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TRIVIAL FIXED POINT SUBALGEBRAS OF THE ROTATION ALGEBRA

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Abstract.

Here we prove that the fixed point subalgebras of the rotation algebra $\mathscr{A}_\theta$ under parabolic and hyperbolic automorphisms induced by SL(2, Z) in the standard representation are trivial.

In this note we study the fixed point subalgebras of the rotation algebra $\mathscr{A}_\theta$, the universal C^*-algebra generated by two unitaries U and V satisfying $VU = \rho UV$ with $\rho = e^{2\pi i\theta}$ and $0 \leqq \theta < 1$, induced by SL(2, Z), where any $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbf{Z})$ induces the automorphism τ_A of $\mathscr{A}_\theta$,

$$\tau_A(U) = e^{ac\pi i\theta} U^a V^c, \tau_A(V) = e^{bd\pi i\theta} U^b V^d,$$

see [11]. We will prove the results we announced in [5], concerning the fixed point subalgebras of the infinite order automorphisms of $\mathscr{A}_\theta$, namely the parabolic and the hyperbolic ones (See Theorem 2.0.8 of [5]). These results are valid for any $\theta \in [0, 1)$. If $A \in \mathrm{SL}(2, \mathbf{Z})$, we shall denote by $\mathscr{A}_\theta^A$ the fixed point subalgebra of the automorphism τ_A of $\mathscr{A}_\theta$.

Many new results concerning the fixed point subalgebras associated to elements of SL(2, Z) have been found recently. In [1] and [2] Bratteli, Elliott, Evans and Kishimoto started studying $\mathscr{A}_\theta{}^{-I_2}$, $\theta \in [0, 1)$. Subsequently Kumjian [10] computed the K-theory of $\mathscr{A}_\theta{}^{-I_2}$ for θ irrational. In [6], [7], [8] and [9] we give a characterization of the fixed point subalgebra associated to any finite order, i.e. elliptic, element of SL(2, Z), respectively in the rational and irrational case. Very recently in [3] Bratteli and Kishimoto have shown that $A_\theta{}^{-I_2}$ is an AF algebra, while in [4] Elliott and Evans have shown that $\mathscr{A}_\theta$ is an inductive limit of direct sums of two circle algebras (both results require θ to be irrational). There are still many open problems. For example an interesting question to ask is if the fixed point subalgebras of the elliptic elements are AF for θ irrational. Unfortunately

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the techniques in [3] and [4] are not directly applicable in these more general examples.

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We now describe the situation for the (parabolic and hyperbolic) infinite order elements of SL(2, Z). Recall the following propsition from [5].

PROPOSITION. *If A is an infinite order element of* SL(2, Z), *then* $|\mathrm{Trace}(A)| \geqq 2$. *Moreover if* $A \in \mathrm{SL}(2, \mathbf{Z})$, $|\mathrm{Trace}(A)| = 2$, *and* $A^n \neq I_2$ *for any* $n \in \mathbf{Z} \setminus \{0\}$, *then A is conjugate in* SL(2, Z) *to* $\pm W^n$ *for some* $n \in \mathbf{Z} \setminus \{0\}$, *where* $W = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

THEOREM. 1. *If* $A \in \mathrm{SL}(2, \mathbf{Z})$, $A \neq I_2$, *and* $\mathrm{Trace}(A) = 2$, *then* $\mathscr{A}_\theta^A \cong C(S^1)$.
2. *If* $A \in \mathrm{SL}(2, \mathbf{Z})$, $A \neq -I_2$, *and* $\mathrm{Trace}(A) = -2$, *then* $\mathscr{A}_\theta^A \cong C([-2, +2])$.
3. *If* $A \in \mathrm{SL}(2, \mathbf{Z})$, *and* $|\mathrm{Trace}(A)| > 2$, *then* $\mathscr{A}_\theta^A \cong \mathbf{C}$.

For the proof of 3. of this theorem in the irrational case see also [11].

PROOF. 1. Since A is conjugate to W^k for some $k \in \mathbf{Z} \setminus \{0\}$, we consider the case in which

$$A = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}, \tau_A(U) = U, \tau_A(V) = e^{\pi i \theta k} U^k V, k \in \mathbf{Z}, k \neq 0.$$

Let x be a fixed point of τ_A and take its Fourier expansion,

$$x = \sum_{n,m \in \mathbf{Z}} c_{n,m} U^n V^m.$$

So,

$$\tau_A(x) = \sum c_{n,m} \tau_A(U^n V^m) = \sum c_{n,m} e^{\pi i \theta k m^2} U^{n+mk} V^m.$$

Therefore, by recursion,

$$\tau_A^K(x) = \sum c_{nm} e^{K \pi i \theta k m^2} U^{n+Kmk} V^m, \forall K \in \mathbf{N}.$$

But if $x \in \mathscr{A}_\theta$ is a fixed point of τ_A, it follows by equating coefficients that,

$$c_{n,m} = c_{n+Kmk,m} e^{-K \pi i \theta k m^2}, \forall K \in \mathbf{N}.$$

Thus,

$$\|c_{n,m}\| = \|c_{n+Kmk,m}\|.$$

But by Riemann-Lebesgue's lemma, we have $c_{n,m} \to 0$ as the indices tend to infinity, so $c_{n,m} = 0$ for all $m \neq 0$. Therefore $x = \sum c_n U^n$, i.e. the fixed point subalgebra of τ_A is $C(S^1)$. This proves 1.

2. Since A is conjugate to $-W^k$ for some $k \in \mathbb{Z} \setminus \{0\}$, we now consider the case in which $A = \begin{pmatrix} -1 & k \\ 0 & -1 \end{pmatrix}$, $k \neq 0$. Then A induces the automorphism τ_A of $\mathscr{A}_\theta$ defined by $\tau_A(U) = U^{-1}$, $\tau_A(V) = e^{-\pi i\theta k} U^k V^{-1}$. Let x be a fixed point of τ_A and let,

$$x = \sum_{n,m \in \mathbb{Z}} c_{n,m} U^n V^m,$$

be the Fourier decomposition of x. Then one computes by recursion,

$$\tau_A^{2K}(x) = \sum c_{n,m} \rho^{-m^2 kK} U^{n-2Kmk} V^m, \ \forall K \in \mathbb{N},$$

$$\tau_A^{2K+1}(x) = \sum c_{n,m} e^{-(2K+1)\pi i\theta k m^2} U^{(2K+1)mk-n} V^{-m}, \ \forall K \in \mathbb{N}.$$

Since $x \in \mathscr{A}_\theta$ is a fixed point of τ_A, one obtains by equating coefficients,

$$c_{n,m} = c_{n-2Kmk,m} \rho^{m^2 kK}, \text{ and}$$

$$c_{n,m} = c_{(2K+1)mk-n,-m} e^{(2K+1)\pi i\theta k m^2}, \ \forall K \in \mathbb{N}.$$

Therefore $c_{n,m} = 0$ for all $m \neq 0$ and $c_{n,0} = c_{-n,0}$ for all n. Thus, $x = \sum_{n \geqq 0} c_{n,0}(U^n + U^{-n}) - c_{0,0}$, so that $\mathscr{A}_\theta{}^A \cong C^*(U + U^*) \cong C[-2, +2]$. This proves 2.

3. Now suppose that,

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where $|T| > 2$, $T = \text{Trace}(A)$. Note that $(\tau_A)^K = \tau_{(A^K)}$, and so we will compute $\tau_{(A^K)}$. By induction on $K \geqq 1$,

$$A^K = \begin{pmatrix} S_K a - S_{K-1} & S_K b \\ S_K c & S_K d - S_{K-1} \end{pmatrix},$$

where $S_0 = 0$, $S_1 = 1$ and $S_{K+1} = S_K T - S_{K-1}$. Note that b and c cannot be zero since the requirement that $|a + d| > 2$ and $ad = 1$ is impossible in $\mathbb{Z}$. Similarly, if $K \geqq 1$,

$$A^{-K} = \begin{pmatrix} S_K d - S_{K-1} & -S_K b \\ -S_K c & S_K a - S_{K-1} \end{pmatrix}.$$

So if $x = \sum_{n,m \in \mathbb{Z}} c_{n,m} U^n V^m$ is a fixed point of τ_A, then using τ_{A^K}, one computes, as in cases 1 and 2, that

$$\|c_{n,m}\| = \|c_{(S_K d - S_{K-1})n - S_K bm,\ -S_K cn + (S_K a - S_{K-1})m}\|$$
$$= \|c_{S_K X - S_{K-1} Y, S_K X' - S_{K-1} Y'}\|, \forall K \geqq 1,$$

where $X = dn - bm$, $X' = -cn + am$, $Y = n$ and $Y' = m$. Note that X, X', Y and Y' are integers and if $(n, m) \neq (0, 0)$, then (X, Y), $(X', Y') \neq (0, 0)$. Now,

$$S_K = \frac{1}{\sqrt{T^2 - 4}}\left[\left(\frac{T + \sqrt{T^2 - 4}}{2}\right)^K - \left(\frac{T - \sqrt{T^2 - 4}}{2}\right)^K\right], |T| > 2,$$

so assuming $(X, Y) \neq (0, 0)$ we have,

$$S_K X - S_{K-1} Y = \frac{1}{2\sqrt{T^2 - 4}}\left(\frac{T + \sqrt{T^2 - 4}}{2}\right)^{K-1}[(T + \sqrt{T^2 - 4})X - 2Y] -$$
$$\frac{1}{2\sqrt{T^2 - 4}}\left(\frac{T - \sqrt{T^2 - 4}}{2}\right)^{K-1}[(T - \sqrt{T^2 - 4})X - 2Y].$$

But as $|T| > 2$, at least one of the numbers $\frac{T \pm \sqrt{T^2 - 4}}{2}$ has absolute value greater than one, and since the product of these numbers is one, the other has absolute value less than one. Since T is an integer with $|T| > 2$, $T \pm \sqrt{T^2 - 4}$ are irrational and hence $(T \pm \sqrt{T^2 - 4})X - 2Y \neq 0$. Thus $\lim_{K \to \infty} |S_K X - S_{K-1} Y| = \infty$.

In combination with the above relation for $\|c_{n,m}\|$, this implies $\|c_{n,m}\| = 0 \, \forall (n, m) \neq (0, 0)$. So any fixed point is a constant, i.e., $\mathscr{A}_\theta{}^A = \mathbf{C}$.

This ends the proof of the Theorem.

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