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SCATTERING AND SPECTRAL THEORY FOR STARK HAMILTONIANS IN ONE DIMENSION

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Introduction.

Let $H_a = -d^2/dx^2 + x$ and let $H = H_a + V$, where V is a function. We are going to study the scattering theory for the pair (H_a, H) and some related inverse problems. There is an extensive literature on these topics and we refer to Avron and Herbst [2], Ben-Artzi [4], Herbst [6], Jensen [3], Rejto and Sinha [12] for some background material. Most of these authors used the time-dependent approach combined with commutator estimates. Under some general conditions (see [4]) they were able to prove the completeness of the wave operators for (H_a, H).

Here we adopt the time-independent approach making use of the Fourier-Airy transformation. We prove first of all a limiting absorption principle (see Agmon [1] and Chapter 14 in Hörmander [8]) for the free Stark Hamiltonian H_a in Section 1. A similar limiting absorption principle for H is obtained in Section 2 under some decay conditions on V introduced in Proposition 2.2, and the absence of discrete spectrum of H is proved. By the results obtained in Sections 1 and 2 we are able to prove an eigenfunction expansion theorem for H in Section 3 (Theorem 3.3). In Section 4 we consider some problems in the inverse scattering theory for (H_a, H). Here, the conditions on V are more restrictive, since we assume that the function $(1 + |x|)V(x)$ is integrable. This condition is assumed in the inverse scattering theory for the Schrödinger equations on the line. We adopt here the approach used in Melin [11] which was previously developed by L. D. Faddeev (see the references in [11]). A similar approach to inverse scattering for the Stark Hamiltonians appeared also in a recent note by Kachalov and Kurylev [10]. However, their conditions on the potentials are much stronger than ours. It is clear that we obtain better estimates here.

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1. The Fourier-Airy transformation and the resolvent of the free Stark Hamiltonian.

Let $\mathscr{S}(\mathsf{R})$ be the space of all smooth functions on R which together with all their derivatives are rapidly decreasing. The Fourier transformation

$$Fu(\xi) = \hat{u}(\xi) = \int e^{-ix\xi}u(x)\,dx$$

is a homeomorphism on $\mathscr{S}(\mathsf{R})$. By duality it can be extended to a homeomorphism on $\mathscr{S}'(\mathsf{R})$. The restriction of $(2\pi)^{-1/2}F$ to $\mathscr{L}^2(\mathsf{R})$ is a unitary operator.

The Fourier-Airy transformation F_a is given by

$$F_a = F^{-1}GF \tag{1.1}$$

where $Gu(\xi) = e^{-i\xi^3/3}u(\xi)$. This is a homeomorphism on $\mathscr{S}(\mathsf{R})$ and $\mathscr{S}'(\mathsf{R})$ by the preceding remarks and its restriction to $\mathscr{L}^2(\mathsf{R})$ is a unitary operator. Moreover, it follows from (1.1) that F_a is a convolution operator:

$$F_a u = \tilde{u} = \check{a} * u, \quad u \in \mathscr{S}(\mathsf{R}), \tag{1.2}$$

where $\check{a}(x) = a(-x)$ and Airy's function $a(x)$ is the inverse Fourier transform of $e^{i\xi^3/3}$. Since $F_a^{-1} = F^{-1}G^{-1}F$ where G^{-1} is the multiplication by $e^{i\xi^3/3}$, it follows that

$$F_a^{-1}u = a * u, \quad u \in \mathscr{S}(\mathsf{R}). \tag{1.3}$$

We now recall some facts about Airy's function which can be found in Hörmander [7] pp. 213–215.

Airy's function $a(z)$ is an entire function which is real when z is real. It satisfies the following differential equation

$$-a''(z) + za(z) = 0, \tag{1.4}$$

and

$$a(0) = 3^{-1/6}\Gamma(1/3)/2\pi, \quad a'(0) = -3^{1/6}\Gamma(2/3)/2\pi. \tag{1.5}$$

Let $\omega = e^{i2\pi/3}$. Then $z \mapsto a(\omega z)$ and $z \mapsto a(\omega^2 z)$ also satisfy the equation (1.4). We have the relation

$$a(z) + \omega a(\omega z) + \omega^2 a(\omega^2 z) = 0 \tag{1.6}$$

and $\{a(z), a(\omega z)\}$ are linearly independent solutions of the equation (1.4).

In order to describe the growth properties of a at infinity it is convenient to introduce the function

$$\varphi(z) = 2z^{3/2}/3, \quad |\arg z| < \pi. \tag{1.7}$$

Here and in the following we set $z^\alpha = e^{\alpha \log z}, \alpha \in \mathbf{C}$, where $\log z = \log|z| + i \arg z$ is the determination of the logarithm which is defined outside the closed negative real axis and has real values on the positive real axis. We note that $\Re\varphi(z)$ extends to a continuous function on the whole complex plane and that $\Re\varphi(z) = \Re\varphi(\bar{z})$. The following simple lemma will be needed.

LEMMA 1.1. *Let* $\omega = e^{i2\pi/3}$. *Then*

$$\Re\varphi(\omega z) \leqq \Re\varphi(z), \quad \Re\varphi(\omega z) \leqq \Re\varphi(\omega^2 z), \quad -\pi/3 \leqq \arg z \leqq \pi; \tag{1.8}$$

$$|\Re\varphi(x + iy) + |x|^{1/2}|y|| \leqq C|y|^3(|x| + |y|)^{-3/2}, \quad x < 0, \quad y \in \mathbf{R}; \tag{1.9}$$

$$|\Re\varphi(\omega(x + iy)) + |x|^{1/2}y| \leqq C|y|^3(|x| + |y|)^{-3/2}, \quad x < 0, \quad y \in \mathbf{R}; \tag{1.10}$$

$$|\Re\varphi(x + iy) - \varphi(x)| \leqq C|y|^2(x + |y|)^{-1/2}, \quad x > 0, \quad y \in \mathbf{R}; \tag{1.11}$$

where C is some positive constant.

PROOF. When proving (1.8) we may assume for reasons of homogeneity that $z = e^{i\alpha}$ where $-\pi/3 \leqq \alpha \leqq \pi$.

If $-\pi/3 < \alpha < \pi/3$, then $\varphi(\omega z) = -\varphi(z)$ and $\varphi(\omega^2 z) = -\varphi(z)$, and the estimates (1.8) hold since $3\Re\varphi(z)/2 = \cos(3\alpha/2) \geqq 0$. On the other hand, if $\pi/3 < \alpha < \pi$, then $\varphi(\omega z) = \varphi(z)$ and $\varphi(\omega^2 z) = -\varphi(z)$, and the estimates (1.8) follow since $3\Re\varphi(z)/2 = \cos(3\alpha/2) \leqq 0$.

When proving (1.9)–(1.11) we may assume for reasons of homogeneity that $|x| = 1$, and it suffices then to prove the estimates when $|y|$ is small. Then (1.9) follows since

$$\begin{aligned}\Re\varphi(-1 + iy) &= \Re\varphi(-1 + i|y|) = 2\Re(e^{3\pi i/2}(1 - i|y|)^{3/2})/3 \\ &= 2\Re(-i(1 - 3i|y|/2))/3 + O(y^3) = -|y| + O(y^3).\end{aligned}$$

The estimate (1.11) follows by expanding $(1 + iy)^{3/2}$ in a Taylor series at $y = 0$. Finally (1.10) follows from (1.9) since $\varphi(\omega z) = \pm\varphi(z)$ when $\Re z < 0$ and $\pm\Im z > 0$. This completes the proof of Lemma 1.1.

LEMMA 1.2. *There are positive constants* $C_k, k = 0, 1, \ldots$, such that

$$|a^{(k)}(z)| \leqq C_k(1 + |z|)^{k/2 - 1/4} e^{-\Re\varphi(z)}, \quad z \in \mathbf{C}. \tag{1.12}$$

PROOF. According to (7.6.20) of Hörmander [7] we have the following asymptotic expansion at infinity:

(1.13) $$a(z)e^{\varphi(z)} \sim (2\pi)^{-1}z^{-1/4}\sum_{k=0}^{\infty}(-9)^{-k}\Gamma(3k+1/2)z^{-3k/2}/(2k)!$$

in any sector $\Gamma_\varepsilon = \{z: |\arg z| \leqq \pi - \varepsilon\}$ where ε is a small positive number. Since the left side of (1.13) is analytic in the sector and $\varphi^{(k)}(z) = O(|z|^{-k+3/2})$, $k = 0, 1, \ldots,$ at infinity the estimates (1.12) hold in Γ_ε. It suffices therefore to prove now that there are estimates

$$|a^{(k)}(\omega z)| \leqq C_k(1+|z|)^{k/2-1/4}e^{-\Re\varphi(\omega z)},$$

when z is in a small conic neighborhood Γ of the half ray from the origin through the point $e^{i\pi/3}$. Since z and $\omega^2 z$ are in a set where (1.12) holds, the result follows then from (1.6) and (1.8).

We have now made all preparations necessary for our estimates of the resolvent of the free Stark Hamiltonian.

Set $Du = -idu/dx$ and $Mu(x) = xu(x)$ when $u \in \mathscr{S}'(\mathsf{R})$. Then

$$P_a = D^2 + M$$

is a continuous operator on $\mathscr{S}'(\mathsf{R})$. We note that

(1.14) $$F_a P_a F_a^{-1} = M$$

since

$$F_a P_a = F^{-1}G(M^2 - D)F = -F^{-1}DGF = MF^{-1}GF = MF_a.$$

It follows directly from (1.14) and the mapping properties of F_a already discussed that the restriction H_a (the free Stark Hamiltonian) of P_a to

$$\mathscr{D}(H_a) = \{u: u \in \mathscr{L}^2, P_a u \in \mathscr{L}^2\}$$

is a self-adjoint operator on $\mathscr{L}^2(\mathsf{R})$. It is unitarily equivalent to M. Hence its spectrum is the whole line and it is simple and absolutely continuous. It also follows from (1.14) that H_a is the closure of P_a restricted to $\mathscr{S}(\mathsf{R})$, and this implies that $C_0^\infty(\mathsf{R})$ is dense in $\mathscr{D}(H_a)$ under the graph norm.

We let $R_a(\lambda) = (H_a - \lambda)^{-1}$ be the resolvent of H_a when $\Im\lambda \neq 0$, and $R_a(x, y; \lambda)$ denotes its integral kernel. (It follows from (1.14) that $F_a R_a(\lambda) F_a^{-1} = (M-\lambda)^{-1}$. Hence $R_a(\lambda)$ is continuous from $\mathscr{S}$ to $\mathscr{S}$ and from $\mathscr{S}'$ to $\mathscr{S}'$.)

Proposition 1.3. *$R_a(x, y; \lambda)$ is the unique continuous function on* $\mathsf{R} \times \mathsf{R} \times (\mathsf{C}\setminus\mathsf{R})$ *which satisfies the following conditions:*

(1.15) $$R_a(x, y; \lambda) = i2\pi e^{-i\pi/3}a(x-\lambda)a(\omega(y-\lambda)), \quad \Im\lambda > 0, \quad \mathrm{x} > \mathrm{y};$$

(1.16) $$R_a(x, y; \lambda) = R_a(y, x; \lambda);$$

(1.17) $$R_a(x, y; \bar{\lambda}) = \overline{R_a(y, x; \lambda)} = \overline{R_a(x, y; \lambda)}.$$

PROOF. Let $\Im\lambda > 0$ and $K(\lambda)$ be the integral operator with the kernel $K(x, y; \lambda)$ defined by (1.15) and (1.16). Since $-a''(x-\lambda) + (x-\lambda)a(x-\lambda) = 0$ and

$$a'(x-\lambda)a(\omega(x-\lambda)) - \omega a(x-\lambda)a'(\omega(x-\lambda)) =$$
$$a'(0)a(0) - \omega a(0)a'(0) = -(2\pi i)^{-1}e^{i\pi/3}$$

by (1.5), it follows that $(P_a - \lambda)K(\lambda)u = u$ when $u \in C_0^\infty(\mathrm{R})$. Since $K(\lambda)u(x)$ is rapidly decreasing at infinity in view of Lemma 1.1 it follows that $K(\lambda)u \in \mathscr{D}(H_a)$. Hence $K(\lambda)u = R_a(\lambda)u$ when $u \in C_0^\infty(\mathrm{R})$. The proposition follows now if we observe that $R_a(\lambda)^* = R_a(\bar{\lambda})$.

It follows from (1.12), (1.15) and the fact that $\varphi(\omega z) = -\varphi(z)$ when $\Im z < 0$ that

$$(1.18)\quad |R_a(x, y; \lambda)| \leqq C(1 + |x-\lambda|)^{-1/4}(1 + |y-\lambda|)^{-1/4}e^{\Re\varphi(y-\lambda) - \Re\varphi(x-\lambda)},$$
$$\Im\lambda > 0, \quad x > y,$$

where C is some positive constant. We shall need the following estimates for the exponent in the right side of (1.18).

LEMMA 1.4. *Let λ be a complex number. Then $x \mapsto \Re\varphi(x-\lambda)$ is an increasing function and*

$$\text{(i)}\qquad \Re\varphi(x-\lambda) - \Re\varphi(y-\lambda) \geqq \varphi(x - \Re\lambda) - \varphi(y - \Re\lambda),$$
$$0 \leqq y - \Re\lambda \leqq x - \Re\lambda;$$

$$\text{(ii)}\quad \Re\varphi(x-\lambda) - \Re\varphi(y-\lambda) \geqq |\Im\lambda|\,(\varphi'(\Re\lambda - y + |\Im\lambda|) - \varphi'(\Re\lambda - x + |\Im\lambda|)),$$
$$y - \Re\lambda \leqq x - \Re\lambda \leqq 0.$$

PROOF. The function $x \mapsto \Re\varphi(x-\lambda)$ increases since its derivative is non-negative.

When proving (i) and (ii) we may assume that $\Re\lambda = 0$, $\Im\lambda \neq 0$, and since $\varphi(z)$ is homogeneous, $\Re\varphi(z) = \Re\varphi(\bar{z})$, we may assume also that $\Im\lambda = -1$. We have thus

$$\Re\varphi(x-\lambda) - \Re\varphi(y-\lambda) = \int_y^x \Re\varphi'(s+i)\,ds = \int_y^x \left(\frac{\sqrt{s^2+1}+s}{2}\right)^{1/2} ds.$$

If $0 \leqq y \leqq x$, then the last term is not less than

$$\int_y^x s^{1/2}\,ds = \varphi(x) - \varphi(y),$$

which gives (i). If $y \leqq x \leqq 0$, then the same term equals

$$\int_y^x \left(\frac{1}{2(\sqrt{s^2+1}+|s|)}\right)^{1/2} ds$$

which is not less than

$$\int_y^x (1+|s|)^{-1/2}\, ds/2 = \varphi'(1-y) - \varphi'(1-x).$$

This completes the proof of Lemma 1.4.

We shall use (1.18) and Lemma 1.4 to establish the continuity properties for the resolvent between some weighted $\mathscr{L}^p$ spaces. For an integral operator K with a locally integrable kernel we shall use the notation

$$\|K\|_{\mathscr{M}} = \max\left\{\operatorname*{ess\,sup}_x \int |K(x,y)|\,dy, \quad \operatorname*{ess\,sup}_y \int |K(x,y)|\,dx\right\}.$$

If this number is finite, then K is a continuous operator on $\mathscr{L}^p$ with norm $\leqq \|K\|_{\mathscr{M}}$ when $1 \leqq p \leqq \infty$. We also introduce the weighted $\mathscr{L}^p$ spaces:

$$\mathscr{L}^p_\alpha = \{u\colon \langle x\rangle^\alpha u(x) \in \mathscr{L}^p\},$$

where α is real and $\langle x\rangle = 1 + |x|$. We shall need to consider such spaces with different exponents p and different weights $\langle x\rangle^\alpha$ on the positive and negative axis. Let θ be the Heaviside function on the positive real axis, and set $\check{\theta}(x) = \theta(-x)$.

Lemma 1.5. *Let α, β be real numbers and let $K = K_{\alpha,\beta,\sigma,\tau}$ be the integral operator with the kernel*

$$K_{\alpha,\beta,\sigma,\tau}(x,y) = \theta(x-y)\langle x\rangle^\alpha \langle y\rangle^\beta e^{\Re\varphi(y-\sigma-i\tau) - \Re\varphi(x-\sigma-i\tau)},$$

when $\sigma, \tau \in \mathrm{R}$. If $I \subset \mathrm{R}\backslash\{0\}$ and $J \subset \mathrm{R}$ are compact sets and $\theta_\pm$ are bounded measurable functions such that $\theta_\pm(x) = \theta(\pm x)$ for $|x|$ large, then

$$\sup_{\sigma\in J, \tau\in \mathrm{R}} \|\theta_+ K_{\alpha,\beta,\sigma,\tau}\theta_+\|_{\mathscr{M}} < \infty, \quad \alpha+\beta \leqq 1/2; \tag{1.19}$$

$$\sup_{\sigma\in J, \tau\in I} \|\theta_+ K_{\alpha,\beta,\sigma,\tau}\theta_-\|_{\mathscr{M}} < \infty; \tag{1.20}$$

$$\sup_{\sigma\in J, \tau\in I} \|\theta_- K_{\alpha,\beta,\sigma,\tau}\theta_-\|_{\mathscr{M}} < \infty, \quad \alpha+\beta \leqq -1/2; \tag{1.21}$$

$$\sup_{\sigma\in J, \tau\in \mathrm{R}} \|\theta_+ K_{\alpha,\beta,\sigma,\tau}\theta_-\|_{\mathscr{L}^1_\beta \to \mathscr{L}^\infty_\gamma} < \infty, \quad \gamma \in \mathrm{R}; \tag{1.22}$$

$$\sup_{\sigma\in J, \tau\in \mathrm{R}} \|\theta_- K_{\alpha,\beta,\sigma,\tau}\theta_-\|_{\mathscr{L}^1_\beta \to \mathscr{L}^\infty_{-\alpha}} < \infty. \tag{1.23}$$

Proof. Since $\langle x+\sigma\rangle^\alpha \langle y+\sigma\rangle^\beta$ is of the same order of magnitude as $\langle x\rangle^\alpha\langle y\rangle^\beta$ when $x, y \in \mathrm{R}$ and σ is in a compact set, it is no restriction for us to assume that $J = \{0\}$.

When proving (1.19) we may assume that $\tau = 0$ in view of Lemma 1.4 and since $K(x, y)$ is rapidly decreasing when y stays in a compact set and x tends to infinity we may also assume that $\theta_+(x) = \theta(x - 1)$. We have for $x \geqq 1$

$$\int_1^x \langle y\rangle^\beta e^{\varphi(y)}\,dy \leqq \int_0^{x/2} \langle y\rangle^\beta e^{\varphi(y)}\,dy + C\langle x\rangle^{\beta-1/2}\int_{x/2}^x \varphi'(y)e^{\varphi(y)}\,dy$$
$$\leqq C_1(1 + \langle x\rangle^{\beta+1})e^{\varphi(x/2)} + C\langle x\rangle^{\beta-1/2}e^{\varphi(x)}.$$

Since $e^{\varphi(x/2)-\varphi(x)}$ is rapidly decreasing at infinity it follows that

$$\int_1^x |K(x, y)|\,dy \leqq C_2\langle x\rangle^{\alpha+\beta-1/2}, \quad x \geqq 1.$$

When $y \geqq 1$ we also have

$$\int_y^\infty \langle x\rangle^\alpha e^{-\varphi(x)}\,dx \leqq C\langle y\rangle^{\alpha-1/2}\int_y^{4y} \varphi'(x)e^{-\varphi(x)}\,dx + \int_{4y}^\infty \langle x\rangle^\alpha e^{-\varphi(x)}\,dx$$
$$\leqq C\langle y\rangle^{\alpha-1/2}e^{-\varphi(y)} + \int_{4y}^\infty \langle x\rangle^\alpha e^{-\varphi(x)}\,dx \leqq C_1\langle y\rangle^{\alpha-1/2}e^{-\varphi(y)}.$$

Hence we have proved that

$$\int_y^\infty |K(x, y)|\,dx \leqq C_2\langle y\rangle^{\alpha+\beta-1/2}, \quad y \geqq 1,$$

and (1.19) follows.

When proving (1.20) we may assume that $\theta_-(y) = \theta(-y)$ and since $e^{\Re\varphi(y-i\tau)}$ is rapidly decreasing at $-\infty$ in view of (1.9), again by Lemma 1.4, we may assume also that $\theta_+ = \theta$. Then we write

$$\Re\varphi(x - i\tau) - \Re\varphi(y - i\tau) = \Re\varphi(x - i\tau) - \Re\varphi(-i\tau) + \Re\varphi(-i\tau) - \Re\varphi(y - i\tau)$$

when $x \geqq 0 \geqq y$. The right side can be estimated from below by a constant plus

$$\varphi(x) + c\,|y|^{1/2},$$

where c is a positive constant which depends on I only. Hence

$$|K(x, y)| \leqq C\langle x\rangle^\alpha\langle y\rangle^\beta e^{-\varphi(x)-c|y|^{1/2}}, \quad y \leqq 0 \leqq x$$

and (1.20) follows.

In the proof of (1.21) we may first replace $\theta_- K\theta_-$ by $\theta_- K\check{\theta}$ and applying (1.20) we find then that θ_- in the last expression may be replaced by $\check{\theta}$. An application of Lemma 1.4 allows us to estimate $\Re\varphi(x - i\tau) - \Re\varphi(y - i\tau)$ from below by a negative constant plus $c(\varphi'(|y|) - \varphi'(|x|))$, where $c > 0$ depends on I only. Hence if G is the operator obtained from $\check{\theta}K\check{\theta}$ when (x, y) is replaced by $(-y, -x)$ in the kernel, then $x, y \geqq 0$ in the kernel of G and

$$|G(x, y)| \leqq C\theta(x-y)\langle x\rangle^{\beta}\langle y\rangle^{\alpha}e^{-c(\varphi'(x)-\varphi'(y))}.$$

Arguing as in the proof of (1.19) we conclude therefore that if $\alpha + \beta \leqq -1/2$, then there is a bound for the $\mathscr{M}$-norm of G which depends only on I, J, α and β. Hence (1.21) holds.

In the proof of (1.22) we may assume that $\theta_+ = \theta$ and $\theta_- = \check{\theta}$. Then (1.22) is obvious if we estimate $K(x, y)$ for $y < 0 < x$ by $\langle x\rangle^{\alpha}\langle y\rangle^{\beta}e^{-\varphi(x)}$. Finally (1.23) follows from the estimate $|K(x, y)| \leqq \langle x\rangle^{\alpha}\langle y\rangle^{\beta}$. This completes the proof of Lemma 1.5.

The preceding lemma together with (1.18) motivate the following definition.

DEFINITION 1.6. *Let α, β be real numbers and let $1 \leqq p, q \leqq \infty$. Then $(\mathscr{L}^p_{\alpha}, \mathscr{L}^q_{\beta})$ is the space of all functions u such that $\theta_-u \in \mathscr{L}^p_{\alpha}$* and *$\theta_+u \in \mathscr{L}^q_{\beta}$ whenever $\theta_{\pm}$ are bounded measurable functions such that $\theta_{\pm}(x) = \theta(\pm x)$ for $|x|$ large.*

We note that $(\mathscr{L}^p_{\alpha}, \mathscr{L}^q_{\beta})$ is a Fréchet space under the semi-norms $\|\theta_-u\|_{\mathscr{L}^p_{\alpha}} + \|\theta_+u\|_{\mathscr{L}^q_{\beta}}$, and it is invariant under translations. In the following we shall use the notation $\mathbf{C}_{\pm} = \{\lambda: \lambda \in \mathbf{C}, \pm\Im\lambda > 0\}$.

THEOREM 1.7. (i) *If $\lambda \in \mathbf{C}\backslash\mathbf{R}$ and α, β are arbitrary real numbers, then $R_a(\lambda)$ is continuous from $(\mathscr{L}^2_{\alpha}, \mathscr{L}^2_{\beta})$ to $(\mathscr{L}^2_{\alpha}, \mathscr{L}^2_{\beta+1})$.*

(ii) *If $B \subset (\mathscr{L}^1_{-1/4}, L^2_{\beta})$ is bounded and $J \subset \mathbf{C}$ is bounded, then $\{R_a(\lambda)u: \lambda \in J, \Im\lambda \neq 0, u \in B\}$ is bounded in $(\mathscr{L}^{\infty}_{1/4}, \mathscr{L}^2_{\beta+1})$.*

(iii) *If X_{β} is the space of all continuous linear operators from $(\mathscr{L}^1_{-1/4}, \mathscr{L}^2_{\beta})$ to $(\mathscr{L}^{\infty}_{1/4}, \mathscr{L}^2_{\beta+1})$, then the mapping $\lambda \mapsto R_a(\lambda)$ is strongly continuous from $\mathbf{C}_{\pm}$ to X_{β}.*

(iv) *If $\eta(x)$ is any bounded measurable function tending to zero at $-\infty$, then the mapping $\lambda \mapsto \eta R_a(\lambda)$ is strongly continuous from $\bar{\mathbf{C}}_{\pm}$ to X_{β}.*

PROOF. (i) Let $\theta_{\pm}$ be as in Definition 1.6 and let $\theta^{\gamma}_{\pm}$ be the operator which is the multiplication by $\theta_{\pm}(x)\langle x\rangle^{\gamma}$. We want to prove that the operators

$$\theta^{\alpha}_-R_a(\lambda)\theta^{-\alpha}_-,\quad \theta^{\alpha}_-R_a(\lambda)\theta^{-\beta}_+,\quad \theta^{\beta+1}_+R_a(\lambda)\theta^{-\alpha}_-,\quad \theta^{\beta+1}_+R_a(\lambda)\theta^{-\beta}_+ \tag{1.24}$$

have finite $\mathscr{M}$-norms. In view of (1.17) and (1.18) it suffices to prove the corresponding statement obtained when $\Im\lambda > 0$ and $R_a(\lambda)$ is replaced by the operators $K_{\pm}(\lambda)$ with the kernels

$$K_+(x, y) = \theta(x-y)\langle x\rangle^{-1/4}\langle y\rangle^{-1/4}e^{\Re\varphi(y-\lambda)-\Re\varphi(x-\lambda)}, \tag{1.25}$$

$$K_-(x, y) = K_+(y, x). \tag{1.26}$$

The assertion is then a consequence of Lemma 1.5.

(ii) When proving this assertion we may again replace $R_a(\lambda)$ by $K_\pm(\lambda)$ as above. This time we want to give estimates independent of λ for the norms

$$\|\theta_- K(\lambda)\theta_-\|_{\mathscr{L}^1_{-1/4}\to\mathscr{L}^\infty_{1/4}},$$

$$\|\theta_- K(\lambda)\theta_+\|_{\mathscr{L}^2_\beta\to\mathscr{L}^\infty_{1/4}},$$

$$\|\theta_+ K(\lambda)\theta_-\|_{\mathscr{L}^1_{-1/4}\to\mathscr{L}^2_{\beta+1}},$$

$$\|\theta_+ K(\lambda)\theta_+\|_{\mathscr{L}^2_\beta\to\mathscr{L}^2_{\beta+1}}$$

when $K(\lambda) = K_\pm(\lambda)$ and λ is in a bounded set in C. The statement for $K_+(\lambda)$ is then again a consequence of Lemma 1.5 and the statement for $K_-(\lambda)$ follows by passing to the adjoint operators.

(iii) It follows from (ii) that it suffices to prove that $\mathsf{C}\backslash\mathsf{R}\ni\lambda\mapsto R_a(\lambda)u$ is continuous with values in $(\mathscr{L}^\infty_{1/4}, \mathscr{L}^2_{\beta+1})$ when $u\in C_0^\infty(\mathsf{R})$. The assertion follows therefore if we recall Proposition 1.3. By that proposition $R_a(\lambda)u(x)$ is a continuous function of $(\lambda, x)\in(\mathsf{C}\backslash\mathsf{R})\times\mathsf{R}$. Moreover, we have $R_a(\lambda)u(x) = f_+(\lambda)a(x-\lambda)$ for large positive x and $R_a(\lambda)u(x) = f_-(\lambda)a(\omega(x-\lambda))$ for large negative x, when $\Im\lambda > 0$, with $f_\pm$ being continuous on $\bar{\mathsf{C}}_+$. Similar argument applies when $\Im\lambda < 0$.

Finally (iv) follows from the proof of (iii) since we have the estimates $|a(\omega^{\pm1}(x-\lambda))| \leqq C\langle x\rangle^{-1/4}$ when $x\leqq 0$ and λ belongs to a bounded set in $\bar{\mathsf{C}}_\pm$. This completes the proof of Theorem 1.7.

REMARK 1.8. It follows from Lemma 1.2 and Proposition 1.3 that the operator kernel $\langle x\rangle^{-1/2}D_xR_a(x,y;\lambda)$ of the operator $\langle M\rangle^{-1/2}DR_a(\lambda)$, where $\langle M\rangle^{-1/2}$ is the multiplication by $\langle x\rangle^{-1/2}$, satisfies the same kind of estimates as $R_a(x,y;\lambda)$. Hence the operators $DR_a(\lambda)$ satisfy the conclusions of Theorem 1.7 if we replace $(\mathscr{L}^2_\alpha, \mathscr{L}^2_{\beta+1})$ and $(\mathscr{L}^\infty_{1/4}, \mathscr{L}^2_{\beta+1})$ by $(\mathscr{L}^2_{\alpha-1/2}, \mathscr{L}^2_{\beta+1/2})$ and $(\mathscr{L}^\infty_{-1/4}, \mathscr{L}^2_{\beta+1/2})$ in the statement of the theorem.

In the next section we shall also use some results about the mapping properties of the resolvent from $\mathscr{L}^2$ to $\mathscr{L}^\infty$.

PROPOSITION 1.9. *Let $J\subset\mathsf{C}\backslash\mathsf{R}$ be compact and $\theta_\pm$ be as in Definition* 1.6. *Then there is a positive constant C so that*

$$\|\theta_-\langle\cdot\rangle^{1/4}R_a(\lambda)u\|_{\mathscr{L}^\infty} + \|\theta_+\langle\cdot\rangle^{3/4}R_a(\lambda)u\|_{\mathscr{L}^\infty} \leqq C\,\|u\|_{\mathscr{L}^2},\quad \lambda\in J,\quad u\in\mathscr{L}^2.$$

PROOF. It is no restriction to assume that $\theta_\pm(x) = \theta(\pm x)$ and that $J\subset\mathsf{C}_+$. Write $\lambda = \sigma + i\tau$ and let $K_{\alpha,\beta,\sigma,\tau}$ be as in Lemma 1.5. It suffices for us to prove the corresponding estimate obtained when $\theta_-\langle\cdot\rangle^{1/4}R_a(\lambda)$ is replaced by the operators $\theta_-K_{0,-1/4,\sigma,\tau}$, $\theta_-K^*_{-1/4,0,\sigma,\tau}$ and $\theta_+\langle\cdot\rangle^{3/4}R_a(\lambda)$ is replaced by the operators $\theta_+K_{1/2,-1/4,\sigma,\tau}$, $\theta_+K^*_{-1/4,1/2,\sigma,\tau}$. Hence we want to have bounds independent of λ and x for the expressions

$$\int \theta_-(x)\,|K_{0,\,-1/4,\sigma,\tau}(x,y)|^2\,dy,$$

$$\int \theta_+(x)\,|K_{1/2,\,-1/4,\sigma,\tau}(x,y)|^2\,dy$$

and the corresponding expressions obtained when $K_{0,\,-1/4,\sigma,\tau}(x,y)$ is replaced by $K_{-1/4,0,\sigma,\tau}(y,x)$ and $K_{1/2,-1/4,\sigma,\tau}(x,y)$ is replaced by $K_{-1/4,1/2,\sigma,\tau}(y,x)$. Since $|K_{\alpha,\beta,\sigma,\tau}|^2$ is the same as $K_{2\alpha,2\beta,\sigma,\tau}$ after φ has been replaced by 2φ, the result follows from the proof of Lemma 1.5. This completes the proof of Proposition 1.9.

We shall conclude this section by making some additional remarks about the solutions of the equation

$$(P_a - \sigma)u = f \tag{1.27}$$

when σ is real and $f \in (\mathscr{L}^1_{-1/4}, \mathscr{L}^2_{-1/2})$. It follows from (iv) of Theorem 1.7 that $R_a(\sigma \pm i0)f$ are solutions in $(\mathscr{L}^\infty_{1/4}, \mathscr{L}^2_{1/2})$ of the equation. If $u \in \mathscr{S}'(\mathsf{R})$ is another solution we have

$$u(x) = c_\pm a(x-\sigma) + R_a(\sigma \pm i0)f(x), \tag{1.28}$$

where $c_\pm$ are some constants depending on σ. In fact, any distribution solution of the equation $(M-\sigma)g(x) = 0$ is proportional to $\delta(x-\sigma)$. It follows therefore from (1.3) and (1.14) that $h(x)$ is proportional to $a(x-\sigma)$ if $(P_a - \sigma)h(x) = 0$ and $h \in \mathscr{S}'(\mathsf{R})$.

An application of (1.2), (1.3) and (1.14) shows that the integral kernel of $R_a(\sigma + i0) - R_a(\sigma - i0)$ is $(2\pi i)a(x-\sigma)a(y-\sigma)$. When proving this we may assume that $\sigma = 0$, for $(P_a u)(\cdot - \sigma) = (P_a - \sigma)u(\cdot - \sigma)$. Then $(2\pi i)\delta(x)\delta(y)$ is the integral kernel of the operator $(M - i0)^{-1} - (M + i0)^{-1}$. The statement follows since $F_a^{-1}\delta = a$ and $F_a\delta = \check{a}$.

If $u \in \mathscr{L}^\infty_{1/4}$ and $f \in \mathscr{L}^1_{-1/4}$ we shall use the notation (u, f) for the integral $\int u(x)\overline{f(x)}\,dx$. We note here that the Fourier-Airy transform $\tilde{f}$ of $f \in (\mathscr{L}^1_{-1/4}, \mathscr{L}^2_\gamma)$ is a continuous function for any real γ.

PROPOSITION 1.10. *Let σ be real and let $u \in \mathscr{S}'(\mathsf{R})$ be a solution of the equation* (1.27) *with $f \in (\mathscr{L}^1_{-1/4}, \mathscr{L}^2_{-1/2})$. Then $u \in (\mathscr{L}^\infty_{1/4}, \mathscr{L}^2_{1/2})$,* (1.28) *holds, and*

$$-\Im(u,f) = (|c_+|^2 - |c_-|^2)/4\pi, \tag{1.29}$$

$$c_+ - c_- = -i2\pi\tilde{f}(\sigma). \tag{1.30}$$

PROOF. We may assume that $\sigma = 0$. We have already proved that (1.28) holds. Hence $u \in (\mathscr{L}^\infty_{1/4}, \mathscr{L}^2_{1/2})$ since a is in this space. It follows from (1.28) that

$$0 = (c_+ - c_-)a(x) + (R_a(i0) - R_a(-i0))f(x)$$

so that

$$c_+ - c_- = -i2\pi\tilde{f}(0)$$

which is (1.30). By (1.28) it follows that

$$2u = (c_+ + c_-)a + (R_a(i0) + R_a(-i0))f$$

so that

$$2\Im(u, f) = \Im((c_+ + c_-)\overline{\tilde{f}(0)}).$$

By (1.30) the right side equals

$$-\Im(i(c_+ + c_-)(\overline{c_+} - \overline{c_-})/2\pi) = (|c_-|^2 - |c_+|^2)/2\pi.$$

This proves (1.29), hence Proposition 1.10.

COROLLARY 1.11. *If one of c_+ and c_- vanishes, then*

$$\tilde{f}(\sigma) = 0 \quad \Leftrightarrow \quad c_+ = c_- = 0 \quad \Leftrightarrow \quad \Im(u, f) = 0.$$

Let $f \in (\mathscr{L}^1_{-1/4}, \mathscr{L}^2_\beta)$ for some real β. Then $\langle x \rangle^{1/4} R_a(\sigma \pm i0) f(x)$ is bounded near $-\infty$ by (iv) of Theorem 1.7. The following result is more precise.

PROPOSITION 1.12. *Let $f \in (\mathscr{L}^1_{-1/4}, \mathscr{L}^2_\beta)$. Then*

$$\langle x \rangle^{1/4} |R_a(\sigma + i0)f(x) - i2\pi e^{-i\pi/3}\tilde{f}(\sigma)a(\omega(x - \sigma))| \to 0,$$

$$\langle x \rangle^{1/4} |R_a(\sigma - i0)f(x) + i2\pi e^{i\pi/3}\tilde{f}(\sigma)a(\omega^2(x - \sigma))| \to 0,$$

as $x \to -\infty$.

PROOF. We may assume that $\sigma = 0$, and by the continuity we may also assume that $f \in C_0^\infty(\mathsf{R})$. It follows then from Proposition 1.3 that

$$R_a(i0)f(x) = i2\pi e^{-i\pi/3}\tilde{f}(0)a(\omega x), \quad R_a(-i0)f(x) = -i2\pi e^{i\pi/3}\tilde{f}(0)a(\omega^2 x)$$

when x is negative and large. This completes the proof of Proposition 1.12.

The following result will be needed in the next section.

PROPOSITION 1.13. *Let σ, N be real numbers and let $f \in \mathscr{L}^1_{-1/4}(-\infty, N)$. Then the equation (1.27) has a unique solution u such that $\langle x \rangle^{1/4} u(x) \to 0$ as $x \to -\infty$. This solution satisfies the estimate*

$$\langle x \rangle^{1/4} |u(x)| \leqq C_N \int_{-\infty}^{x} \langle y \rangle^{-1/4} |f(y)| \, dy, \quad x < N,$$

where C_N is a positive constant which is independent of f.

PROOF. We may assume that $\sigma = 0$. Since

$$G(x, y) = i2\pi e^{-i\pi/3}\theta(x - y)(a(x)a(\omega y) - a(y)a(\omega x))$$

is a fundamental solution of P_a, it follows that $u(x) = \int G(x, y)f(y)\,dy$ is a solution of the equation (1.27) with $\sigma = 0$ and satisfies the properties listed above.

To prove the uniqueness we assume that u satisfies the equation $P_a u = 0$ and $\langle x\rangle^{1/4}u(x) \to 0$ as $x \to -\infty$. We can write

$$u(x) = C_+ a(\omega x) + C_- a(\omega^{-1}x),$$

where $C_\pm$ are some constants. Since $\varphi(\omega^{\pm 1}x) = \mp i\varphi(|x|)$ when $x < 0$ it follows from (1.13) that

$$A_+ C_+ e^{i\varphi(|x|)} + A_- C_- e^{-i\varphi(|x|)} \to 0, \quad x \to -\infty,$$

where $A_\pm$ ($\neq 0$) are some numbers. Hence $C_\pm = 0$. This completes the proof of Proposition 1.13.

2. The resolvent of a short range perturbation of H_a.

We shall first find a relatively weak condition on a potential $V(x)$ which makes $H_a + V$ a self-adjoint operator on $\mathscr{D}(H_a)$.

PROPOSITION 2.1. *Let $V(x) \in \mathscr{L}^2_{\mathrm{loc}}(\mathrm{R})$ be a real function and assume that*

$$V(x) = \theta(-x)(V_1(x) + V_2(x)) + \theta(x)(V_3(x) + V_4(x))$$

with $V_1(x) \in \mathscr{L}^\infty$, $V_2(x) \in \mathscr{L}^2_{-1/4}$, $\lim_{x\to\infty} V_3(x)/x = 0$ and $V_4(x) \in \mathscr{L}^2_{-3/4}$. Then $H_a + V$ is a self-adjoint operator with the domain $\mathscr{D}(H_a)$.

PROOF. By Kato's theorem and the fact that $C_0(\mathrm{R})$ is dense in any $\mathscr{L}^2_\gamma$ it suffices to prove that $\mathscr{D}(H_a)$ equipped with the graph norm is continuously embedded in the Banach space of all functions $u(x)$ such that $\check{\theta}(x)\langle x\rangle^{1/4}u(x) \in \mathscr{L}^\infty$, $\check{\theta}u \in \mathscr{L}^2$, $\theta(x)\langle x\rangle^{3/4}u(x) \in \mathscr{L}^\infty$, $\theta(x)xu(x) \in \mathscr{L}^2$.

Since $R_a(i\tau)$ is a homeomorphism from $\mathscr{L}^2$ to $\mathscr{D}(H_a)$ when $\tau \neq 0$, the result follows from Theorem 1.7 and Proposition 1.9.

PROPOSITION 2.2. *Let V satisfy the conditions of Proposition 2.1 with $V_1 = 0$. We assume in addition that $\check{\theta}V \in \mathscr{L}^1_{-1/2}$. Let $U(\lambda) = VR_a(\lambda)$ when $\lambda \in \mathrm{C}_\pm$. Then the mappings $\mathrm{C}_\pm \ni \lambda \mapsto U(\lambda)$ are strongly continuous from $\mathrm{C}_\pm$ to the space X of all bounded linear operators on $(\mathscr{L}^1_{-1/4}, \mathscr{L}^2)$, and each $U(\lambda)$ is a compact operator on that space. The operator $I + U(\lambda)$ is a bijection on that space for each $\lambda \in \mathrm{C}_\pm$ and the mappings $\mathrm{C}_\pm \ni \lambda \mapsto (I + U(\lambda))^{-1} \in X$ are strongly continuous.*

PROOF. Let $K_\pm \subset \mathrm{C}_\pm$ be compact sets and $B \subset (\mathscr{L}^1_{-1/4}, \mathscr{L}^2)$ be a bounded set. We shall first prove that the sets

$$\{VR_a(\lambda)f\colon \lambda\in K_\pm,\quad f\in B\}$$

are bounded in $(\mathscr{L}^1_{-1/4}, \mathscr{L}^2)$. Since $\{R_a(\lambda)f\colon \lambda\in K_\pm, f\in B\}$ are bounded in $(\mathscr{L}^\infty_{1/4}, \mathscr{L}^2_1)$ by Theorem 1.7, it suffices for us to prove that $\{\theta R_a(\lambda)f\colon \lambda\in K_\pm, f\in B\}$ are bounded in $\mathscr{L}^\infty_{3/4}$. We shall use the estimate

$$\left|\frac{d}{dx}(\langle x\rangle^{3/2}u^2(x))\right| \leqq C(\langle x\rangle^{1/2}|u(x)|^2 + (\langle x\rangle^{1/2}|u'(x)|)(\langle x\rangle |u(x)|)),$$

where $u = R_a(\lambda)f$, $\lambda\in K_\pm$, $f\in B$, $x > 0$. Then it follows from Remark 1.8 that we have a bound for the norm in $\mathscr{L}^1(\mathsf{R}_+)$ of the right side. Hence we have a bound for $\langle x\rangle^{3/4}u(x)$ in $\mathscr{L}^\infty(\mathsf{R}_+)$.

When proving that $\lambda\mapsto U(\lambda)\in X$ is strongly continuous and that $U(\lambda)$ is compact, we may in view of the estimate above assume that V is compactly supported, and then the first assertion follows from (iv) of Theorem 1.7. The second assertion follows from Remark 1.8, which shows that $\{R_a(\lambda)f\colon f\in B\}$ is an equicontinuous family of functions when restricted to the support of V. Therefore the sets

$$\{VR_a(\lambda)f\colon \lambda\in K_\pm,\quad f\in B\}$$

are relatively compact in $(\mathscr{L}^1_{-1/4}, \mathscr{L}^2)$.

Next we prove that $I + U(\lambda)$ is a bijection. Since $U(\lambda)$ is compact, it suffices to prove that $f = 0$ if $f\in(\mathscr{L}^1_{-1/4}, \mathscr{L}^2)$ and $f = -VR_a(\lambda)f$. We note that $f\in\mathscr{L}^2$ since $R_a(\lambda)f\in(\mathscr{L}^\infty_{1/4}, \mathscr{L}^2_1)$. Set $u = R_a(\lambda)f$. If $\Im\lambda \neq 0$, then $u\in\mathscr{D}(H_a) = \mathscr{D}(H)$ and $Hu = \lambda u$. Hence $f = u = 0$ since H is self-adjoint. When λ is real we have to apply another argument. Assume for example that

$$f = -VR_a(\lambda + i0)f$$

with $f\in(\mathscr{L}^1_{-1/4}, \mathscr{L}^2)$. Assume also, as we may, that $\lambda = 0$. Then $u = R_a(i0)f\in(\mathscr{L}^\infty_{1/4}, \mathscr{L}^2_1)$. Since $(u, f) = -(u, Vu)$ is real, it follows from Corollary 1.11 that $\tilde{f}(0) = 0$, so that by Proposition 1.12

$$\langle x\rangle^{1/4}|u(x)| \to 0$$

as $x\to -\infty$. Since $P_a u = f\in\mathscr{L}^1_{-1/4}(-\infty, N)$ for any N, it follows from Proposition 1.13 that

$$\langle x\rangle^{1/4}|u(x)| \leqq C_N\int_{-\infty}^x \langle y\rangle^{-1/4}|V(y)u(y)|\,dy,\quad x < N,$$

where C_N is a positive constant. Since $\theta V\in\mathscr{L}^1_{-1/2}$ it follows that $u = 0$, hence $f = 0$. A similar argument applies when $R_a(i0)$ is replaced by $R_a(-i0)$.

Finally we prove that $\bar{\mathsf{C}}_\pm \ni \lambda\mapsto (I + U(\lambda))^{-1}\in X$ are strongly continuous mappings. Let B and $K_\pm$ be as above. We claim that the sets

$$D_{\pm} = \{(I + U(\lambda))^{-1}f\colon \lambda \in K_{\pm},\quad f \in B\}$$

are bounded in $(\mathscr{L}^1_{-1/4}, \mathscr{L}^2)$. Assume for example that D_+ is not bounded. Then there are a number $\sigma \in \mathbf{R}$ and sequences $f_j \in B, \lambda_j \in K_+$ so that

$$\varrho_j \equiv \|\check{\theta}(\cdot - \sigma)(I + U(\lambda_j))^{-1}f_j\|_{\mathscr{L}^1_{-1/4}} + \|\theta(\cdot - \sigma)(I + U(\lambda_j))^{-1}f_j\|_{\mathscr{L}^2} \to \infty$$

as $j \to \infty$. It is no restriction to assume that $\sigma = 0$. Set $g_j = (I + U(\lambda_j))^{-1}f_j/\varrho_j$. Then

$$g_j + U(\lambda_j)g_j = f_j/\varrho_j. \tag{2.1}$$

It follows from the proof of Theorem 1.7 together with Remark 1.8 that $\{\check{\theta}R_a(\lambda_j)g_j\}$ is bounded in $\mathscr{L}^\infty_{1/4}$, $\{\theta R_a(\lambda_j)g_j\}$ is bounded in $\mathscr{L}^2_1$ and $\{R_a(\lambda_j)g_j\}$ is bounded in $\mathscr{L}^\infty_{\text{loc}}$. Hence $\{U(\lambda_j)g_j\}$ is bounded in $(\mathscr{L}^1_{-1/4}, \mathscr{L}^2)$. It follows therefore from (2.1) that $\{g_j\}$ is bounded in that space. But then $\{U(\lambda_j)g_j\}$ is precompact in $(\mathscr{L}^1_{-1/4}, \mathscr{L}^2)$, and it follows from (2.1) that we may take out a subsequence g_{j_k} of g_j which converges to an element $g\,(\neq 0)$ in $(\mathscr{L}^1_{-1/4}, \mathscr{L}^2)$. Since $g + U(\lambda)g = 0$ if λ is a limit point of $\{\lambda_{j_k}\}$, we obtain a contradiction. Similar argument shows that D_- is bounded.

It is now easy to complete the proof. Let $\lambda_0 \in \mathbf{C}_+$ and $\lambda_j \in \mathbf{C}_+$ be a sequence which converges to λ_0. Let $h_j = (I + U(\lambda_j))^{-1}f$, where $f \in (\mathscr{L}^1_{-1/4}, \mathscr{L}^2)$. Since

$$h_j + U(\lambda_j)h_j = f$$

it follows that $\{h_j\}$ is precompact in $(\mathscr{L}^1_{-1/4}, \mathscr{L}^2)$. Since $h = (I + U(\lambda_0))^{-1}f$ is the only limit point of $\{h_j\}$, we conclude that $h_j \to h$ in $(\mathscr{L}^1_{-1/4}, \mathscr{L}^2)$. Hence

$$\mathbf{C}_+ \ni \lambda \mapsto (I + U(\lambda))^{-1}f \in (\mathscr{L}^1_{-1/4}, \mathscr{L}^2)$$

is continuous. This completes the proof of Proposition 2.2, since the proof for the corresponding statement with $\mathbf{C}_+$ replaced by $\mathbf{C}_-$ is the same.

We shall need the following result about the resolvent $R(\lambda) = R_V(\lambda)$ of $H_a + V$.

PROPOSITION 2.3. *Assume that V satisfies the assumptions of Proposition* 2.2. *If $\lambda \in \mathbf{C}\setminus\mathbf{R}$ then $U(\lambda)$ is compact on $\mathscr{L}^2$, $I + U(\lambda)$ is a bijection on that space, and*

$$R(\lambda) = R_a(\lambda)(I + U(\lambda))^{-1}. \tag{2.2}$$

PROOF. It follows from the proof of Proposition 2.1 that $U(\lambda)$ is continuous on $\mathscr{L}^2$. When proving that it is compact we may assume that V is compactly supported. Then it follows from Remark 1.8 that $\{R_a(\lambda)f\colon f \in B\}$ is equicontinuous when restricted to the support of V, if B is the unit ball in $\mathscr{L}^2$. Hence $U(\lambda)$ is compact on $\mathscr{L}^2$. The resolvent equation

$$R_a(\lambda) = R(\lambda) + R(\lambda)VR_a(\lambda) = R(\lambda) + R_a(\lambda)VR(\lambda) \tag{2.3}$$

shows that $R_a(\lambda) = R(\lambda)(I + U(\lambda))$. Hence the bijectiveness of $I + U(\lambda)$ and (2.2) follow. This completes the proof of Proposition 2.3.

It follows from (2.2), Theorem 1.7 and Proposition 2.2 that $R(\lambda)$ is defined when $\lambda \in \mathbf{C}_\pm$ and it is continuous from $(\mathscr{L}^1_{-1/4}, \mathscr{L}^2)$ to $(\mathscr{L}^\infty_{1/4}, \mathscr{L}^2_1)$ there.

THEOREM 2.4. *Let V satisfy the assumptions in Proposition 2.2, $R(\lambda)$ be the resolvent of $H = H_a + V$ and η be a bounded measurable function tending to zero at $-\infty$. Then the mappings $\mathbf{C}_\pm \quad \lambda \mapsto \eta R(\lambda)$ extend to strongly continuous mappings from $\mathbf{C}_\pm$ to the space X_0 of bounded linear operators from $(\mathscr{L}^1_{-1/4}, \mathscr{L}^2)$ to $(\mathscr{L}^\infty_{1/4}, \mathscr{L}^2_1)$. Moreover, the spectrum of H is absolutely continuous.*

PROOF. It follows from (2.2), Theorem 1.7 and Proposition 2.2 that $\eta R(\lambda) = \eta R_a(\lambda)(I + U(\lambda))^{-1}$, $\lambda \in \mathbf{C}_\pm$, are strongly continuous families of operators in X_0 multiplied to the right by strongly continuous families of operators on $(\mathscr{L}^1_{-1/4}, \mathscr{L}^2)$. This proves the first part of the theorem.

The statement that H has an absolutely continuous spectrum follows from Proposition 4.1 in Cycon-Froese-Kirsch-Simon [5] and the fact that

$$\sup_{\lambda \in K} \|R(\lambda)f\|_{\mathscr{L}^2(I)} < \infty$$

for any $f \in C_0^\infty(\mathbf{R})$, any compact set K in $\mathbf{C}_\pm$ and any bounded interval $I \subset \mathbf{R}$. This completes the proof of Theorem 2.4.

The preceding results motivate the following definition.

DEFINITION 2.5. *The space $\mathscr{V}$ is defined as the set of functions $V(x)$ satisfying the assumptions of Proposition 2.2. We call $H = H_a + V$ a short range perturbation of H_a if $V \in \mathscr{V}$.*

The following result shows that $R(\lambda) = R_V(\lambda)$ depends continuously on V.

THEOREM 2.6. *Let $V \in \mathscr{V}$ and $(V_j)_1^\infty$ be a sequence in $\mathscr{V}$ such that $|V_j| \leqq |V|$ and $V_j \to V$ uniformly on any compact set. If $K_\pm \subset \mathbf{C}_\pm$ are compact sets and $B \subset (\mathscr{L}^1_{-1/4}, \mathscr{L}^2)$ is bounded, then*

$$\{R_{V_j}(\lambda)f\colon \lambda \in K_\pm, \quad j \in \mathbf{Z}_+, \quad f \in B\}$$

are bounded in $(\mathscr{L}^\infty_{1/4}, \mathscr{L}^2_1)$. Moreover, if η is a bounded measurable function tending to 0 at $-\infty$ and $\lambda \in \mathbf{C}_\pm$, then $\eta R_{V_j}(\lambda)$ tends to $\eta R_V(\lambda)$ strongly in the space X_0 of continuous operators from $(\mathscr{L}^1_{-1/4}, \mathscr{L}^2)$ to $(\mathscr{L}^\infty_{1/4}, \mathscr{L}^2_1)$.

PROOF. Set $U_j(\lambda) = V_j R_a(\lambda)$ and

$$\tilde{D}_\pm = \{(I + U_j(\lambda))^{-1} f\colon \lambda \in K_\pm, \quad j \in \mathbf{Z}_+, \quad f \in B\}.$$

The argument given in the proof of Proposition 2.2 for the boundedness of the sets $D_\pm$ in $(\mathscr{L}^1_{-1/4}, \mathscr{L}^2)$ shows also that $\tilde{D}_\pm$ are bounded in the same space. The first part of the theorem follows since $R_{V_j}(\lambda) = R_a(\lambda)(I + U_j(\lambda))^{-1}$. The argument given at the end of the proof of Proposition 2.2 shows that $(I + U_j(\lambda))^{-1}f \to (I + VR_a(\lambda))^{-1}f$ in $(\mathscr{L}^1_{-1/4}, \mathscr{L}^2)$ for every f in that space. Hence $\eta R_{V_j}(\lambda) \to \eta R_V(\lambda)$ strongly in $(\mathscr{L}^1_{-1/4}, \mathscr{L}^2)$. This completes the proof of Theorem 2.6.

3. The distorted Fourier-Airy transformations.

It was proved in Theorem 2.4 that $H = H_a + V$ has an absolutely continuous spectrum, if H is a short range perturbation of H_a. The distorted Fourier-Airy transformations, which we are about to construct, will give explicit diagonalizations of H. The operators $U(\lambda) = VR_a(\lambda)$ were introduced in the preceding section and their main properties were proved in Propositions 2.2 and 2.3.

DEFINITION 3.1. *Let $f \in C_0(\mathsf{R})$. The distorted Fourier-Airy transforms of f are given by*

$$F_\pm f(\sigma) = F_a(I + U(\sigma \pm i0))^{-1} f(\sigma).$$

The definition makes sense since $(I + U(\sigma \pm i0))^{-1}f$ is a continuous function of σ with values in $(\mathscr{L}^1_{-1/4}, \mathscr{L}^2)$, and F_a is continuous from that space to the space of bounded continuous functions on R.

PROPOSITION 3.2. *Let*

$$b_\pm(x, \sigma) = a(x - \sigma) - R(\sigma \pm i0)V_\sigma(x) \tag{3.1}$$

with $V_\sigma(x) = V(x)a(x - \sigma)$. Then $b_\pm(x, \sigma)$ are continuous functions on $\mathsf{R} \times \mathsf{R}$. They satisfy the differential equation

$$(P_a + V(x) - \sigma)b_\pm(x, \sigma) = 0 \tag{3.2}$$

and

$$\overline{\lim_{x \to -\infty}} \langle x \rangle^{1/4} |b_\pm(x.\sigma)| > 0. \tag{3.3}$$

If $f \in C_0(\mathsf{R})$, then

$$F_\pm f(\sigma) = \int b_\pm(x, \sigma) f(x)\, dx. \tag{3.4}$$

PROOF. When σ stays in a compact set we have a bound (independent of σ) for $V_\sigma(x)$ in $(\mathscr{L}^1_{-1/4}, \mathscr{L}^2)$. Since $\sigma \mapsto V_\sigma(\cdot)$ is a continuous function of σ with values in that space an application of Theorem 2.4 shows that

$$\sup_{x\in I} |R(\sigma' \pm i0)V_{\sigma'}(x) - R(\sigma \pm i0)V_\sigma(x)| \to 0$$

when $\sigma' \to \sigma$ and I is a bounded interval. The continuity of $b_\pm$ follows now from the continuity of $b_\pm$ as functions of x and (3.2) follows by letting $P_a + V - \sigma$ act on both sides of (3.1).

By (2.2) and Proposition 1.12 we have

$$b_\pm(x,\sigma) - a(x-\sigma) \pm i2\pi e^{\mp i\pi/3}(F_\pm V_\sigma)(\sigma)a(\omega^{\pm 1}(x-\sigma)) = o(|x|^{-1/4}), \quad x \to -\infty. \tag{3.5}$$

Hence (3.3) follows then from (3.5) and (1.13) (see also the proof of Proposition 1.13).

When proving (3.4) we can write

$$\int b_\pm(x,\sigma)f(x)\,dx = F_a f(\sigma) - \langle R(\sigma \pm i0)V_\sigma, f\rangle = F_a f(\sigma) - \langle V_\sigma, R(\sigma \pm i0)f\rangle. \tag{3.6}$$

Indeed, since $f \in C_0$ it suffices by Theorem 2.4 and by a simple argument of approximation to prove the second identity when $V_\sigma(x) = V(x)a(x-\sigma)$ is replaced by a function in C_0, and then the identity follows from (1.16). Now the right side of (3.6) is equal to $F_a f_{\sigma\pm i0}(\sigma)$ where

$$f_{\sigma\pm i0} = (I - VR(\sigma \pm i0))f = (I - U(\sigma \pm i0)(I + U(\sigma \pm i0))^{-1})f = (I + U(\sigma \pm i0))^{-1}f.$$

Hence $F_a f_{\sigma\pm i0}(\sigma) = F_\pm f(\sigma)$. This completes the proof of (3.4) and therefore also of Proposition 3.2.

THEOREM 3.3. *The distorted Fourier-Airy transformations $F_\pm$ extend to unitary operators on $\mathscr{L}^2$ and $F_\pm H = MF_\pm$ if M is the operator $Mu(\sigma) = \sigma u(\sigma)$ with domain equal to the set of all the $u \in \mathscr{L}^2$ such that $\sigma u(\sigma) \in \mathscr{L}^2$.*

PROOF. We first show that

$$\Im(R(\sigma \pm i0)f, f) = \pm\pi |F_\pm(\sigma)|^2, \quad f \in C_0. \tag{3.7}$$

Let $f_{\sigma\pm i0} = (I + U(\sigma \pm i0))^{-1}f$ be as before, so that $R(\sigma \pm i0)f = R_a(\sigma \pm i0)f_{\sigma\pm i0}$. It follows from Proposition 1.10 that

$$R(\sigma \pm i0)f(x) = c_\pm(\sigma)a(x-\sigma) + R_a(\sigma \mp i0)f_{\sigma\pm i0}(x)$$

with

$$c_\pm(\sigma) = \pm i2\pi F_\pm f(\sigma), \tag{3.8}$$

$$|c_\pm|^2 = \pm 4\pi \Im(R(\sigma \pm i0)f, f_{\sigma \pm i0}). \tag{3.9}$$

Since $f - f_{\sigma\pm i0} = U(\sigma \pm i0)f_{\sigma\pm i0} = VR_a(\sigma \pm i0)f_{\sigma \pm i0}$ and $R(\sigma \pm i0)f = R_a(\sigma \pm i0)f_{\sigma\pm i0}$, the equation (3.9) holds if $f_{\sigma\pm i0}$ is replaced by f. Therefore (3.7) follows from (3.8) and (3.9).

It follows from (3.7) that

$$(\phi(H)f, g) = \int \phi(\sigma)F_\pm f(\sigma)\overline{F_\pm g(\sigma)}\, d\sigma \tag{3.10}$$

when $f, g, \phi \in C_0$. Taking $g = f$ and letting $0 \leqq \phi_j$ be a sequence in C_0 that increases to 1 we conclude that $F_\pm$ extend to isometric transformations on $\mathscr{L}^2$, and (3.10) is true for arbitrary $f, g \in \mathscr{L}^2$, $\phi \in C_0$. Let $f \in \mathscr{L}^2$. Since $f \in \mathscr{D}(H_a)$ if and only if $\overline{\lim}_{j\to\infty} \|H\phi_j(H)f\| < \infty$, it follows that $f \in \mathscr{D}(H_a)$ if and only if $F_\pm f \in \mathscr{D}(M)$. Moreover,

$$F_\pm Hf(\sigma) = \sigma F_\pm f(\sigma), \quad f \in \mathscr{D}(H_a). \tag{3.11}$$

In fact, if $f \in C_0^\infty(\mathsf{R})$ then (3.11) follows from (3.2), (3.4) and a partial integration in the integral

$$F_\pm Hf(\sigma) = \int b_\pm(x, \sigma)Hf(x)\, dx,$$

and (3.11) holds then in general since $C_0^\infty(\mathsf{R})$ is dense in $\mathscr{D}(H_a)$ equipped with the graph norm.

It remains to prove that $F_\pm$ are onto, or equivalently that $F_\pm^*$ are injective. Assume that h is in the kernel of F_+^*. Since by (3.11) we have $(F_+^*(\phi h), g) = (h, F_+(\bar\phi(H)g))$ when $\phi \in C_0(\mathsf{R})$, $g \in \mathscr{L}^2$, it follows that $F_+^*(\phi h) = 0$ when $\phi \in C_0(\mathsf{R})$. In view of (3.4) this implies that $\overline{b_+(x,\sigma)}h(\sigma) = 0$ for every x and almost every σ. Hence $h = 0$ by (3.3). Similar arguments show that F_- is onto. This completes the proof of Theorem 3.3.

We shall now introduce the scattering matrix using the time-independent approach.

THEOREM 3.4. *Let*

$$S(\sigma) = 1 - i2\pi F_+ V_\sigma(\sigma), \tag{3.12}$$

where $V_\sigma(x) = V(x)a(x - \sigma)$. Then $S(\sigma)$ is a continuous function satisfying $|S(\sigma)| = 1$ and

$$F_+ f(\sigma) = S(\sigma)F_- f(\sigma), \quad f \in \mathscr{L}^2. \tag{3.13}$$

PROOF. It follows from the equations $F_\pm H = MF_\pm$ that $F_+F_-^*M = MF_+F_-^*$. Hence the unitary operator $F_+F_-^*$ commutes with multiplications by functions and we conclude that there is a measurable function $S(\sigma)$ of modulus 1 such that $F_+f(\sigma) = S(\sigma)F_-f(\sigma)$, $f \in \mathscr{L}^2$. By the definition of $b_\pm(x,\sigma)$ and (3.4) we have $b_+(x,\sigma) = S(\sigma)b_-(x,\sigma)$.

It follows from (3.5) that

$$b_+(x,\sigma) \equiv a(x-\sigma) - i2\pi e^{-i\pi/3}F_+V_\sigma(\sigma)a(\omega(x-\sigma)),$$

$$S(\sigma)b_-(x,\sigma) \equiv S(\sigma)a(x-\sigma) + i2\pi e^{i\pi/3}S(\sigma)F_-V_\sigma(\sigma)a(\omega^2(x-\sigma))$$

modulo functions which are $o(|x|^{-1/4})$ at $-\infty$. It follows from Proposition 1.13 and (1.6) then that

$$\begin{aligned}(S(\sigma)-1)a(x-\sigma) &= i2\pi F_+V_\sigma(\sigma)(\omega a(\omega(x-\sigma)) + \omega^2 a(\omega^2(x-\sigma)))\\ &= -i2\pi F_+V_\sigma(\sigma)a(x-\sigma)\end{aligned}$$

so that

$$S(\sigma) - 1 = -i2\pi F_+V_\sigma(\sigma).$$

This proves (3.12) from which follows the continuity of $S(\sigma)$. This completes the proof of Theorem 3.4.

4. Some problems in inverse scattering theory.

Let H_a be as before and let V be a short range perturbation of H_a. It was proved in Sections 2 and 3 that $H = H_a + V$ is a self-adjoint operator on the domain of H_a. The distorted Fourier-Airy transformations $F_\pm$ introduced in Definition 3.1 diagonalize H so that $F_\pm H = MF_\pm$, where M is the multiplication operator in (1.14). Since $F_aH_a = MF_a$ it follows that $F_\pm^* F_a$ are solutions of the following equation

$$HA = AH_a. \tag{4.1}$$

In the time-dependent scattering theory the operators $F_\pm^* F_a$ can be identified with the usual wave operators. In the following we shall find another solution of (4.1) which differs from the operators $F_\pm^* F_a$ when V is not identically equal to 0, since its distribution kernel will be supported in a half-plane. The distribution kernel $A(x,y)$ of a solution A of (4.1) satisfies the equation

$$(\partial_x^2 - \partial_y^2 + y - x)A(x,y) = V(x)A(x,y), \tag{4.2}$$

provided that the right side is a distribution. In order to construct a solution of the equation (4.2) we need to construct an inverse of the operator in the left side of (4.2).

LEMMA 4.1. *Assume that $x' \leqq y'$. Then the equation*

$$(4.3) \qquad (\partial_x^2 - \partial_y^2 + y - x)u(x, y) = \delta(x - x', y - y')$$

has a unique bounded and measurable solution with support in the set where $x \leqq x'$.

PROOF. Since the operator on the left side of (4.3) is invariant under any translation $(x, y) \mapsto (x + h, y + h)$, we may assume that $y' = 0$. In order to construct a solution we make the *Ansatz*

$$u(x, y) = \tfrac{1}{2}E(x - x', y)\psi(q(x, y, x')),$$

where $E(x, y)$ is the characteristic function of the set where $|y| \leqq |x|$ and $x \leqq 0$, $\psi(z)$ is a function to be determined and

$$q(x, y, x') = \left(\left(\frac{x-y}{2}\right)^2 - \left(\frac{x'}{2}\right)^2\right)\left(\frac{x+y-x'}{2}\right).$$

Since

$$(\partial_x - \partial_y)E(x - x', y) = -2\theta(-y)\delta(x' - x + y)$$

and

$$(\partial_x + \partial_y)E(x - x', y) = -2\theta(y)\delta(x' - x - y),$$

it follows that

$$(\partial_x - \partial_y)u(x, y) = -\theta(-y)\delta(x' - x + y)\psi(0) + E(x - x', y)\psi'(q)\left(\frac{x-y}{2}\right)\left(\frac{x+y-x'}{2}\right)$$

and

$$(\partial_x^2 - \partial_y^2)u(x, y) = \psi(0)\delta(x - x', y) + E(x - x', y)\psi''(q)\left(\frac{x-y}{2}\right)q + E(x - x', y)\psi'(q)\left(\frac{x-y}{2}\right).$$

Hence we have

$$(\partial_x^2 - \partial_y^2 + y - x)u(x, y) = \psi(0)\delta(x - x', y) + E(x - x', y)\left(\frac{x-y}{2}\right)(q\psi''(q) + \psi'(q) - \psi(q)).$$

In order to have a solution of (4.3) we shall therefore choose ψ so that $\psi(0) = 1$ and

$$(4.4) \qquad z\psi''(z) + \psi'(z) - \psi(z) = 0.$$

Since $x' \leqq 0$ and $|y| \leqq x' - x$ in the support of $u(x, y)$, it follows that $q(x, y, x') \leqq 0$ there. In order for u to be bounded, we choose ψ so that ψ is bounded on the negative axis. Hence we can take

$$\psi(z) = \sum_{k=0}^{\infty} z^k/(k!)^2.$$

Then $\psi(x) = J_0(2|x|^{1/2})$ when $x \leqq 0$, where $J_0(z)$ is the Bessel function of the first kind of order 0. This is an even function which is bounded on the real line. In fact,

$$J_0(x) = \left(\frac{2}{\pi|x|}\right)^{1/2} \cos(|x| - \pi/4)(1 + O(|x|^{-2}))$$

as $|x| \to \infty$ (see Whittaker -Watson p. 368). From this follows the estimate

$$|u(x, y)| \leqq C(1 + |q|)^{-1/4} \tag{4.5}$$

for some positive constant C.

Since the half-plane $x \leqq x'$ is non-characteristic for the operator $\partial_x^2 - \partial_y^2 + y - x$ which has analytic coefficients, the uniqueness follows. This completes the proof of Lemma 4.1.

Let $x' \leqq y'$ and let $G(x, y; x', y')$ be the function constructed in Lemma 4.1, i.e.

$$G(x, y; x', y') = \tfrac{1}{2}E(x - x', y - y')\psi(q(x - y', y - y', x' - y')), \tag{4.6}$$

where E and q are defined as in the proof of Lemma 4.1. The estimate (4.5) implies that

$$|G(x, y; x', y')| \leqq CE(x - x', y - y')(1 + |q(x - y', y - y', x' - y')|)^{-1/4}, \quad x' \leqq y', \tag{4.7}$$

and this implies in particular that $x \leqq y$, $x \leqq x'$ when $(x, y, x', y') \in \operatorname{supp} G$ and $x' \leqq y'$.

Since y is bounded from below in the support of G when the other variables are kept fixed, we can define the partial Fourier-Airy transform

$$\tilde{G}(x, \sigma; x', y') = \int a(y - \sigma)G(x, y; x', y')\, dy.$$

It follows from (1.14) and (4.3) that

$$(-\partial_x^2 + x - \sigma)\tilde{G}(x, \sigma; x', y') = -a(y' - \sigma)\delta(x - x'). \tag{4.8}$$

Since $\tilde{G}(x, \sigma; x', y') = 0$ when $x > x'$, this implies that

$$\tilde{G}(x, \sigma; x', y') = a(y' - \sigma)T(x - \sigma, x' - \sigma), \tag{4.9}$$

where

$$T(x, x') = (2\pi i)e^{-i\pi/3}\theta(x' - x)(a(x)a(\omega x') - a(x')a(\omega x)).$$

In fact, the right side of (4.9) is also a solution of (4.8), and it vanishes when $x > x'$.

We shall construct a solution of the equation (4.2) under some conditions on $V(x)$ that are more restrictive than the conditions considered in Section 2. We are going to require that $x \leqq y$ in the support of A. Writing

$$A(x, y) = \delta(x - y) + R(x, y)$$

we have then the following equation

$$(\partial_x^2 - \partial_y^2 + y - x)R(x, y) = V(x)\delta(x - y) + V(x)R(x, y).$$

An integral version is given by

$$R(x, y) = R_0(x, y) + \iint G(x, y; x', y')V(x')R(x', y')\, dx'\, dy'. \tag{4.10}$$

Here we have used the notation

$$R_0(x, y) = R_{0,V}(x, y) = \theta(y - x)\int_{(x+y)/2}^{\infty} V(z)\psi((x - y)^2(x + y - 2z)/8)\, dz/2 \tag{4.11}$$

for the expression which can formally be written as

$$\iint G(x, y; x', y')V(x')\delta(x' - y')\, dx'\, dy'.$$

If we denote the second term on the right side of (4.10) by $LR(x, y) = L_V R(x, y)$, then (4.10) takes the form

$$(I - L)R = R_0. \tag{4.12}$$

Hence we have to invert the operator $I - L$ on some suitable space. In order to be able to give a meaning to (4.12) and to solve that equation we shall assume that

$$\|V\|_{\mathscr{L}} = \int_{-\infty}^{\infty} (1 + |x|)|V(x)|\, dx < \infty. \tag{4.13}$$

The space of real-valued measurable functions which satisfy (4.13) is denoted by $\mathscr{L}$, and we let $\mathscr{L}^+$ be the Fréchet space of real-valued measurable functions V on R such that $\|V\|_z = \int_z^\infty (1 + |x|)|V(x)|\, dx < \infty$ for any real z. Following Melin [11] we introduce the following space.

DEFINITION 4.2. We let $\mathscr{R}$ be the set of real functions $R(x, y)$ on R^2 such that
(i) $R(x, y)$ tends to zero when $x + y \to +\infty$ and $x, y \geqq C$,

(ii) the restrictions of $R(x, y)$ to the sets where $x < y$ and $x > y$ extend to continuous functions on R^2,

(iii) if

$$R_*(z, w) = \sup_{x \geqq z, y \geqq z, x+y \geqq 2w} |R(x, y)|, \tag{4.14}$$

then

$$\|R\|_z = \max_{z \leqq w} R_*(z, w) + 2\int_z^\infty R_*(z, w)dw < \infty$$

for any real z.

We let $\mathscr{N} \subset \mathscr{R}$ be the subspace consisting of all $R(x, y) \in \mathscr{R}$ such that $R(x, y) = 0$ when $x > y$. (The spaces $\mathscr{R}$, $\mathscr{N}$ are denoted by V^+, $\mathscr{N}^+$ in Melin [11].) We also let $\mathscr{I} + \mathscr{N}$ be the set of distributions $\delta(x - y) + R(x, y)$ where $R(x, y) \in \mathscr{N}$, and $\mathscr{I} + \mathscr{R}$ is defined in a similar way. The space $\mathscr{R}$ is a Fréchet space under the semi-norms $\|R\|_z$ above, and the mapping $R(x, y) \mapsto R^*(x, y) = R(y, x)$ is a homeomorphism on $\mathscr{R}$. In what follows we shall always identify a continuous operator $K: C_0^\infty(\mathsf{R}) \to \mathscr{D}'(\mathsf{R})$ with its distribution kernel $K(x, y) \in \mathscr{D}'(\mathsf{R} \times \mathsf{R})$. We have continuous composition operations $\mathscr{N} \times \mathscr{R} \ni (R_1, R_2) \mapsto R_1 R_2,\ R_2 R_1^* \in \mathscr{R}$, and $\mathscr{I} + \mathscr{N}$ is a topological group (Proposition 3.4 in Melin [11]). If $R(x, y) \in \mathscr{N}$, then $|R(x, y)| \leqq R_*(x/2, y/2)$ when $x \geqq 0$. Since $\int_z^\infty R_*(z_0, w)\,dw \to 0$ when $z \to \infty$ and z_0 is fixed, and $R_*(z, w)$ is decreasing in each of its arguments, it follows that

$$\|R\|_z \to 0, \ z \to +\infty. \tag{4.15}$$

LEMMA 4.3. (i) *If $V \in \mathscr{L}^+$, then $I - L_V$ is a linear homeomorphism on $\mathscr{N}$.*

(ii) *The mapping $\mathscr{L}^+ \times \mathscr{N} \ni (V, R) \mapsto (I - L_V)^{-1} R \in \mathscr{N}$ is continuous.*

(iii) *The mapping $\mathscr{L}^+ \ni V \mapsto R_{0,V} \in \mathscr{N}$ is continuous.*

(iv) *If $V \in \mathscr{L}$ and $R_V = (I - L_V)^{-1} R_{0,V}$, then $V(x)R_V(x, y) \in \mathscr{L}^1(\mathsf{R}^2)$ and the mapping $\mathscr{L} \ni V \mapsto V(x)R_V(x, y) \in \mathscr{L}^1(\mathsf{R}^2)$ is continuous.*

PROOF. Since $|G(x, y; x', y')| \leqq CE(x - x', y - y')$, the proof is the same as that of Lemma 4.2 of Melin [11].

We shall need the following simple result.

LEMMA 4.4. (i) *Assume that $f(x, y)$ is a measurable function with support in the set where $x \leqq y$ and that*

$$\iint_{x \geqq x_0} |f(x, y)|\,dx\,dy < \infty$$

for every x_0. Then

$$Jf(x,y) = \iint G(x,y;x',y')f(x',y')\,dx'\,dy'$$

is a continuous function and

$$(\partial_x^2 - \partial_y^2 + y - x)Jf(x,y) = f(x,y).$$

(ii) *If* $R(x,y) \in \mathcal{N}$ *and* $(\partial_x^2 - \partial_y^2 + y - x)R(x,y) = 0$, *then* $R = 0$.

PROOF. (i) Since $x' \geqq x$ in the support of G, it follows that the restriction of Jf to the set where $x \geqq x_0$ depends only on the restriction of f to that set. A simple argument of approximation, which uses the fact that $C_0^\infty(\mathsf{R}^2)$ is dense in $\mathcal{L}^1(\mathsf{R}^2)$, allows us therefore to assume that $f \in C_0^\infty(\mathsf{R}^2)$, and then (i) follows from (4.3).

(ii) Let

$$\tilde{R}(x,\sigma) = \int a(y-\sigma)R(x,y)dy$$

be the Fourier-Airy transform with respect to the last variable. Then $(\partial_x^2 - x + \sigma)\tilde{R}(x,\sigma) = 0$, and it follows from (4.15) that $\tilde{R}(x,\sigma)/a(x-\sigma) \to 0$ as $x \to +\infty$. From this observation and (1.13) we conclude that $\tilde{R}(x,\sigma) = 0$. Hence $R = 0$. This completes the proof of Lemma 4.4.

THEOREM 4.5. *If* $V \in \mathcal{L}^+$, *then there is a unique solution* $A(x,y) = \delta(x-y) + R_V(x,y)$ *in* $\mathcal{I} + \mathcal{N}$ *of the equation* (4.2). *Moreover,* $A(x,y) - \delta(x-y) - R_0(x,y)$ *is continuous.*

PROOF. An application of Lemma 4.3 shows that $R_0(x,y) \in \mathcal{N}$ and that there is a unique $A(x,y) = \delta(x-y) + R(x,y) \in \mathcal{I} + \mathcal{N}$ such that

$$A(x,y) = \delta(x-y) + R_0(x,y) + L_V R(x,y).$$

Since $(\partial_x^2 - \partial_y^2 + y - x)R_0(x,y) = V(x)\delta(x-y)$ and $(\partial_x^2 - \partial_y^2 + y - x)(L_V R)(x,y) = V(x)R(x,y)$ by Lemma 4.4, it follows that $A(x,y)$ solves (4.2), and $L_V R(x,y)$ is continuous in view of the previous lemma. If $A'(x,y) = \delta(x-y) + R'(x,y)$ is another solution of (4.2) in $\mathcal{I} + \mathcal{N}$ and $S(x,y) = R'(x,y) - L_V R'(x,y) - R_0(x,y)$, then $S(x,y) \in \mathcal{N}$ and it follows from (i) of Lemma 4.4 that $(\partial_x^2 - \partial_y^2 + y - x)S(x,y) = 0$. The second part of that Lemma shows that $S = 0$, and since $I - L_V$ is injective on $\mathcal{N}$ it follows finally that $R' = R$. This completes the proof of Theorem 4.5.

The next result shows that $V(x)$ is uniquely determined from $A(x,y)$.

PROPOSITION 4.6. *Let* $A(x,y) = \delta(x-y) + R_V(x,y)$ *be the solution of* (4.2), *where* $V \in \mathcal{L}^+$. *Then*

$$R_V(x, x + 0) = 2^{-1} \int_x^\infty V(z)\, dz.$$

PROOF. This is an immediate consequence of the fact that $R_V(x, y) - R_{0,V}(x, y)$ is continuous and supported in the set where $x \leqq y$.

If $V \in \mathscr{L}$ and $A(x, y) = \delta(x - y) + R_V(x, y)$ is as above, then the Fourier-Airy transform

$$\tilde{A}(x, \sigma) = a(x - \sigma) + \int a(y - \sigma) R_V(x, y)\, dy$$

of $A(x, y)$ with respect to the second variable is a solution of the equation (3.2). We observe that (see the proof of Lemma 4.4)

$$\tilde{A}(x, \sigma)/a(x - \sigma) \to 1, \quad x \to +\infty. \tag{4.16}$$

PROPOSITION 4.7. *Assume $V \in \mathscr{L} \cap \mathscr{V}$. There are bounded continuous functions $\tilde{N}_\pm(\sigma) \neq 0$ so that*

$$\tilde{A}(x, \sigma) = \tilde{N}_\pm(\sigma) \overline{b_\pm(x, \sigma)}, \tag{4.17}$$

and $\tilde{N}_\pm(\sigma) \to 1$ as $|\sigma| \to \infty$.

PROOF. Assume first that V is compactly supported. Then every solution of (3.2) which is in $\mathscr{L}^2$ at $+\infty$ must be proportional to $\tilde{A}(x, \sigma)$. Since $b_\pm(x, \sigma)$ are in $\mathscr{L}^2$ at $+\infty$ by Lemma 2.4 and (3.1), it follows that $b_\pm(x, \sigma)$ are proportional to $\tilde{A}(x, \sigma)$.

In the general case we can take a sequence V_j of compactly supported functions such that $|V_j| \leqq |V|$ and $V_j \to V$ uniformly on any compact set. Since $V_j(x) a(x - \sigma)$ tends to $V(x) a(x - \sigma)$ in $(\mathscr{L}^1_{-1/4}, \mathscr{L}^2)$, it follows from Theorem 2.6 that the corresponding solutions $b_{\pm,j}$ of (3.2) converges to $b_\pm$ in $\mathscr{L}^2_{\mathrm{loc}}(\mathsf{R})$. From Lemma 4.3 follows that $\tilde{A}_{V_j}(x, \sigma)$ converges locally uniformly to $\tilde{A}_V(x, \sigma)$. If I is a compact interval on which $\tilde{A}(x, \sigma)$ does not vanish, then $b_{\pm,j}(x, \sigma)/\tilde{A}_{V_j}(x, \sigma)$ converges in $\mathscr{L}^2(I)$. Since these functions do not depend on x, we conclude that $b_\pm$ and $\tilde{A}$ are proportional to each other. Hence (4.17) holds for some functions $\tilde{N}_\pm(\sigma) \neq 0$. Since $b_\pm(x, \sigma)$ depend continuously on σ by Proposition 3.2 and the corresponding statement for $\tilde{A}(x, \sigma)$ is obvious, it is true that $\tilde{N}_\pm(\sigma)$ are continuous functions.

In order to prove that $\tilde{N}_\pm(\sigma)$ are bounded and to study their behaviour at infinity we introduce the functions

$$Y(\sigma) = 2\pi i e^{-i\pi/3} \int a(\omega(x-\sigma)) V(x) \tilde{A}(x,\sigma) dx,$$

$$Z(\sigma) = 2\pi i \int a(x-\sigma) V(x) \tilde{A}(x,\sigma) dx.$$

The integrals in the right side are convergent since $V(x)A(x,y)$ is a bounded measure on R^2 (Lemma 4.3) and $a(\omega(x-\sigma))a(y-\sigma)$, $a(x-\sigma)a(y-\sigma)$ are bounded functions of (x,y) when $x \leqq y$. These statements follow from (1.15), (1.18) and Lemma 1.4 from which it also follows that

(4.18)

$$|Y(\sigma)| + |Z(\sigma)| \leqq C \int\int (1+|x-\sigma|)^{-1/4}(1+|y-\sigma|)^{-1/4} |V(x)A(x,y)| dx dy,$$

where $|V(x)A(x,y)|$ should be interpreted as a measure in the right side. It follows from this that $Y(\sigma)$ and $Z(\sigma)$ are continuous functions which tend to 0 at infinity.

The functions $Y(\sigma)$ and $Z(\sigma)$ can be used to describe the asymptotic behavior of $\tilde{A}(x,\sigma)$ when x tends to $-\infty$. The equations (4.10) and (4.12) show that

$$R(x,y) = \int\int G(x,y;x',y') V(x') A(x',y') dx' dy', \tag{4.19}$$

if again we treat the leading part of $V(x')A(x',y')$ as a measure. When x is kept fixed then we have a bound from below for x' by x in the support of the integrand. Since $\int |G(x,y;x',y')| \, dy \leqq C|x-x'|$ and

$$\int\int_{x \leqq x'} |x' V(x') A(x',y')| dx' dy' < \infty,$$

it follows that $G(x,y;x',y') V(x') A(x',y')$ is a bounded measure on R^3 for every x. If we multiply (4.19) by $a(y-\sigma)$ and integrate with respect to y, we find therefore that

$$\tilde{A}(x,\sigma) - a(x-\sigma) = \int\int \tilde{G}(x,\sigma;x',y') V(x') A(x',y') dx' dy'.$$

By (4.9) the right side can be written as

$$\int T(x-\sigma, x'-\sigma) V(x') \tilde{A}(x',\sigma) dx',$$

and it follows that

$$\begin{aligned} &\tilde{A}(x,\sigma) - a(x-\sigma) \\ &= (2\pi i) e^{-i\pi/3} a(x-\sigma) \int_x^\infty a(\omega(x'-\sigma)) V(x') \tilde{A}(x',\sigma) dx' \end{aligned}$$

$$\text{(4.20)} \qquad - (2\pi i)e^{-i\pi/3} a(\omega(x-\sigma)) \int_x^\infty a(x'-\sigma) V(x') \tilde{A}(x',\sigma) dx'.$$

Since $\int |V(x')\tilde{A}(x',\sigma)| dx' < \infty$, it follows that

(4.21)

$$\tilde{A}(x,\sigma) = (1 + Y(\sigma))a(x-\sigma) - e^{-i\pi/3} Z(\sigma) a(\omega(x-\sigma)) + o(|x|^{-1/4}), \quad x \to -\infty.$$

Comparing (4.21) and (3.5) we conclude that

$$\tilde{A}(x,\sigma) = (1 + Y(\sigma))b_+(x,\sigma).$$

Also taking into account that $\tilde{A}(x,\sigma)$ is real-valued, we conclude from (4.21) and (3.5) that

$$\tilde{A}(\sigma,x) = (1 + \overline{Y(\sigma)})b_-(x,\sigma).$$

Hence (4.17) holds with

$$\tilde{N}_+(\sigma) = 1 + \overline{Y(\sigma)},$$

$$\text{(4.22)} \qquad \tilde{N}_-(\sigma) = 1 + Y(\sigma).$$

This completes the proof of Proposition 4.7.

PROPOSITION 4.8. *The following relations hold:*

$$\text{(4.23)} \qquad S(\sigma) = \frac{1 + \overline{Y(\sigma)}}{1 + Y(\sigma)},$$

$$\text{(4.24)} \qquad Z(\sigma) = Y(\sigma) - \overline{Y(\sigma)}.$$

PROOF. The equation (4.23) follows from (4.17) and (4.22) since $|S(\sigma)| = 1$ and $b_+(x,\sigma) = S(\sigma)b_-(x,\sigma)$. Since $Z(\sigma)(1 + Y(\sigma))^{-1} = 2\pi i(F_+ V_\sigma)(\sigma)$ by (3.5) and (4.21), and $S(\sigma) - 1 = -2\pi i(F_+ V_\sigma)(\sigma)$, it is true that

$$Z(\sigma)(1 + Y(\sigma))^{-1} = 1 - S(\sigma) = \frac{Y(\sigma) - \overline{Y(\sigma)}}{1 + Y(\sigma)}.$$

Hence (4.24) holds.

We have already seen that $Y(\sigma)$ and $Z(\sigma)$ are continuous functions which tend to 0 at $\pm\infty$. In order to obtain more precise estimates about Y and Z we prove the following lemma.

LEMMA 4.9. *Assume that $V \in \mathscr{L}$. Then there is a positive constant C so that*

$$\text{(4.25)} \qquad \langle x \rangle^{-\alpha - 1/4} \int |R_V(x,y)| \langle y \rangle^\alpha dy \leqq C, \quad 0 \leqq \alpha \leqq 1/2.$$

PROOF. We shall first prove that

$$\int |G(x,y;x',y')|\,dy \leqq C_0\langle x'-x\rangle^{1/2}\langle y'-x\rangle^{-1/4}, \quad x' \leqq y'. \tag{4.26}$$

It follows from (4.7) that the left side can be estimated from above by a constant times

$$\int_{|y|\leqq\tau} (1+|q(x-y',y,-\varrho)|)^{-1/4}\,dy \leqq C_1 \int_{|y|\leqq\tau} (|(y+\tau+\varrho)^2-\varrho^2||y-\tau|)^{-1/4}\,dy,$$

where $\tau = x'-x \geqq 0$ and $\varrho = y'-x' \geqq 0$. The integral in the right side is equal to

$$\tau^{1/4}\int_{|y|\leqq 1} (y+1+2\varrho/\tau)^{-1/4}(1-y^2)^{-1/4}\,dy \leqq C_1'(1+\tau)^{1/2}(1+\tau+\varrho)^{-1/4},$$

and this proves (4.26) since $\tau+\varrho = y'-x$.

Since (4.19) holds, it follows that

$$\int |R(x,y)|\,dy \leqq C_0\int \langle x'-x\rangle^{1/4}|V(x')|\,dx'$$
$$+ C_0 \iint_{x\leqq x'\leqq y'} \langle x'-x\rangle^{1/2}\langle y'-x\rangle^{-1/4}|V(x')R(x',y')|\,dx'\,dy'.$$

In the right side we may estimate $\langle x'-x\rangle^{1/2}\langle y'-x\rangle^{-1/4}$ from above by $\langle x'-x\rangle^{1/4}$. Set $h_\beta(x) = \int |R(x,y)|\langle y\rangle^\beta dy$. Then there is a positive constant C_2 so that $h_0(x) \leqq C_2$ when $x \geqq 0$, and when $x \leqq 0$ it follows from the inequality above that

$$h_0(x) \leqq C_0\int \langle x'-x\rangle^{1/4}|V(x')|\,dx' + C_0\int_{x\leqq x'\leqq 0}\langle x'-x\rangle^{1/4}|V(x')|h_0(x')dx'$$
$$+ C_0C_2\int_{0\leqq x'}\langle x'-x\rangle^{1/4}|V(x')|\,dx'.$$

The first term may be estimated from above by a constant times $\langle x\rangle^{1/4}$. In the second term in the right side we may estimate $\langle x'-x\rangle^{1/4}$ from above by $\langle x\rangle^{1/4}$, and we know that $\int |V(x')|h_0(x')dx' < \infty$. It follows therefore that the second term in the right side may be estimated from above by a constant times $\langle x\rangle^{1/4}$. In the third term in the right side we may estimate $\langle x'-x\rangle^{1/4}$ from above by $\langle x\rangle^{1/4}+\langle x'\rangle^{1/4}$, and since $\int |x'||V(x')|\,dx' < \infty$, we conclude that $\langle x\rangle^{-1/4}h_0(x)$ is a bounded function. Hence (4.25) holds when $\alpha = 0$.

When proving (4.25) in general we apply (4.19) and (4.26) again. Since

$$\int |G(x, y; x', y')| \langle y \rangle^\alpha \, dy \leqq \int |G(x, y; x', y')| (\langle x' - x \rangle^\alpha + \langle y' \rangle^\alpha) dy,$$

the following estimate follows

$$\int |R(x, y)| \langle y \rangle^\alpha \, dy \leqq C \int (\langle x \rangle + \langle x' \rangle)^{\alpha + 1/4} |V(x')| \, dx'$$

$$+ C \iint_{x \leqq x' \leqq y'} (1 + |x' - x| + |y'|)^\alpha \langle x' - x \rangle^{1/2} \langle y' - x \rangle^{-1/4} |V(x') R(x', y')| dx' dy'$$

The first term in the right side can be estimated from above by a constant times $\langle x \rangle^{\alpha + 1/4}$, and the contribution to the second term from an integration over a set where $\langle y' \rangle (\langle x \rangle + \langle x' \rangle)^{-1}$ is large can be estimated from above by a positive constant times

$$\int \langle x' - x \rangle^{1/2} |V(x')| h_{\alpha - 1/4}(x') dx'.$$

The contribution from the set where $\langle y' \rangle (\langle x \rangle + \langle x' \rangle)^{-1}$ is not large can be estimated from above by a constant times

$$\iint_{x \leqq x' \leqq y'} (\langle x \rangle + \langle x' \rangle)^\alpha \langle x - x' \rangle^{1/4} |V(x') R(x', y')| \, dx' \, dy'$$

$$\leqq \int_{x \leqq x'} (\langle x \rangle + \langle x' \rangle)^\alpha \langle x - x' \rangle^{1/4} |V(x')| h_0(x') \, dx'.$$

When x is positive then $h_0(x')$ in the right side is bounded and the right side can be estimated from above by a constant times $\langle x \rangle^{\alpha + 1/4}$. When $x < 0$ we can estimate the right side from above by a constant times

$$\int_{x \leqq x' \leqq 0} \langle x \rangle^{\alpha + 1/4} |V(x')| h_0(x') \, dx' + \int_{0 \leqq x'} (\langle x \rangle + \langle x' \rangle)^{\alpha + 1/4} |V(x')| \, dx'.$$

This can again be estimated from above by a constant times $\langle x \rangle^{\alpha + 1/4}$. It follows from these estimates that

$$h_\alpha(x) \leqq C \langle x \rangle^{\alpha + 1/4} + C \int \langle x' - x \rangle^{1/2} |V(x')| h_{\alpha - 1/4}(x') dx'.$$

If $\alpha = 1/4$, then it follows that

$$h_\alpha(x) = h_{1/4}(x) \leqq C \langle x \rangle^{1/2} + C \int \langle x' - x \rangle^{1/2} \langle x' \rangle^{1/4} |V(x')| dx' \leqq C' \langle x \rangle^{1/2}.$$

Hence the estimate (4.25) is true when $\alpha = 1/4$. If $\alpha = 1/2$, then it follows that

$$h_\alpha(x) \leqq C\langle x\rangle^{3/4} + C\int \langle x' - x\rangle^{1/2}|V(x')|\langle x'\rangle^{1/2}dx' \leqq C'\langle x\rangle^{3/4}.$$

Hence (4.25) holds when $\alpha = 0$ and $\alpha = 1/2$, and therefore it holds for $0 \leqq \alpha \leqq 1/2$. This completes the proof of Lemma 4.9.

PROPOSITION 4.10. *Assume that $V \in \mathscr{L}$. Then*

$$|Y(\sigma)| + |Z(\sigma)| \leqq C\langle\sigma\rangle^{-1/2}.$$

PROOF. This is an immediate consequence of (4.18) and Lemma 4.9.

PROPOSITION 4.11. *Assume that $V \in \mathscr{L}$. Then*

$$|\tilde{A}(x,\sigma) - a(x-\sigma)| \leqq a(x-\sigma)\|R_V\|_x,\quad x - \sigma \geqq 0, \tag{4.27}$$

and there is a positive constant C so that

$$|\tilde{A}(x,\sigma) - a(x-\sigma)| \leqq C\langle x-\sigma\rangle^{-1/4}\langle\sigma\rangle^{-1/2}e^{-\Re\varphi(x-\sigma)}, \tag{4.28}$$

where φ is defined in (1.7).

PROOF. The first statement is obvious, since $a(x)$ is a positive decreasing function when $x > 0$. In order to prove (4.28) we first observe that it follows from (4.25) with $\alpha = 1/4$ that there is a positive constant C so that

$$|\tilde{A}(x,\sigma) - a(x-\sigma)| \leqq \int |a(y-\sigma)R_V(x,y)|\,dy \leqq C\langle x\rangle^{1/2}\left(\sup_{y\geqq x}\langle y\rangle^{-1/4}|a(y-\sigma)|\right).$$

Since the last factor in the right side can be estimated from above by a constant times $\langle\sigma\rangle^{-1/4}e^{-\Re\varphi(x-\sigma)}$, it follows that $|\tilde{A}(x,\sigma)| \leqq C\langle x\rangle^{1/2}\langle\sigma\rangle^{-1/4}e^{-\Re\varphi(x-\sigma)}$ for some other positive constant C. An application of (4.20) together with the definition of $Y(\sigma)$ gives

$$\begin{aligned}|\tilde{A}(x,\sigma) - (1 + Y(\sigma))a(x-\sigma)| \leqq C&\int_{-\infty}^{x} B(x-\sigma, x'-\sigma)|V(x')\tilde{A}(x',\sigma)|\,dx' \\ &+ C\int_{x}^{\infty} B(x'-\sigma, x-\sigma)|V(x')\tilde{A}(x',\sigma)|\,dx',\end{aligned}$$

where $B(x,y) = \theta(x-y)|a(x)a(\omega y)|$ can be estimated from above by a constant times $\langle x\rangle^{-1/4}\langle y\rangle^{-1/4}e^{\Re\varphi(y)-\Re\varphi(x)}$. Also estimating $\tilde{A}(x',\sigma)$ from above by a constant times $\langle x'\rangle^{1/2}\langle\sigma\rangle^{-1/4}e^{-\Re\varphi(x'-\sigma)}$ we obtain (4.28) since $\langle x\rangle^{3/4}V(x)$ is integrable and $\Re\varphi(x)$ increases. This completes the proof of Proposition 4.11.

PROPOSITION 4.12. *Assume that $V \in \mathscr{L}$. Set*

$$Y_0(\sigma) = 2\pi i e^{-i\pi/3} \int a(\omega(x-\sigma))a(x-\sigma)V(x)dx,$$

$$Z_0(\sigma) = 2\pi i \int a^2(x-\sigma)V(x)dx.$$

Then there is a positive constant C so that

$$|Y(\sigma) - Y_0(\sigma)| + |Z(\sigma) - Z_0(\sigma)| \leqq C\langle\sigma\rangle^{-1}.$$

PROOF. This is an immediate consequence of (4.28).

We assume now $V \in \mathscr{L} \cap \mathscr{V}$ again. Let $\tilde{N}_\pm(\sigma)$ be as before and let $\tilde{N}_\pm$ be the multiplication by $\tilde{N}_\pm(\sigma)$. Then $N_\pm = F_a^* \tilde{N}_\pm F_a$ are linear homeomorphisms on $\mathscr{L}^2$ by Proposition 4.7. We set

$$W = (N_+^* N_+)^{-1} = (N_-^* N_-)^{-1}, \tag{4.29}$$

where the second identity follows from (4.22). It follows therefore that W is a linear homeomorphism on $\mathscr{L}^2$, and the following result shows then that the same is true for A.

PROPOSITION 4.13. *If $V \in \mathscr{L} \cap \mathscr{V}$, then*

$$A = F_\pm^* F_a N_\pm, \tag{4.30}$$

$$I = AWA^*. \quad \textit{(Gel'fand-Levitan-Marchenko equation)} \tag{4.31}$$

PROOF. The formula (4.30) follows from (4.17) since $\tilde{A}(x,\sigma)$ is the integral kernel of AF_a^* and $F_a N_\pm F_a^*$ is the multiplication by $\tilde{N}_\pm(\sigma)$. The formula (4.31) follows now from (4.29) and (4.30). This completes the proof of Proposition 4.13.

In order to be able to solve A from (4.31) we introduce the space $\mathscr{P}$ consisting of all $B \in \mathscr{I} + \mathscr{R}$ such that $B^* = B$ and B has a positive lower bound on $\mathscr{L}^2(z,\infty)$ for any real z (Definition 3.5 in [11]). The following result follows from Proposition 3.6 of [11] together with the fact that $\mathscr{I} + \mathscr{N}$ is a topological group.

PROPOSITION 4.14. *If $A \in \mathscr{I} + \mathscr{N}$, then $A^{-1}(A^*)^{-1} \in \mathscr{P}$ and the mapping*

$$\mathscr{I} + \mathscr{N} \ni A \mapsto A^{-1}(A^*)^{-1} \in \mathscr{P} \tag{4.32}$$

is a homeomorphism.

We shall finish our discussions by giving some comments on the scheme for inverse scattering. Thus assume that $V \in \mathscr{L} \cap \mathscr{V}$ and that the function $S(\sigma)$ is given. Then (4.23) holds and it follows from (1.15) together with the definition of $Y(\sigma)$ that

$$(4.33)\qquad Y(\sigma) = \iint R_a(y, x; \sigma + i0)\, V(x) A(x, y)\, dx\, dy.$$

Hence (1.18) implies that $\tilde{N}_-(\sigma) = 1 + Y(\sigma)$ extends to an analytic function in $\mathbf{C}_+$ which is continuous in the closure of that set, and $\tilde{N}_-(\sigma) \to 1$ when $\sigma \to \infty$ in $\mathbf{C}_+$. It follows therefore from (4.22) and (4.23) that

$$(4.34)\qquad S(\sigma) = \frac{\tilde{N}_+(\sigma)}{\tilde{N}_-(\sigma)}, \quad \sigma \in \mathbf{R},$$

where $\tilde{N}_+(\sigma) = 1 + \overline{Y(\bar{\sigma})}$ extends to an analytic function in $\mathbf{C}_-$ which is continuous in $\bar{\mathbf{C}}_-$ and tends to 1 at infinity. The function $\tilde{N}_\pm(\sigma)$ has no zeros in $\mathbf{C}_\mp$. To see this we set

$$V_t(x) = tV(x), \quad t \in [0, 1],$$

and let $Y_{(t)}(\sigma)$ be the corresponding functions obtained when V is replaced by V_t. It follows from Lemma 4.3 and (4.33) that $Y_{(t)}(\sigma)$ depends continuously on t when $\sigma \in \bar{\mathbf{C}}_+$ and it is uniformly small when σ is large. Since $1 + Y_{(t)}(\sigma) \neq 0$ when σ is real and $Y_{(0)} = 0$, a simple application of the Hurwitz's theorem (p. 171 in [14]) shows that $1 + Y_{(t)}(\sigma) \neq 0$ when $\sigma \in \bar{\mathbf{C}}_+$ and $t \in [0, 1]$. Hence $\tilde{N}_\pm(\sigma)$ have no zeros and we have a representation of $S = \tilde{N}_+/\tilde{N}_-$ as a quotient of functions analytic in $\mathbf{C}_\mp$ with continuous extension to the closures $\mathbf{C}_\mp$ and no zeros there. This representation is of course unique. Hence W is uniquely determined from S. It follows then from Propositions 4.6 and 4.14 that V is also uniquely determined from S.

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