1$ such that $\varkappa$ is H-bilipschitz. Then 3.12 yields

$$(3.15) \quad \begin{aligned} &(1)\ \varkappa\sigma^0_Q \subset \varkappa E_Q, \\ &(2)\ \varkappa T_Q = T(\varkappa\sigma_Q), \\ &(3)\ H^{-1}c_1\lambda_Q \leqq d(\varkappa\sigma_Q) \leqq Hc_2\lambda_Q, \\ &(4)\ F(\varkappa\sigma_Q) \leqq H^2 c_3, \end{aligned}$$

where c_1, c_2 and c_3 are the constants of Lemma 3.12, depending only on K'.

4. The isometries h_Q and h^*_Q.

We assume the situation of Theorem 1.2 and the results and notation of Sections 2 and 3 as explained in 3.14. For example, we have the cube family $\mathscr{A}$ and the estates E_Q, $Q \in \mathscr{A}$, given by the constructions of Sections 2 and 3 when applied to the collapse $\varkappa^{-1}K' \searrow \varkappa^{-1}K$ through $\varkappa^{-1}\Delta = e_0 \dots e_p$.

Let $L > 1$ and let $f : |K| \to Y$ be an L-BL map. Since $|K'|$ has the BLEP in

(R^n, Y), Theorem 1.2 will follow if we can extend f to $|K'|$. To be more precise, it suffices to find a number $L_0 = L_0(K') > 1$ such that if $L \leqq L_0$, then f has an L_1-BL extension $g:|K'| \to Y$ with $L_1 = L_1(L, K') \to 1$ as $L \to 1$. We are going to do this by finding a number $q_0 = q_0(K') > 0$ and for every $q \in]0, q_0]$ two numbers $1 < L(q, K') \leqq L_1(q, K')$ with $L_1(q, K') \to 1$ as $q \to 0$ and such that if $L \leqq L(q, K')$, then f has an $L_1(q, K')$-BL extension $g:|K'| \to Y$. The auxiliary parameter q will not be needed before Lemma 4.18.

In the rest of the proof (to the end of Section 5) we replace R^n by $T(|K'|)$. Then n depends on $|K'|$.

We plan to extend f to $|K'| \setminus |K| = \Delta \setminus |K|$ by making use of the fact that $\Delta \setminus |K| \subset \varkappa A$; see 2.8(1) and 3.14. For this we shall use two families of isometries h_Q and h_Q^*, $Q \in \mathscr{A}$. Before defining them we make a useful normalization.

4.1. Normalization. Using auxiliary isometries of l_2, we may assume that $\mathsf{R}^n \subset Y$ and $fv_0 = f(0) = 0$. Since $|K|$ is compact, we can approximate f by an isometry. Applying [V, 3.1] we can find an isometry $h: \mathsf{R}^n \to Y$ such that $h(0) = 0$ and

$$|h - f|_{|K|} \leqq \delta(L, n) d(|K|),$$

where $\delta(L, n)$ is increasing in L and $\delta(L, n) \to 0$ as $L \to 1$. Extending h to a bijective isometry $h: Y \to Y$ and replacing f by $h^{-1}f$, we see that it suffices to consider the case where h is the identity map of Y. Then we have $f(0) = 0$ and

$$|f - \mathrm{id}|_{|K|} \leqq \delta(L, n) d(|K|), \tag{4.2}$$

where id is the identity map. Later on, we let id also denote various other inclusions.

4.3. The isometries h_Q. We shall associate to each $Q \in \mathscr{A}$ an isometry h_Q: $\varkappa T_Q \to Y$ approximating f on $\varkappa E_Q$. First we set $h_Q = \mathrm{id}: \varkappa T_Q \to Y$ for $Q \in \mathscr{A}_0 \cup \mathscr{A}_1$. Next assume that $Q \in \mathscr{A}_k$, $k \geqq 2$. Since $\varkappa$ is H-BL, Remark 3.8.2 implies that $d(\varkappa E_Q) \leqq H(7n\sqrt{2} + 2)\lambda_Q$. Applying again the approximation theorem [V, 3.1] we find an isometry $h_Q: \varkappa T_Q \to Y$ such that

$$|h_Q - f|_{\varkappa E_Q} \leqq \delta(L, n) H(7n\sqrt{2} + 2)\lambda_Q. \tag{4.4}$$

Write

$$\varepsilon(L, K') = \delta(L, n) \max\{2d(|K|), H(7n\sqrt{2} + 2)\}. \tag{4.5}$$

Then (4.2) and (4.4) imply

$$|h_Q - f|_{\varkappa E_Q} \leqq \varepsilon(L, K')\lambda_Q \tag{4.6}$$

for all $Q \in \mathscr{A}$.

A linear map $\varphi: E \to F$ between inner products spaces is called *orthogonal* if it preserves the inner product. Each isometry h: $E \to F$ can be written as $hx = \varphi x + h(0)$, where φ is orthogonal, called the *orthogonal part* of h. In the present situation, we let $\varphi_Q: \varkappa T_Q \to Y$ denote the orthogonal part of h_Q.

For $Q \in \mathscr{A}$ we set

$$E_Q^* = E_Q \cup \varkappa^{-1} \Delta, \; T_Q^* = T(E_Q^*), \; m_Q^* = \dim T_Q^*.$$

Recalling (3.10) we have $T_Q \subset T_Q^*$. Since K_Q contains one of the simplexes $e_0 \dots \hat{e}_j \dots e_p$, we have either $T_Q = T_Q^*$ or $m_Q^* = m_Q + 1$. We want to extend the isometries $h_Q: \varkappa T_Q \to Y$ to isometries $h_Q^*: \varkappa T_Q^* \to Y$ in such a way that if $Q, R \in \mathscr{A}$ intersect, then h_Q^* and h_R^* do not differ much in $\varkappa[Q \cup R]$. To accomplish this, we shall extend the maps $\varphi_Q: \varkappa T_Q \to Y$ to suitable orthogonal maps $\varphi_Q^*: \varkappa T_Q^* \to Y$ in Lemma 4.18. Before that we make some preparations.

If $Q, R \in \mathscr{A}$, we set $T_{QR} = T_Q \cap T_R$ and $\varphi_{QR} = \varphi_Q | \varkappa T_{QR}: \varkappa T_{QR} \to Y$. If $\varphi: E \to F$ is a linear map between normed spaces, we let $|\varphi|$ denote the usual norm of φ:

$$|\varphi| = \sup \{|\varphi x| : x \in E, \; |x| = 1\}.$$

4.7. LEMMA. *For every $t > 0$ there is a number $\bar{L}(t, K') > 1$ such that if $L \leqq \bar{L}(t, K')$, then $|\varphi_{QR} - \varphi_{RQ}| \leqq t$ whenever $Q, R \in \mathscr{A}$ intersect.*

PROOF. Let $t > 0$, and suppose that Q, $R \in \mathscr{A}$ with $Q \cap R \neq \emptyset$. We may assume that $k(R) \geqq k(Q)$. Since $\varphi_S = \mathrm{id}$ for all $S \in \mathscr{A}_0 \cup \mathscr{A}_1$, we may assume that $k(R) \geqq 2$. Obviously we have $k(R) \leqq k(Q) + 1$. We consider two cases:

Case 1. $k(R) = k(Q) + 1$. By 3.9(2) we have $E_R \subset E_Q$, and hence $T_{QR} = T_R$. Consider the simplex σ_R given by Lemma 3.12. If $u \in \varkappa \sigma_R^0$, then $u \in \varkappa E_R \subset \varkappa E_Q$. By (4.6) we get the estimate

$$|h_Q u - h_R u| \leqq |h_Q u - fu| + |fu - h_R u| \leqq 2\varepsilon(L, K')\lambda_Q.$$

Fix $v \in \varkappa \sigma_R^0$. From [V, 2.11] and from (3.15) it follows that for each $x \in \varkappa T_R$ we have

$$\begin{aligned} |h_Q x - h_R x| &\leqq 2\varepsilon(L, K')\lambda_Q(1 + M_1 |x - v| / d(\varkappa \sigma_R)) \\ &\leqq 2\varepsilon(L, K')\lambda_Q + 4\varepsilon(L, K') M_1 H c_1^{-1} |x - v|, \end{aligned}$$

where

$$M_1 = M_1(K') = 4 + 6H^2 c_3 n(1 + H^2 c_3)^{n-1}.$$

Let $x \in \varkappa T_R$, $|x| = 1$. Since $\varphi_S(x) = h_S(x + v) - h_S(v)$ for $S \in \{Q, R\}$, we get

$$\begin{aligned} |\varphi_Q x - \varphi_R x| &\leqq |h_Q(x + v) - h_R(x + v)| + |h_Q v - h_R v| \\ &\leqq 4\varepsilon(L, K')\lambda_Q + 4\varepsilon(L, K') M_1 H c_1^{-1}. \end{aligned}$$

We choose $\bar{L} = \bar{L}(t, K') > 1$ in such a way that $4(1 + 2M_1 H c_1^{-1})\varepsilon(\bar{L}, K') \leqq t$. If $L \leqq \bar{L}$, the estimates above and the fact that $\lambda_Q \leqq 1/2$ imply that

$$|\varphi_{QR} - \varphi_{RQ}| \leqq \varepsilon(L, K')(2 + 4M_1 H c_1^{-1}) \leqq t/2 < t.$$

Case 2. $k(R) = k(Q) = k$. Let $Q \lhd S \in \mathscr{J}_{k-1}$. Then $S \in \mathscr{A}$ by 2.8(2), and hence the maps φ_{SQ} and φ_{SR} are defined. Suppose that $L \leqq \bar{L}$, where $\bar{L}$ is as in Case 1. By 2.8(4) we have $R \cap S \neq \emptyset$. Hence we can apply the argument of Case 1 to obtain the estimate

$$|\varphi_{QR} - \varphi_{RQ}| \leqq |\varphi_Q - \varphi_{SQ}| + |\varphi_{SR} - \varphi_R| \leqq t.$$

4.8. REMARK. We may assume that the function $t \mapsto \bar{L}(t, K')$ given by Lemma 4.7 is strictly increasing on $]0, \infty[$ and that $\bar{L}(t, K') \to 1$ as $t \to 0$. For Lemma 4.18 below, let $\bar{L}(t, K')$ have these properties.

4.9. INTERPOLATION. We next introduce the interpolation technique, which will be our main tool for controlled extension of the maps φ_Q to orthogonal maps φ_Q^*.

In 4.9-4.17 we let E and F be real inner product spaces with inner product written as $x \cdot y$ and the induced norm as $|x|$. We also assume that $1 \leqq \dim E < \infty$.

We begin with a technical lemma:

4.10. LEMMA. *Let $(v_j)_{j \in J}$ be a finite family of unit vectors in F, and let $s = \# J \geqq 1$. Suppose that $|v_i - v_j| \leqq \delta \leqq \sqrt{2}$ for all $i, j \in J$, and set*

$$v_K = \frac{1}{k} \sum_{j \in K} v_j,\ k = \# K,$$

for every nonempty $K \subset J$. Then

(1) $$|v_K|^2 \geqq 1 - \delta^2/2.$$

Moreover, if $L, K \subset J$ and $L \cap K \neq \emptyset$, then

(2) $$|v_K - v_L| \leqq (1 - s^{-1})\delta.$$

In particular, we have $|v_j - v_K| \leqq (1 - s^{-1})\delta$ for all $j \in K$.

PROOF. First observe that

$$2v_i \cdot v_j = |v_i|^2 + |v_j|^2 - |v_i - v_j|^2 \geqq 2 - \delta^2.$$

Using this we obtain

$$k^2 |v_K|^2 = \sum_{j \in K} |v_j|^2 + \sum_{\substack{i, j \in K \\ i \neq j}} v_i \cdot v_j \geqq k + k(k-1)(2 - \delta^2)/2 \geqq k^2(1 - \delta^2/2),$$

which gives (1).

To prove (2), assume that $\# K = k \leqq l = \# L$, and set $u_\alpha = v_i$, $w_\alpha = v_j$ for $\alpha = (i,j) \in K \times L$. Then

$$|v_K - v_L| = \frac{1}{kl}\left|\sum_{\alpha\in K\times L} u_\alpha - \sum_{\beta\in K\times L} \omega_\beta\right|.$$

Choose an index $i_0 \in K \cap L$. Then $w_\beta = v_{i_0}$ for all $\beta \in K \times \{i_0\}$ and $u_\alpha = v_{i_0}$ for all $\alpha \in \{i_0\} \times L$. Since $k \leqq l$, there is a permutation ψ of $K \times L$ satisfying $\psi[K \times \{i_0\}] \subset \{i_0\} \times L$. Then we have $u_{\psi(\beta)} = w_\beta$ for all $\beta \in K \times \{i_0\}$. Hence we obtain

$$|v_K - v_L| \leqq \frac{1}{kl}\sum_{\beta\in K\times L} |u_{\psi(\beta)} - w_\beta| \leqq \frac{(kl-k)\delta}{kl} \leqq (1 - s^{-1})\delta.$$

4.11. Suppose that E_0 is a linear subspace of E and that the orthogonal complement $E_0^\perp$ of E_0 in E is one-dimensional. Let e and $-e$ be the two unit vectors of $E_0^\perp$.

Let $\varphi : E_0 \to F$ be orthogonal, and let $\Psi = (\psi_j)_{j\in J}$ be a finite nonempty family of orthogonal maps $\psi_j : E \to F$. Set $\psi_j^0 = \psi_j | E_0$ and let $0 \leqq \delta \leqq 1/2$. We say that the pair (φ, Ψ) satisfies the *interpolation conditions* with the constant δ if

$$|\psi_i - \psi_j| \leqq \delta,\ |\varphi - \psi_j^0| \leqq \delta^2 \tag{4.12}$$

for all $i,j \in J$. Assuming this, we present a method which gives an orthogonal extension $\varphi^* : E \to F$ of φ.

Define a linear map $\psi : E \to F$ by

$$\psi = \frac{1}{s}\sum_{j\in J} \psi_j,\ s = \# J.$$

Let $q : F \to \varphi E_0$ be the orthogonal projection, and set $a = q\psi e$. We prove in Lemma 4.13 below that $a \neq \psi e$. Hence we can define a unit vector b of F orthogonal to φE_0 by

$$b = \frac{\psi e - a}{|\psi e - a|}.$$

There is a unique orthogonal map $\varphi^* : E \to F$ with $\varphi^* e = b$ and $\varphi^* | E_0 = \varphi$. If we choose the other possibility $-e$ instead of e, the process above gives the same map φ^*. Thus φ^* depends only on the pair (φ, Ψ) satisfying (4.12). We say that φ^* is obtained by *interpolation* from (φ, Ψ).

4.13. Lemma. *In the situation described in* 4.11 *we have* $|a| \leqq \delta^2$, $a \neq \psi e$, *and* $|\varphi^* e - \psi e| \leqq \delta^2\sqrt{5}/2$.

Proof. We assume that $\dim E \geqq 2$; the case $\dim E = 1$ is easier. Choose $x \in E_0$ with $|x| = 1$ and $a = |a|\varphi x$. Since $e \cdot x = 0$ and $a = (\psi e \cdot \varphi x)\varphi x$, (4.12) implies that

$$|a| = \psi e \cdot \varphi x = \frac{1}{s} \sum_{j \in J} [\psi_j e \cdot (\varphi x - \psi_j x) + \psi_j e \cdot \psi_j x]$$

$$\leqq \frac{1}{s} \sum_{j \in J} |\psi_j e| \, |\varphi x - \psi_j x| \leqq \frac{1}{s} \sum_{j \in J} \delta^2 = \delta^2.$$

On the other hand, Lemma 4.10(1) implies that $|\psi e| > \delta^2$, because $\delta \leqq 1/2$. Hence $a \neq \psi e$.

To prove the last inequality of the lemma, consider the vector

$$c = \psi e - a = \psi e - q\psi e.$$

Then

$$|c| \leqq |\psi e| \leqq 1, \; c = |c| \varphi^* e, \; |\psi e|^2 = |a|^2 + |c|^2.$$

Applying 4.10(1) and the estimate $|a| \leqq \delta^2$ we get

$$|c|^2 = |\psi e|^2 - |a|^2 \geqq 1 - \delta^2/2 - \delta^4.$$

Since $\delta \leqq 1/2$, an easy computation shows that

$$|c| \geqq 1 - \delta^2/4 - \delta^4.$$

Since $\varphi^* e - \psi e = (1 - |c|)\varphi^* e - a$, we obtain

$$|\varphi^* e - \psi e|^2 = (1 - |c|)^2 + |a|^2 \leqq (\delta^2/4 + \delta^4)^2 + \delta^4 \leqq 5\delta^4/4,$$

where the last inequality follows from $\delta \leqq 1/2$ by direct computation.

In the next lemma we prepare for 4.18 by deriving some estimates for the extensions obtained by interpolation.

4.14. LEMMA. *Let $0 \leqq \delta \leqq 1/2$ and let $\Psi = (\psi_j)_{j \in J}$ be a finite family of orthogonal maps $\psi_j : E \to F$ satisfying $|\psi_j - \psi_i| \leqq \delta$ for all $i, j \in J$. Let $J_1, J_2 \subset J$ with $J_1 \cap J_2 \neq \emptyset$, and set $s = \# J$, $\Psi_1 = (\psi_j)_{j \in J_1}$, $\Psi_2 = (\psi_j)_{j \in J_2}$. Suppose that $\varphi_1, \varphi_2 : E_0 \to F$ are orthogonal maps such that $|\varphi_1 - \varphi_2| \leqq \delta^2$ and such that the pairs (φ_i, Ψ_i), $i = 1, 2$, satisfy (4.12). Let $\varphi_i^* : E \to F$ be obtained by interpolation from (φ_i, Ψ_i). Then*

(1) $|\varphi_1^* - \varphi_2^*| \leqq (1 - s^{-1})\delta + 3\delta^2$,

(2) $|\varphi_i^* - \psi_j| \leqq (1 - s^{-1})\delta + 3\delta^2/2$

for all $j \in J_i$, $i = 1, 2$.

PROOF. Let $i \in \{1, 2\}$ and set

$$\bar{\psi}_i = \frac{1}{s_i} \sum_{j \in J_i} \psi_j, \; s_i = \# J_i.$$

Then Lemma 4.13 gives

$$|\varphi_i^* e - \bar{\psi}_i e| \leqq \delta^2 \sqrt{5}/2. \tag{4.15}$$

Let u be a unit vector in E. We can write $u = \lambda v + \mu e$, where $v \in E_0$, $|v| = 1$, and $\lambda^2 + \mu^2 = 1$. By (4.12) and (4.15) we obtain

$$|\varphi_i^* u - \bar{\psi}_i u| \leqq |\lambda|\, |\varphi_i v - \bar{\psi}_i v| + |\mu|\, |\varphi_i^* e - \bar{\psi}_i e| \leqq (|\lambda| + |\mu| \sqrt{5}/2)\delta^2.$$

By the Schwarz inequality this yields

$$|\varphi_i^* - \bar{\psi}_i| \leqq 3\delta^2/2.$$

This and 4.10(2) now imply (1):

$$|\varphi_1^* - \varphi_2^*| \leqq |\varphi_1^* - \bar{\psi}_1| + |\bar{\psi}_1 - \bar{\psi}_2| + |\bar{\psi}_2 - \varphi_2^*| \leqq (1 - s^{-1})\delta + 3\delta^2.$$

To prove (2), let $i \in \{1, 2\}$ and $j \in J_i$. The last statement of Lemma 4.10 implies that $|\psi_j - \bar{\psi}_i| \leqq (1 - s^{-1})\delta$. Hence

$$|\varphi_i^* - \psi_j| \leqq |\varphi_i^* - \bar{\psi}_i| + |\bar{\psi}_i - \psi_j| \leqq 3\delta^2/2 + (1 - s^{-1})\delta.$$

4.16. INTERPOLATION AND RESTRICTION. Consider the situation described in 4.11. Let $\varphi^* : E \to F$ be the map obtained by interpolation from (φ, Ψ). Let $E' \subset E$ be a linear subspace with $E' \not\subset E_0$. Then the linear subspace $E_0' = E' \cap E_0$ is a hyperplane in E'. We set $\psi_j' = \psi_j | E'$, $\Psi' = (\psi_j')_{j \in J}$, and $\varphi' = \varphi | E_0'$. Clearly, the pair (φ', Ψ') also satisfies the interpolation conditions (4.12). Let $\varphi'^* : E' \to F$ be the extension of φ' obtained by interpolation from (φ', Ψ'). The maps φ'^* and $\varphi^* | E'$ are not always equal, but they do not differ too much:

4.17. LEMMA. *In the situation above we have* $|\varphi'^* - (\varphi^* | E')| \leqq 3\delta^2$.

PROOF. As in 4.11 we choose a unit vector $e' \in E' \cap (E_0')^\perp$ and define

$$\psi' = \frac{1}{s} \sum_{j \in J} \psi_j' = \psi | E'.$$

Let $q' : F \to \varphi' E_0'$ be the orthogonal projection, and set

$$a' = q' \psi' e', \ b' = \frac{\psi' e' - a'}{|\psi' e' - a'|}.$$

Then $\varphi'^* e' = b'$. Write $e' = \lambda e + \mu v$ with $v \in E_0$, $|v| = 1$, and $\lambda^2 + \mu^2 = 1$. From 4.13 we get

$$|\varphi'^* e' - \psi' e'| \leqq \delta^2 \sqrt{5}/2, |\psi e - \varphi^* e| \leqq \delta^2 \sqrt{5}/2.$$

Since $\psi' = \psi | E'$, these estimates, (4.12) and the Schwarz inequality give

$$\begin{aligned}|\varphi'^* e' - \varphi^* e'| &\leqq |\varphi'^* e' - \psi' e'| + |\psi' e' - \varphi^* e'| \\ &\leqq \delta^2 \sqrt{5}/2 + |\lambda|\,|\psi e - \varphi^* e| + |\mu|\,|\psi v - \varphi v| \\ &\leqq \delta^2 \sqrt{5}/2 + |\lambda|\delta^2 \sqrt{5}/2 + |\mu|\delta^2 < 3\delta^2.\end{aligned}$$

Since $\varphi'^*|E'_0 = \varphi^*|E'_0$, this implies the lemma.

After these preparations we are ready to extend the orthogonal parts φ_Q of the isometries h_Q, $Q \in \mathscr{A}$, chosen in 4.3. This is done in the following central lemma. To formulate it properly, we need the auxiliary parameter q mentioned in the beginning of Section 4.

4.18. LEMMA. *Let $q_1 = 2^{1-p}/9$. There exists a stictly increasing function $q \mapsto L'(q) = L'(q, K')$ from $]0, q_1]$ into $]1, \infty[$ satisfying the following two conditions:*

(1) *$L'(q) \to 1$ as $q \to 0$.*

(2) *If $0 < q \leqq q_1$ and if $L \leqq L'(q)$, then the maps $\varphi_Q : \varkappa T_Q \to Y$ have orthogonal extensions $\varphi_Q^* : \varkappa T_Q^* \to Y$ with the following properties:*

(a) *If $Q \lhd R$, then $|\varphi_{QR}^* - \varphi_{RQ}^*| \leqq q - q^2$, where $\varphi_{QR}^* = \varphi_Q^*|\varkappa T_{QR}^*$, $T_{QR}^* = T_Q^* \cap T_R^*$.*

(b) *If $k(Q) = k(R)$ and $Q \cap R \neq \emptyset$, then $|\varphi_{QR}^* - \varphi_{RQ}^*| \leqq q$.*

PROOF. Let $0 < q \leqq q_1$ and consider the function $\bar{L}(t, K')$ chosen in 4.8. We define

$$L'(q) = L'(q, K') = \bar{L}(q^2, K').$$

Then $L'(q)$ is strictly increasing in q and satisfies (1).

Suppose that $L \leqq L'(q)$. It remains to construct the extensions φ_Q^* satisfying (a) and (b). By 4.7 we already have

$$|\varphi_{QR} - \varphi_{RQ}| \leqq q^2 \tag{4.19}$$

whenever $Q, R \in \mathscr{A}$ intersect.

If $Q \in \mathscr{A}_0 \cup \mathscr{A}_1$, then $\varphi_Q = \mathrm{id}$, and we also define $\varphi_Q^* = \mathrm{id} : \varkappa T_Q^* \to Y$. Then (a) and (b) are trivially true for Q, $R \in \mathscr{A}_0 \cup \mathscr{A}_1$.

Let $k \geqq 1$, and assume inductively that φ_Q^* is defined for all $Q \in \mathscr{A}$ with $k(Q) \leqq k$ so that (a) and (b) are true. Let $Q \in \mathscr{A}_{k+1}$. If $T_Q = T_Q^*$, we of course set $\varphi_Q^* = \varphi_Q$.

Suppose that $T_Q^* \neq T_Q$. Then $m_Q^* = m_Q + 1$ and $v_Q = \{j\}$ for some $j \in \{1, \dots, p\}$. For notation, see 2.4. Since $k + 1 \geqq 2$, Q does not meet any of the planes $x_i = 2$, $1 \leqq i \leqq p$. By Lemma 2.12 there is a unique Rubik (p, r)-box $\Gamma_Q \subset \mathscr{J}_{k+1}^j$ with $Q \in \Gamma_Q$, $\Gamma'_Q \subset \mathscr{P}_{k+1}^j$ and r minimal. In fact, 2.12 gives $\Gamma'_Q = \{R \in \mathscr{P}_{k+1}^j : R \cap Q \neq \emptyset\}$. The family

$$\mathscr{D}(Q) = \{S \in \mathscr{J}_k^j : P_S^j \in \Gamma'_Q\}$$

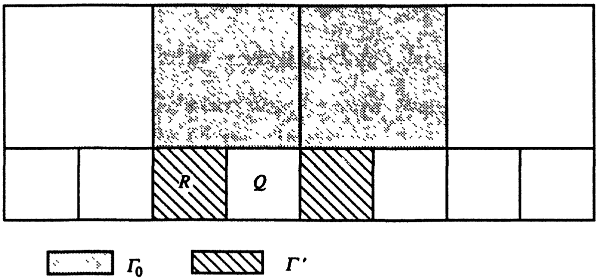

Figure 5.

consists of all predecessors of the members of Γ'_Q. Clearly $S \cap Q \neq \emptyset$ for all $S \in \mathscr{D}(Q)$. Hence we have $\mathscr{D}(Q) \subset \mathscr{A}_k$ by 2.8(3), and thus the maps φ_S^* are defined for all $S \in \mathscr{D}(Q)$. Moreover, by Lemma 3.9(2) we also have $E_Q \subset E_S$ and hence $T_Q^* \subset T_S^*$ for $S \in \mathscr{D}(Q)$. Thus we can define the family $\Psi_Q = (\psi_S)_{S \in \mathscr{D}(Q)}$ by

$$\psi_S = \varphi_S^* \mid \varkappa T_Q^* = \varphi_{SQ}^*.$$

Moreover, if $R, S \in \mathscr{D}(Q)$, then $R \cap S \neq \emptyset$, and the inductive hypothesis implies that

$$|\psi_R - \psi_S| \leqq q.$$

By (4.19) we also have

$$|\varphi_Q - (\psi_S | \varkappa T_Q)| = |\varphi_{QS} - \varphi_{SQ}| \leqq q^2.$$

Hence the interpolation conditions (4.12) hold for the pair (φ_Q, Ψ_Q) with $J = \mathscr{D}(Q)$, $\delta = q$, $E_0 = \varkappa T_Q$, $E = \varkappa T_Q^*$ and $F = Y$. We let $\varphi_Q^* : \varkappa T_Q^* \to Y$ be the orthogonal extension of φ_Q obtained by interpolation from (φ_Q, Ψ_Q).

It remains to verify the conditions (a) and (b). To prove (a), let $Q \in \mathscr{A}_{k+1}$ and let $Q \lhd R$. If $T_Q^* = T_Q$, then $T_Q^* = T_Q \subset T_R = T_R^*$. By (4.19) we obtain

$$|\varphi_{QR}^* - \varphi_{RQ}^*| = |\varphi_{QR} - \varphi_{RQ}| \leqq q^2 < q - q^2.$$

Assume that $T_Q^* \neq T_Q$. Since $R \in \mathscr{D}(Q)$ and since $\# \mathscr{D}(Q) \leqq 2^{p-1}$, Lemma 4.14(2) implies

$$|\varphi_{QR}^* - \varphi_{RQ}^*| = |\varphi_Q^* - \psi_R| \leqq (1 - 2^{1-p})q + 3q^2/2 \leqq q - q^2,$$

because $q \leqq q_1$. Hence (a) is true.

To prove (b), let Q, $R \in \mathscr{A}_{k+1}$, $Q \cap R \neq \emptyset$. If $p = 1$, (b) is trivially true. Assume that $p \geqq 2$. Writing

$$\mathscr{E} = \{Q \in \mathscr{A} : T_Q = T_Q^*\} \tag{4.20}$$

we can divide the proof of (b) into three cases.

Case 1. $Q, R \in \mathscr{E}$. In this case we can apply (4.19):

$$|\varphi^*_{QR} - \varphi^*_{RQ}| = |\varphi_{QR} - \varphi_{RQ}| \leqq q^2 \leqq q.$$

Case 2. $Q, R \in \mathscr{A} \setminus \mathscr{E}$. We prove in Lemma 4.22 below that in this case we have $v_Q = \{j\} = v_R$ for some $j \in \{1, \ldots, p\}$. By Lemma 2.13 there is a Rubik $(p, p-1)$-box $\Gamma \subset \mathscr{J}^j_{k+1}$ containing Q and R such that there is $\Gamma_0 \subset \mathscr{A}_k$ with $\Gamma' = \{P^j_S : S \in \Gamma_0\}$. See Figure 5. Since $\Gamma'_Q = \{P^j_S : S \in \mathscr{D}(Q)\} \subset \Gamma'$, we have $\mathscr{D}(Q) \subset \Gamma_0$. Similarly $\mathscr{D}(R) \subset \Gamma_0$. By 2.13 we then have $S_1 \cap S_2 \neq \emptyset$ for all $S_1, S_2 \in \mathscr{D}(Q) \cup \mathscr{D}(R)$. We next show that $\mathscr{D}(Q) \cap \mathscr{D}(R) \neq \emptyset$. Suppose that this is false. Then $\Gamma'_Q \cap \Gamma'_R = \emptyset$, and since $\Gamma'_R = \{S \in \mathscr{P}^j_{k+1} : S \cap R \neq \emptyset\}$, we obtain $S \cap R = \emptyset$ for all $S \in \Gamma'_Q$. This implies that $d(Q, R) \geqq \lambda_Q$, which is a contradiction, since $Q \cap R \neq \emptyset$. Thus $\mathscr{D}(Q) \cap \mathscr{D}(R) \neq \emptyset$.

Set $E' = \varkappa T^*_{QR}$ and $E'_0 = \varkappa T_{QR}$.

We show that $E'_0 = E' \cap E_0$ or, equivalently, that $T_Q \cap T_R = T_Q \cap T^*_R$. Clearly $T_Q \cap T_R \subset T_Q \cap T^*_R$. Hence it suffices to show that $T_Q \cap T^*_R \subset T_R$. By the second part of Lemma 4.22 below, we have $T_Q \subset T_R$ or $T_R \subset T_Q$. The former case is trivial; assume that $T_R \subset T_Q$. Since T^*_R is spanned by $T_R \cup \{e_j\}$ and since $e_j \notin T_Q$, we obtain $T_Q \cap T^*_R = T_R$.

We now have the situation of 4.16. We write $\psi'_S = \psi_S \mid E'$ for $S \in \mathscr{D}(Q) \cup \mathscr{D}(R)$, and

$$\Psi'_Q = (\psi'_S)_{S \in \mathscr{D}(Q)},\ \Psi'_R = (\psi'_S)_{S \in \mathscr{D}(R)},\ \varphi'_Q = \varphi_Q \mid E'_0,\ \varphi'_R = \varphi_R \mid E'_0.$$

By (4.19), we have $|\varphi'_Q - \varphi'_R| \leqq q^2$. We let φ'^*_Q and φ'^*_R be the extensions obtained by interpolation from (φ'_Q, Ψ'_Q) and (φ'_R, Ψ'_R), respectively. From Lemma 4.17 we get

$$(4.21) \qquad |(\varphi^*_Q \mid E') - \varphi'^*_Q| \leqq 3q^2,\ |(\varphi^*_R \mid E') - \varphi'^*_R| \leqq 3q^2.$$

If $S, U \in \mathscr{D}(Q) \cup \mathscr{D}(R)$, then $S \cap U \neq \emptyset$ and $|\psi'_S - \psi'_U| \leqq q$. Hence we can apply Lemma 4.14(1) with the substitution $J \mapsto \mathscr{D}(Q) \cup \mathscr{D}(R)$, $J_1 \mapsto \mathscr{D}(Q)$, $J_2 \mapsto \mathscr{D}(R)$, $E \mapsto E'$, $E_0 \mapsto E'_0$, $F \mapsto Y$, $\varphi_1 \mapsto \varphi'_Q$, $\varphi_2 \mapsto \varphi'_R$, $\Psi_1 \mapsto \Psi'_Q$, $\Psi_2 \mapsto \Psi'_R$, $\delta \mapsto q$, $\varphi^*_1 \mapsto \varphi'^*_Q$, $\varphi^*_2 \mapsto \varphi'^*_R$. We get

$$|\varphi'^*_Q - \varphi'^*_R| \leqq (1 - 2^{1-p})q + 3q^2.$$

This and (4.21) imply the desired estimate

$$\begin{aligned} |\varphi^*_{QR} - \varphi^*_{RQ}| &= |(\varphi^*_Q \mid E') - (\varphi^*_R \mid E')| \\ &\leqq |(\varphi^*_Q \mid E') - \varphi'^*_Q| + |\varphi'^*_Q - \varphi'^*_R| + |\varphi'^*_R - (\varphi^*_R \mid E')| \\ &\leqq 9q^2 + (1 - 2^{1-p})q \leqq q, \end{aligned}$$

because $q \leqq q_1$.

Case 3. $Q \in \mathscr{A} \setminus \mathscr{E}$, $R \in \mathscr{E}$. Let $Q \lhd S \in \mathscr{A}_k$. Then $S \cap R \neq \emptyset$ by 2.8(4), and hence

$E_R \subset E_S$ by 3.9(2). Since $R \in \mathscr{E}$, this implies that $T_R^* = T_R \subset T_S = T_S^*$. If $x \in \varkappa T_{QR}^*$ is a unit vector, then $x \in \varkappa T_S^*$, and we obtain

$$\begin{aligned}|\varphi_{QR}^* x - \varphi_{RQ}^* x| &= |\varphi_Q^* x - \varphi_R^* x| \leqq |\varphi_Q^* x - \varphi_S^* x| + |\varphi_S^* x - \varphi_R^* x| \\ &= |\varphi_{QS}^* x - \varphi_{SQ}^* x| + |\varphi_{SR}^* x - \varphi_{RS}^* x| \leqq |\varphi_{QS} - \varphi_{SQ}| + |\varphi_{SR}^* - \varphi_{RS}^*| \\ &= |\varphi_{QS}^* - \varphi_{SQ}^*| + \varphi_{SR} - \varphi_{RS}| \leqq (q - q^2) + q^2 = q,\end{aligned}$$

where the last inequality follows from (a) and (4.19).

In the proof of Lemma 4.18 we needed the following result:

4.22. LEMMA. *Let $p \geqq 2$ and let $Q, R \in \mathscr{A}_k \setminus \mathscr{E}$ with $Q \cap R \neq \emptyset$. Then we have $v_Q = \{j\} = v_R$ for some $j \in \{1, \ldots, p\}$.* Moreover, $T_Q \subset T_R$ or $T_R \subset T_Q$.

PROOF. Obviously we have $l_Q = 1 = l_R$. This means that $v_Q = \{i\}$ and $v_R = \{j\}$ for some $i, j \in \{1, \ldots, p\}$. We must prove that $i = j$.

By the definitions of $\mathscr{A}$, v_Q, v_R, T_Q and T_R we have $\emptyset \neq Q(5) \cap \varrho \subset Q(7) \cap \varrho \subset \varkappa^{-1}\sigma_i$, $\emptyset \neq R(5) \cap \varrho \subset R(7) \cap \varrho \subset \varkappa^{-1}\sigma_j$. This implies that $Q(7) \cap \varkappa^{-1}\mathring{\sigma}_i \neq \emptyset$ and $R(7) \cap \varkappa^{-1}\mathring{\sigma}_j \neq \emptyset$, where $\mathring{\sigma}_i$ and $\mathring{\sigma}_j$ are the interiors of σ_i and σ_j in $\varkappa T_i$ and $\varkappa T_j$, respectively. By Lemma 2.3(2) we have $Q(5) \subset R(7)$, which now implies that $Q(5) \cap \varrho \subset \varkappa^{-1}\sigma_j$. We get

$$Q(7) \cap \varkappa^{-1}\mathring{\sigma}_j \neq \emptyset,$$

and hence we have $T_i \cup T_j \subset T_Q$. Since $T_Q \neq T_Q^*$, this is possible only if $i = j$.

From now on, we always assume that $0 < q \leqq q_1$ and that $L \leqq L'(q)$ where $L'(q)$ is given by Lemma 4.18. We also let φ_Q^*, $Q \in \mathscr{A}$, be the maps of 4.18(2).

4.23. LEMMA. *If $Q, R \in \mathscr{A}$ with $Q \cap R \neq \emptyset$, then*

$$|\varphi_{QR}^* - \varphi_{RQ}^*| < 2q.$$

PROOF. If $k(Q) = k(R)$, this is a direct consequence of 4.18(b). Suppose that $k(Q) = k(R) + 1$, and let $Q \lhd S \in \mathscr{A}$. Then $S \cap R \neq \emptyset$. Hence we can apply 4.18(2), which gives

$$|\varphi_{SQ}^* - \varphi_{QS}^*| \leqq q - q^2, \; |\varphi_{RS}^* - \varphi_{SR}^*| \leqq q.$$

Since $T_Q^* \subset T_{RS}^*$ by 3.9(2), we obtain

$$\begin{aligned}|\varphi_{QR}^* - \varphi_{RQ}^*| &= |(\varphi_R^* | \varkappa T_Q^*) - \varphi_Q^*| \\ &\leqq |(\varphi_R^* | \varkappa T_Q^*) - (\varphi_S^* | \varkappa T_Q^*)| + |(\varphi_S^* | \varkappa T_Q^*) - \varphi_Q^*| \\ &\leqq |\varphi_{RS}^* - \varphi_{SR}^*| + |\varphi_{SQ}^* - \varphi_{QS}^*| \leqq q + (q - q^2) < 2q.\end{aligned}$$

4.24. THE ISOMETRIES h_Q^*. We close Section 4 by defining the isometries h_Q^*: $\varkappa T_Q^* \to Y$ promised in 4.3. To do this, we choose $v \in \varkappa T_Q$ and set

$$h_Q^* x = h_Q v + \varphi_Q^*(x - v)$$

for $x \in \varkappa T_Q^*$. Clearly h_Q^* is independent of the choice of v. If $Q \in \mathscr{A}_0 \cup \mathscr{A}_1$, we have $\varphi_Q^* = h_Q^* = \mathrm{id}$.

5. The extension.

5.1. THE BASIC PLAN. We continue the discussion directly from Section 4. Thus we have the numbers $0 < q \leqq q_1$, $1 < L \leqq L'(q)$ and the isometries $h_Q^* : \varkappa T_Q^* \to Y$ of 4.24 extending the isometries $h_Q : \varkappa T_Q \to Y$ defined in 4.3. We want to find $q_0 = q_0(K') \in]0, q_1]$ and for every $q \in]0, q_0]$ two numbers $1 < L(q, K') \leqq L_1(q, K') = L_1$ such that $L_1 \to 1$ as $q \to 0$ and such that if $L \leqq L(q, K')$, then the L-BL map $f : |K| \to Y$ has an L_1-BL extension $g : |K'| \to Y$. The exact bounds for $q_0(K')$, $L(q, K')$ and $L_1(q, K')$ will remain somewhat implicit. In the course of the proof we introduce new restrictions of the right type for them whenever needed.

5.2. THE TRIANGULATION $\mathscr{T}$. We construct a triangulation $\mathscr{T}$ of the set $A = \cup \mathscr{A}$ such that the triangulation $\varkappa\mathscr{T}$ satisfies a regularity condition needed in the proof of Lemma 5.16 below.

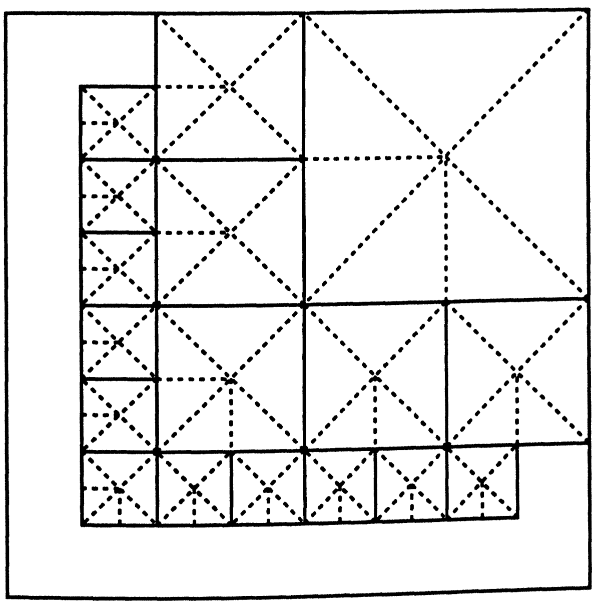

Figure 6. The triangulation $\mathscr{T}$.

Consider the decomposition of A into the closed p-cubes $Q \in \mathscr{A}$ with disjoint interiors. Clearly, there is a 1-dimensional infinite simplicial complex $\mathscr{T}_1$ such that the 1-simplexes of $\mathscr{T}_1$ are the edges of the cubes of $\mathscr{A}$ not containing any other such edge.

If C is a 2-face of some $Q \in \mathscr{A}$, we triangulate C by the cone construction from its center. We get a triangulation $\mathscr{T}_2$ of the union of all 2-faces C of the cubes $Q \in \mathscr{A}$ such that $\mathscr{T}_1$ is a subcomplex of $\mathscr{T}_2$.

Proceeding similarly to faces of higher dimensions, we obtain a finite sequence of simplicial complexes $\mathscr{T}_1 \subset \mathscr{T}_2 \subset \ldots \subset \mathscr{T}_p = \mathscr{T}$ such that each $\mathscr{T}_i$ is a triangulation of the union of all i-faces of the cubes of $\mathscr{A}$. Thus $\mathscr{T}$ is a triangulation of A. See Figure 6.

5.3. The extension g. For each vertex v of $\mathscr{T}$, we choose a cube $Q(v) \in \mathscr{A}$ containing v. We set $h_v = h^*_{Q(v)}: \varkappa T^*_{Q(v)} \to Y$ and let $g_0 : \varkappa A \to Y$ denote the map which is affine in each simplex of $\varkappa\mathscr{T}$ and satisfies $g_0(\varkappa v) = h_v(\varkappa v)$ for each vertex v of $\mathscr{T}$. Since $\Delta \setminus |K| \subset \varkappa A$ by 2.8(1), we can define an extension $g: |K'| \to Y$ of f by letting g agree with f in $|K|$ and with g_0 in $\Delta \setminus |K|$. It remains to prove that g is L_1-BL with $L_1 = L_1(q, K') \to 1$ as $q \to 0$, provided that $q \leqq q_0(K')$ and $L \leqq L(q, K')$.

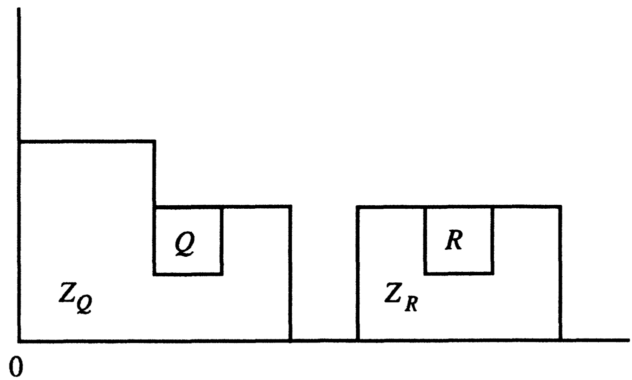

Figure 7. The sets Z_Q.

5.4. The sets Z_Q. For each $Q \in \mathscr{A}$ we let $\mathscr{B}(Q)$ denote the family of all $R \in \mathscr{A}$ such that there is a finite sequence $R = R_0 \lhd R_1 \lhd \ldots \lhd R_s$ in $\mathscr{A}$ with $s \geqq 0$, $k(R_s) = k(Q)$ and $R_s \cap Q \neq \emptyset$. In other words, $\mathscr{B}(Q)$ consists of the cubes $S \in \mathscr{A}_{k(Q)}$ meeting Q, their followers in $\mathscr{A}$, the followers of these in $\mathscr{A}$, etc. The subsets

$$Z_Q = \cup \mathscr{B}(Q)$$

of A, illustrated in Figure 7, have the following properties:

5.5 LEMMA. *Suppose that* $Q \in \mathscr{A}$. *Then*
(1) $Z_Q \subset Q(5)$,
(2) $\bar{Z}_Q \cap \varrho \subset E_Q^0$,
(3) $Q \lhd R$ *implies* $d(Z_Q, A \setminus Z_R) \geqq \lambda_Q$.

PROOF. To prove (1) let $x \in Z_Q$, and choose $R \in \mathscr{B}(Q)$ with $x \in R$. Let $R = R_0 \lhd \ldots \lhd R_s$ be the sequence of $\mathscr{A}$ given by the definition of $\mathscr{B}(Q)$ in 5.4. By 2.3(3) and 2.3(2) we have $R \subset R_s(3) \subset Q(5)$. Hence $x \in Q(5)$.

Since $E_Q^0 = \Delta_Q \cap \varrho$ and $Q(7) \subset \Delta_Q$, (2) follows from (1). If $Q \lhd R$, it is easy to see that $\mathscr{B}(R)$ contains all cubes in $\mathscr{A}$ meeting Q. This implies (3).

5.6. THE APPROXIMATION OF g_0 BY h_Q^* ON $\varkappa Z_Q$. We want to obtain a suitable upper bound for $|g_0 - h_Q^*|_{\varkappa Z_Q}$. To this end, we first choose $L(q) = L(q, K') \in]1, L'(q)]$ in such a way that the function $\varepsilon(L, K')$ given by (4.5) satisfies the restriction

$$\varepsilon(L(q, K'), K') \leqq q. \tag{5.7}$$

Observe that $L(q, K') \to 1$ as $q \to 0$. We may assume that $L(q, K')$ is increasing in $q \in]0, q_1]$. From now on we also assume that $L \leqq L(q, K')$.

5.8. LEMMA. *If* $Q, R \in \mathscr{A}$ *and* $Q \cap R \neq \emptyset$, *then*

$$|h_Q^* - h_R^*|_{\varkappa Z_Q} \leqq M_2 q \lambda_Q, \; |g_0 - h_Q^*|_{\varkappa Z_Q} \leqq 2M_2 q \lambda_Q,$$

where $M_2 = M_2(K')$ *is a positive constant.*

PROOF. By Lemma 3.9(1) we may choose a point $a \in E_Q^0 \cap E_R^0$. Let $y \in \varkappa Z_Q$, and set $x = \varkappa^{-1} y$, $b = \varkappa a$. By 5.5 we have $x \in Q(5)$, and hence a and x are in Δ_Q. By 3.8.2 this implies that $|x - a| \leqq 7p\lambda_Q\sqrt{2}$. Since $\varkappa$ is H-BL, the vector $y - b \in \varkappa T_{QR}^*$ satisfies $|y - b| \leqq 7Hp\lambda_Q\sqrt{2}$. By 4.24 we have

$$h_Q^* y = h_Q b + \varphi_Q^*(y - b), \; h_R^* y = h_R b + \varphi_R^*(y - b).$$

Applying these facts together with (4.6), 4.23 and (5.7) we get

$$\begin{aligned} |h_Q^* y - h_R^* y| &\leqq |h_Q b - h_R b| + |\varphi_{QR}^* - \varphi_{RQ}^*|\,|y - b| \\ &\leqq 3\varepsilon(L, K')\lambda_Q + 14qHp\lambda_Q\sqrt{2} \leqq M_2 q \lambda_Q \end{aligned}$$

with $M_2 = M_2(K') = 3 + 14Hn\sqrt{2}$. This implies the first inequality of the lemma.

To prove the second inequality, let $b \in \varkappa[\mathscr{T}^0 \cap Z_Q]$. Since g_0 and h_Q^* are affine in the simplexes of $\varkappa\mathscr{T}$, it suffices to prove that $|g_0 b - h_Q^* b| \leqq 2M_2 q \lambda_Q$. For this, we choose a sequence $R_0 \lhd \ldots \lhd R_s$ in $\mathscr{A}$ such that $b \in \varkappa R_0$, $k(R_s) = k(Q)$, and $R_s \cap Q \neq \emptyset$. Setting $R_{-1} = Q(b)$ and $R_{s+1} = Q$ we have $b \in \varkappa Z_{R_j}$ and $R_j \cap R_{j+1} \neq \emptyset$ for $-1 \leqq j \leqq s$. Applying the first inequality of the lemma we obtain

$$\begin{aligned}|g_0 b - h_Q^* b| &\leqq |h_{R_{-1}}^* b - h_{R_0}^* b| + \sum_{j=0}^{s} |h_{R_j}^* b - h_{R_{j+1}}^* b| \\ &\leqq M_2 q\left(\lambda_{R_0} + \sum_{j=0}^{s} \lambda_{R_j}\right) = 2M_2 q \lambda_Q\end{aligned}$$

as desired.

5.9. Last preparations. In 5.10–5.16 we complete our machinery before proving in 5.17 that g is L_1-BL. We first derive a simple inequality for two intersecting simplexes:

5.10. Lemma. *Let A and B be two simplexes in R^n with $A \cap B \neq \emptyset$. Then there is a constant $C = C(A, B) \geqq 1$ such that $d(a, A \cap B) \leqq Cd(a, B)$ for all $a \in A$.*

Proof. Using an auxiliary piecewise linear map we may assume that $A \cap B = e_0 \dots e_k$ and that the vertices of A and B are in $\{e_j : 0 \leqq j \leqq n\}$, where as in Section 2, $e_0 = 0$ and $(e_1, \dots, e_n)$ is the standard basis of R^n. It is easy to see that in this case we can choose $C = 1$.

If $|K| \setminus \Delta \neq \emptyset$, we set

$$d_0 = d(\Delta, \cup\{\sigma \in K : \sigma \cap \Delta = \emptyset\}). \tag{5.11}$$

Then $d_0 = d_0(K') > 0$. The following two lemmas give estimates based on the fact that K' is a simplicial complex. We let M_3 and M_4 denote new positive constants depending only on K'.

5.12. Lemma. *If $Q \in \mathscr{A}$, then*

$$d(\varkappa Q \cap \Delta, |K|) \geqq \lambda_Q / M_3.$$

Proof. Suppose that $z \in \varkappa Q \cap \Delta$ and $\tau \in K$. It suffices to show that $d(z, \tau) \geqq \lambda_Q / M'$ with some $M' = M'(\tau, \Delta) > 0$. If $\tau \cap \Delta = \emptyset$, then $d(z, \tau) \geqq d_0 \geqq \lambda_Q d_0$, where d_0 is given in (5.11). Hence we can choose $M' = d_0^{-1}$.

Suppose that $\tau \cap \Delta = \tau' \neq \emptyset$. Since $\varkappa^{-1} z \in Q \in \mathscr{A}$ and since $\varkappa$ is H-BL, we have $d(z, \tau') \geqq \lambda_Q / H$. By 5.10 we get

$$d(z, \tau) \geqq d(z, \tau')/C \geqq \lambda_Q / M'$$

with $M' = CH$, $C = C(\Delta, \tau)$.

5.13. Lemma. *If $Q \in \mathscr{A}$, then*

$$d(\varkappa Z_Q \cap \Delta, |K| \setminus \varkappa E_Q) \geqq \lambda_Q / M_4.$$

Proof. We may assume that $\varkappa = \mathrm{id}$. We shall prove the stronger inequality

$$d(Q(5) \cap \Delta, |K| \setminus E_Q) \geqq \lambda_Q / M_4.$$

Indeed, since $Z_Q \subset Q(5)$ by 5.5, this implies the lemma.

Suppose that $z \in Q(5) \cap \Delta$ and $y \in |K| \setminus E_Q$. It suffices to find an estimate

$$|y - z| \geqq \lambda_Q / M_4. \tag{5.14}$$

Let $y \in \tau \in K$, and set $\tau' = \tau \cap \Delta$. If $\tau' = \emptyset$, we have

$$|y - z| \geqq d_0 \geqq \lambda_Q d_0 = \lambda_Q / M_4$$

with $M_4 = 1/d_0$. Hence we may assume that $\tau' \neq \emptyset$. Recalling 3.5 and (3.6) we set $F = \Delta_Q \cap \tau' = E_Q^0 \cap \tau$. We consider two cases:

Case 1. $F = \emptyset$. By (3.6) this is equivalent to $\tau \notin K_Q$. Since $Q(7) \subset \Delta_Q$, we have $Q(7) \cap \tau' = \emptyset$. Since $z \in Q(5)$, this implies $d(z, \tau') \geqq \lambda_Q$. Thus 5.10 gives

$$|y - z| \geqq d(z, \tau) \geqq d(z, \tau')/C \geqq \lambda_Q / C$$

with $C = C(\Delta, \tau)$. Since K is finite, we obtain (5.14).

Case 2. $F \neq \emptyset$ or, equivalently, $\tau \in K_Q$. Since $y \in |K| \setminus E_Q$, the definition (3.7) of E_Q implies the estimate $d(y, F) > \lambda_Q$.

Subcase 2a. $d(y, \tau') \geqq \lambda_Q/2$. Then 5.10 gives

$$|y - z| \geqq d(y, \Delta) \geqq d(y, \tau')/C \geqq \lambda_Q/(2C)$$

with $C = C(\tau, \Delta)$. Again this yields (5.14).

Subcase 2b. $d(y, \tau') < \lambda_Q/2$. Choose $x \in \tau'$ with $|x - y| < \lambda_Q/2$. Since $d(y, F) > \lambda_Q$, we have $x \in \tau' \setminus F$ and hence $x \notin Q(7)$. Since $z \in Q(5)$, this implies that $|z - x| \geqq \lambda_Q$. Consequently,

$$|y - z| \geqq |z - x| - |x - y| > \lambda_Q/2,$$

which proves (5.14).

We still need one technical lemma before the final conclusions. For $Q \in \mathscr{A}$ set

$$Y_Q = \cup \{R \in \mathscr{A} : R \cap Q \neq \emptyset\}, \quad W_Q = \{\sigma \in \mathscr{T} : \sigma \subset Y_Q\}. \tag{5.15}$$

Then W_Q is a finite simplicial complex with $|W_Q| = Y_Q$.

5.16. LEMMA. *There exists a number $q_2 = q_2(K') > 0$ such that if $q \leqq q_2$ and $Q \in \mathscr{A}$, then $g_0 | \varkappa Y_Q$ is Λ_1-BL with $\Lambda_1 = \Lambda_1(q, K') \to 1$ as $q \to 0$.*

PROOF. Let $Q \in \mathscr{A}$. Then $Y_Q \subset Z_R$ where $R = Q$ if $k(Q) = 0$ and $Q \lhd R$ if $k(Q) \geqq 1$. This and 5.8 give

$$|g_0 - h_Q^*|_{\varkappa Y_Q} \leqq |g_0 - h_R^*|_{\varkappa Z_R} + |h_R^* - h_Q^*|_{\varkappa Z_R} \leqq 6 M_2 q \lambda_Q.$$

We can now apply [V, 2.14] with the substitution $K \mapsto \varkappa W_Q$, $f \mapsto g_0 | \varkappa Y_Q$, $h \mapsto h_Q^* | \varkappa W_Q^0$. This gives a number $\alpha_Q > 0$ such that if $\alpha = 6 M_2 q \lambda_Q \leqq \alpha_Q$, then $g_0 | \varkappa Y_Q$ is Λ-BL with $\Lambda = \Lambda(\alpha, Q) \to 1$ as $\alpha \to 0$. Moreover, the last statement of [V, 2.14] allows us to choose $\alpha_Q = \alpha_0 \lambda_Q$ and $\Lambda = \Lambda_1(q, K')$ where $\alpha_0 =$

$\alpha_0(K') > 0$ and $A_1(q, K') \to 1$ as $q \to 0$. To justify this, observe that by the construction of $\mathscr{T}$ in 5.2, the family $\mathscr{A}$ can be divided into a finite number of classes such that if Q and R belong to the same class, then W_Q is mapped onto W_R by the similarity map $\gamma: x \to \lambda_R/\lambda_Q(x - x_Q) + z_R$. Then $\kappa W_R = u\kappa W_Q$, where u is the similarity map $uy = \kappa\gamma\kappa^{-1}y = \lambda_R/\lambda_Q(y - \kappa z_Q) + \kappa z_R$ with Lipschitz constant $L_u = \lambda_R/\lambda_Q$. Hence the lemma is true with $q_2 = \alpha_0/6M_2$.

From now on, we assume that $q \leqq q_2$.

5.17. THE BILIPSCHITZ PROOF. We are finally ready to prove that the function $g: |K'| \to Y$ constructed in 5.3 is L_1-BL. For this, consider two points $x, y \in |K'|$, $x \neq y$. We must find an estimate

$$|x - y|/L_1 \leqq |gx - gy| \leqq L_1|x - y|, \tag{5.18}$$

where $L_1 = L_1(q, K') \to 1$ as $q \to 0$.

Since $g \| K| = f$ is L-BL and $L \leqq L(q, K')$ by 5.6, (5.18) holds with $L_1 = L(q, K')$ if $x, y \in |K|$. Hence we may assume that $x \in \Delta \setminus |K|$. Choose $Q \in \mathscr{A}$ with $x \in \varkappa Q$. We consider four cases.

Case 1. $y \in \Delta \setminus |K|$. Choose $R \in \mathscr{A}$ with $y \in \varkappa R$. We may assume that $k(Q) \geqq k(R)$. If $x \in \varkappa Y_R$, then 5.16 gives (5.18) with $L_1 = A_1(q, K')$. Thus we may assume that $x \notin \varkappa Y_R$. Set $k = k(R)$ and consider the sequence $R = R_0 \lhd \ldots \lhd R_k = [1, 2]^p$. Then $R_j \in \mathscr{A}$ by 2.8(3). Since $\Delta \setminus \rho \subset \varkappa Z_{R_k}$ by 2.8(1), we can choose the least index j with $x \in \varkappa Z_{R_j}$.

We show that

$$|x - y| \geqq \lambda_{R_j}/4H. \tag{5.19}$$

Since $d(R, \varkappa^{-1}\Delta \setminus Y_R) \geqq \lambda_R/2$ and since $\varkappa$ is H-BL, this is clear if $j \leqq 1$. If $j \geqq 2$, then $y \in \varkappa Z_{R_{j-2}}$, $x \notin \varkappa Z_{R_{j-1}}$, and 5.5(3) gives $|x - y| \geqq \lambda_{R_{j-2}}/H \geqq \lambda_{R_j}/4H$ and proves (5.19).

Applying (5.19) and 5.8 with $Q \mapsto R_j$ we obtain

$$\begin{aligned} |gx - gy| &\leqq |h^*_{R_j}x - h^*_{R_j}y| + |h^*_{R_j}x - gx| + |h^*_{R_j}y - gy| \\ &\leqq |x - y| + 4M_2q\lambda_{R_j} \leqq (1 + 16HM_2q)|x - y|. \end{aligned}$$

In a similar manner we see that

$$|gx - gy| \geqq (1 - 16HM_2q)|x - y|.$$

By restricting q we may assume that $q < 1/(16HM_2)$. Then we get (5.18) with $L_1 = (1 - 16HM_2q)^{-1}$.

Case 2. $y \in \varkappa E_Q$. Now Lemma 5.12 implies $|x - y| \geqq \lambda_Q/M_3$. By (4.6) and (5.7) we have $|gy - h^*_Qy| = |fy - h_Qy| \leqq q\lambda_Q$. Moreover, Lemma 5.8 gives $|gx - h^*_Qx| \leqq 2M_2q\lambda_Q$. These facts imply

$$|gx - gy| \leqq |h_Q^* x - h_Q^* y| + |gx - h_Q^* x| + |gy - h_Q^* y|$$
$$\leqq |x - y|(1 + (1 + 2M_2)M_3 q),$$
$$|gx - gy| \geqq |x - y|(1 - (1 + 2M_2)M_3 q).$$

Again, by restricting q, we get (5.18) with $L_1 = (1 - (1 + 2M_2)M_3 q)^{-1}$.

Case 3. $y \in \varkappa[E_S \setminus E_Q]$, where $S = [1,2]^p$. Now there is a sequence $Q = Q_1 \lhd \ldots \lhd Q_j$ such that $j \geqq 2$ and $y \in \varkappa[E_{Q_j} \setminus E_{Q_{j-1}}]$. Since $x \in \varkappa Q \subset \varkappa Z_{Q_{j-1}}$, Lemma 5.13 implies that $|x - y| \geqq \lambda_{Q_{j-1}}/2M_4 = \lambda_{Q_j}/2M_4$. By Lemma 5.8 we have $|gx - h_{Q_j}^* x| \leqq 2M_2 q \lambda_{Q_j}$. From (4.6) and (5.7) we get $|gy - h_{Q_j}^* y| = |fy - h_{Q_j} y| \leqq q \lambda_{Q_j}$. As in Case 2 we now get the estimates

$$(1 - M_5 q)|x - y| \leqq |gx - gy| \leqq (1 + M_5 q)|x - y|$$

where $M_5 = 2M_4(1 + 2M_2)$. After restricting q we obtain (5.18) with $L_1 = (1 - M_5 q)^{-1}$.

Case 4. $y \in |K| \setminus \varkappa E_S$, $S = [1,2]^p$. Choose $\tau \in K$ with $y \in \tau$. If $\tau' = \tau \cap \Delta = \emptyset$, then (5.11) gives $|x - y| \geqq d_0$. If $\tau' \neq \emptyset$, then (3.7) gives $d(y, \tau') \geqq H^{-1}$, because $\varkappa$ is H-BL. By Lemma 5.10 this implies

$$|x - y| \geqq d(y, \Delta) \geqq 1/(HC)$$

where $C = C(\tau, \Delta)$. In both cases we may write $|x - y| \geqq 1/M_6$ with $M_6 = M_6(K')$. By (4.2), (4.5) and (5.7) we get $|gy - y| = |fy - y| \leqq q$. Since $h_S^* = \mathrm{id}$ and $x \in \varkappa Z_S$, Lemma 5.8 gives $|gx - x| = |gx - h_S^* x| \leqq 2M_2 q$. Hence we get the estimates

$$(1 - M_7 q)|x - y| \leqq |gx - gy| \leqq (1 + M_7 q)|x - y|$$

where $M_7 = (1 + 2M_2)M_6$. After restricting q, this gives (5.18) with $L_1 = (1 - M_7 q)^{-1}$.

5.20. THE CASE $p = 1$, $\{0\}$ ISOLATED. The proof of Theorem 1.2 is now complete except for the special case where $p = 1$ and $\{0\}$ is an isolated simplex of K, which was postponed until this point. In this case we first normalize a given L-BL map $f: |K| \to Y$ by $f(0) = 0$ and $|f - \mathrm{id}|_{|K|} \leqq \delta(L, n)d(|K|)$ as in 4.1. We extend f to $g: |K'| \to Y$ by $g | \Delta = \mathrm{id}$. A straightforward computation shows that if $\delta(L, n)d(|K|) < d(\Delta, |K| \setminus \Delta)$, then g is L_1-BL with

$$L_1 = \max\{L, (1 - \delta(L, n)d(|K|)/d(\Delta, |K| \setminus \Delta))^{-1}\}.$$

Theorem 1.2 is now completely proved. Theorem 1.1 was reduced to Theorem 1.2 in Section 1. Hence Theorem 1.1 is also proved.

5.21. REMARK. Let $X \subset \mathsf{R}^n$ and Y be as in Theorem 1.1. Then the BLEP of X in (R^n, Y) gives the numbers $L_0(X, \mathsf{R}^n, Y)$ and $L_1(L, X, \mathsf{R}^n, Y)$ mentioned in the definition of the BLEP in the introduction. However, the proof shows that they can be chosen to be independent of Y.

6. Raylike polyhedra.

We shall apply Theorem 1.1 to prove the BLEP for some noncompact polyhedra. We say that a set $A \subset \mathsf{R}^n$ is *raylike* with vertex $v \in A$ if $v + t(x - v) \in A$ whenever $x \in A$ and $t \geqq 0$.

6.1. THEOREM. *Suppose that X is a raylike closed polyhedron in* R^n. *Then X has the* BLEP *in* $(\mathsf{R}^n, \mathsf{R}^n)$.

PROOF. We may assume that the vertex of X is the origin. For positive integers k, we let Q_k denote the n-cube $[-k, k]^n$. Then $X_k = X \cap Q_k$ is a compact polyhedron. By Theorem 1.1, X_k has the BLEP in $(\mathsf{R}^n, \mathsf{R}^n)$ Moreover, the sets X_k are mutually similar. From this it easily follows that the numbers $L_0 = L_0(X_k, \mathsf{R}^n)$ and $L_1 = L_1(L, X_k, \mathsf{R}^n)$ of the definition of the BLEP do not depend on k. We show that these can be chosen to be the corresponding numbers for X.

Let $1 \leqq L \leqq L_0$, and let $f\colon X \to \mathsf{R}^n$ be L-BL. Then each $f|X_k$ extends to an L_1-BL map $g_k\colon \mathsf{R}^n \to \mathsf{R}^n$. The family of all g_k is equicontinuous. Moreover, for $x \in \mathsf{R}^n$ we have $|g_k x| \leqq |f(0)| + L_1|x|$ for all k. From the Ascoli theorem it follows that a subsequence of (g_k) converges to a map $g\colon \mathsf{R}^n \to \mathsf{R}^n$, which is the desired L_1-BL extension of f.

6.2. COROLLARY. *Let E and F be affine subspaces of* R^n *with $E \cap F \neq \emptyset$. Then $E \cup F$ has the* BLEP *in* $(\mathsf{R}^n, \mathsf{R}^n)$.

6.3. REMARK. Corollary 6.2 is not true without the condition $E \cap F \neq \emptyset$. For example, the union of two parallel lines does not have the BLEP in $(\mathsf{R}^3, \mathsf{R}^3)$. This is seen by screwing one of the lines slowly around the other; cf. [Gh, 3.3].

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