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THE SPECTRUM ASSOCIATED TO A RADICAL OPERATION

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Introduction.

The Hilbert Nullstellensatz describes the correspondence between the closed subsets of an affine variety with points in an algebraically closed field and the radical ideals in the ring of polynomial functions defined on the variety. To extend this description to varieties with points in not necessarily algebraically closed fields, Laksov [3] introduced the concept of radial operations. Laksov [5] constructed a radical operation and gave a Nullstellensatz that states a correspondence between the closed subsets of an affine variety over a not necessarily algebraically closed field and the invariant ideals under this radical operation. The points in the variety correspond to the maximal elements in the set of invariant ideals under the radical operation.

Given a radical operation defined on a ring it is therefore natural to consider the points corresponding to prime ideals that are invariant for the given radical operation. We shall in this work study these points and show that they form the underlying space of a naturally ringed space.

To this end we give a definition of a radical operation that differs from the one given by Laksov and Westin ([6], §2, page 178) in that it includes two axioms which assure that our association of affine schemes to rings is functorial. Different radical operations can reflect completely different geometries, even when they are defined on the same category. To illustrate, in the case of a real algebra the real radical operation alluded to above (the radical operation associated to the non algebraically closed field of real numbers) describes the geometry of real zeroes whereas the ususal radical operation more resembles the complex geometry of the algebra. We give several examples in this situation (Example 1.7, 1.8, 4.5).

One of the main features in this work is to define the localizations in certain multiplicatively closed sets that will give us the ringed space structure alluded to above.

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Finally we define the category of R-schemes for a given radical operation R defined on a category of rings. Example 4.5 shows that the relationship between the global sections of the sheaf of regular functions on the spectrum of a ring and the ring itself is not as simple as in the case of the usual spectrum. Consequently, the theory for R-schemes seems to be more complicated than the theory for usual schemes.

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1. The functor Spec_R.

In this section we define the notion of a radical operation. When a radical operation R is defined on a category of commutative rings $\mathscr{C}$ we define what it means for R to be functorial. When R is a functorial radical operation we associate to it a contravariant functor Spec_R taking the category $\mathscr{C}$ to the category of topological spaces. The relations between the algebraic properties of a ring A and the topological properties of the space $\mathrm{Spec}_R(A)$ are investigated. We give examples that show that although the definitions are similar the behaviour of Spec_R can be fundamentally different from that of Spec.

The following definition of the radical operation was given by Laksov and Westin [6].

DEFINITION 1.1. Let A be a ring. An operation R which to each ideal I of A associates an ideal $R(I)$ of A is called a *radical operation* if

$$R(I) = \bigcap_{\substack{P \supseteq I \\ R(P)=P \\ P \text{ prime ideal}}} P \quad \text{for any ideal } I \text{ of } A. \tag{1}$$

If there are no prime ideals P satisfying the conditions $P \supseteq I$ and $R(P) = P$ then $R(I) = A$.

If R is a radical operation defined on ideals of any ring in a category $\mathscr{C}$. We say that R is *functorial* if axiom 2 and 3 below hold

(2) for any ring homomorphism φ: $A \to B$ in $\mathscr{C}$ and any prime ideal I of B satisfying $R(I) = I$ we have that $R(\varphi^{-1}(I)) = \varphi^{-1}(I)$;

(3) for any surjection φ: $A \to B$ and any ideal I of A containing $\ker(\varphi)$ we have that $R(\varphi(I)) = \varphi(R(I))$.

DEFINITION 1.2. Given a ring A and a radical operation R on A, The *R-prime spectrum* $\mathrm{Spec}_R(A)$ of a ring A is the set

$$\mathrm{Spec}_R(A) = \{\text{Prime ideals } P \text{ of } A \text{ such that } R(P) = P\}.$$

The prime ideals with the property $R(P) = P$ are called *R-prime ideals*.

Since $\mathrm{Spec}_R(A)$ is a subset of $\mathrm{Spec}(A)$, it inherits the subset topology. For any ideal I of A we denote by $Z_R(I)$ the closed set $\{P \mid P \supseteq I\}$ Furthermore, with $X = \mathrm{Spec}_R(A)$ and $f \in A$ we denote by X_f the open set $\{P \in \mathrm{Spec}_R(A) \mid P \not\ni f\}$. The open sets $\{X_f\}_{f \in A}$ form an open base for the topological space $\mathrm{Spec}_R(A)$.

REMARK 1.3. Starting with an arbitrary subset B of $\mathrm{Spec}(A)$ one can define a radical operation by setting

$$R(I) = \bigcap_{\substack{P \supseteq I \\ P \in B}} P \text{ for any ideal } I \text{ of } A.$$

Any radical operation defined on A arises from some B following this procedure. In general the inclusion $B \subseteq \mathrm{Spec}_R(A)$ is strict. However if A is a ring where all prime ideals are maximal we have equality for any subset B of $\mathrm{Spec}(A)$ (compare Example 2.5).

In the following we fix a functorial radical operation R defined on a category $\mathscr{C}$. We will show that Spec_R is a contravariant functor from $\mathscr{C}$ to the category of topological spaces. It follows from axioms 2 of Definition 1.1 that for each homomorphism $\varphi : A \to B$ in $\mathscr{C}$ we can define a map

$$\varphi^* : \mathrm{Spec}_R(B) \to \mathrm{Spec}_R(A)$$

by setting $\varphi^*(P) = \varphi^{-1}(P)$ for any R-prime ideal P of B. Let $X = \mathrm{Spec}_R(A)$ and $Y = \mathrm{Spec}_R(B)$. The continuity of φ^* is a consequence of the fact that $\varphi^{*-1}(X_f) = Y_{\varphi(f)}$ for any $f \in A$.

It is now straight forward to verify that Spec_R is a functor.

PROPOSITION 1.4. *Let R be a functorial radical operation on a category $\mathscr{C}$. Then Spec_R is a contravariant functor from the category $\mathscr{C}$ to the category of topological spaces.*

Axiom 3 of Definition 1.1 has the following geometric counterpart. This proposition is the only place where axiom 3 is used.

PROPOSITION 1.5. *Let $\varphi : A \to B$ be a surjective homomorphism in $\mathscr{C}$. Then φ^*: $\mathrm{Spec}_R(\mathrm{B}) \to \mathrm{Spec}_R(A)$ is a homeomorphism of $\mathrm{Spec}_R(B)$ onto the closed set $Z_R(\ker \varphi)$.*

Several results relating algebraic properties of the ring A to the topology of the space $\mathrm{Spec}(A)$ have an obvious generalization for arbitrary radical operations. As an example we state the following result.

PROPOSITION 1.6. *Let I be an ideal of A. Then the closed subset $Z_R(I)$ of $\mathrm{Spec}_R(A)$ is irreducible if and only if $R(I)$ is an R-prime ideal.*

The reducibility of Spec(A) and Spec$_R(A)$ are not related. In the following example we give a ring A and a radical operation R such that Spec(A) is reducible while Spec$_R(A)$ is irreducible. For future reference we will state the generalized Hilbert Nullstellensatz (Laksov [5], §1). For the definition of the k-radical see [5] or Example 2.2 below.

HILBERT k-NULLSTELLENSATZ. *Let k be a field. Let R be the k-radical operation. Given an ideal I in the polynomial ring $k[x_1,\ldots,x_n]$, then*

$$R(I) = \{f \in k[x_1,\ldots,x_n] \mid f(a) = 0 \quad \text{for all } a \in k^n \text{ with } g(a) = 0 \text{ for all } g \in I\}.$$

EXAMPLE 1.7. Let R denote the real numbers. Let A denote the ring $\mathsf{R}[x]/(x(x^2+1))$ and let R be the R-radical [L3]. It is clear that the prime spectrum of A is a reducible topological space. However it follows from the Hilbert R-Nullstellensatz that the space Spec$_R(A)$ consists of the single R-prime ideal $(\bar{x})$ and is consequently irreducible.

In the following example we give a ring A such that Spec$_R(A)$ is reducible while Spec(A) is irreducible.

EXAMPLE 1.8. Let R and C denote the real and the complex numbers, respectively. Denote by A the ring $\mathsf{R}[x_1,x_2,x_3]/(x_1^2x_3^2 + x_2^2)$ and let R be the R-radical. Then Spec(A) is irreducible. However using the Hilbert R-Nullstellensatz it is easy to see that x_1x_3 belongs to $R(x_1^2x_3^2 + x_2^2)$ but neither x_1 nor x_3 does. Hence $R(0)$ is not an R-prime ideal in A. It follows from Proposition 1.6 that Spec$_R(A)$ is reducible.

2. Radical polynomial operations.

Most radical operations that occur in practice are defined by polynomials. In this section we show that any radical polynomial operation is functorial. We also show that all open sets in a spectrum associated to a radical polynomial operation are quasi-compact.

DEFINITION 2.1 (Laksov [3], page 78).
(1) Let k be a ring and R a subset of the polynomial ring $k[y_0,y_1,\ldots]$ in a countably infinite set of independent variables $y_0, y_1,\ldots$ over k. Given an ideal I of a k-algebra A we let $R(I)$ be the subset of A defined by

$$R(I) = \{a \in A \mid \text{there exists a polynomial } p(y_0,y_1,\ldots,y_m) \in R \text{ and elements } a_1,\ldots,a_m \in A \text{ such that } p(a,a_1,\ldots,a_m) \in I\}.$$

We say that the operation R on the set of ideals of A is a *polynomial operation* or

sometimes that it is *defined by polynomials*. For simplicity we will use the same notation R for the set of polynomials as for the operation it defines.
(2) If the operation defined by R is a radical operation we say that R is a *radical polynomial operation*.

EXAMPLE 2.2. (1) The usual radical is a polynomial radical operation defined by the set of polynomials $\{1, y_0, y_0^2, \dots\}$.

(2) The k-radical is defined as the union in $k[y_0, y_1, \dots]$ of the following sets for $n = 0, 1, 2, \dots$

$$\{p \in k[y_0, \dots, y_n] \mid \text{if } p(\alpha_0, \dots, \alpha_n) = 0 \text{ for } \alpha_0, \dots, \alpha_n \in k \text{ then } \alpha_0 = 0\}.$$

In the following we fix a radical polynomial operation R defined on the category of k-algebras for a ring k. It turns out that axiom 2 and 3 are formal consequences of the properties of a radical polynomial operation.

PROPOSITION 2.3. *Let R be a radical polynomial operation defined on the category of k-algebras where k is a ring. Then R is functorial.*

PROOF. Axiom 2 of Definition 1.1 follows from Laksov ([3], Proposition 11). Laksov also showed axiom 3 when R is equal to the K-radical ([3], Proposition 11(v), page 82). The proof generalizes to our situation.

The following result generalizes the fact that affine schemes are quasi-compact.

PROPOSITION 2.4. *Suppose that R is a radical polynomial operation on a k-algebra A. Let $I = (g_1, \dots, g_r)$ be a finitely generated ideal of A. Then the open set $\mathrm{Spec}_R(A) \setminus Z_R(I)$ is quasi-compact. In particular we have that X_f and $\mathrm{Spec}_R(A) = X_0$ are quasi-compact.*

PROOF. Let $\{X_{f_\alpha}\}_{\alpha \in \mathscr{A}}$ be a covering of $U = \mathrm{Spec}_R(A) \setminus Z_R(I)$. We want to show that there is a finite subcover.

We have that $Z_R(I) \supseteq Z_R(J)$ where J is the ideal generated by $\{f_\alpha\}_{\alpha \in \mathscr{A}}$. We have equivalently that $I \subseteq R(J)$. By the definition of a radical polynomial operation we have that for each $i = 1, \dots, r$ there are integers n_i and N_i, a polynomial $p_i(y_0, y_1, \dots, y_{n_i}) \in R$ and elements $a_1^{(i)}, \dots, a_{n_i}^{(i)}, b_1^{(i)}, \dots, b_{N_i}^{(i)} \in A$ such that

$$p_i(g_i, a_1^{(i)}, \dots, a_{n_i}^{(i)}) = \sum_{k=1}^{N_i} b_k^{(i)} f_k$$

Now set $N = \max_i \{N_i\}$. We claim that $U \subseteq \cup_{k=1}^{N} X_{f_k}$. To prove the claim let $P \in U$. We have that $P \not\supseteq I = (g_1, \dots, g_r)$. Hence there is some j such that $g_j \notin P$. Since P is an R-prime ideal we must have that

$$p_j(g_j, a_1^{(j)}, \dots, a_{n_j}^{(j)}) = \sum_{k=1}^{N_j} b_k^{(j)} f_k \notin P.$$

We conclude that for some $k \in \{1, \ldots, N_j\}$ we have that $f_k \notin P$ or equivalently that $P \in X_{f_k}$.

Next we shall indicate that there is a radical operation R defined on a ring A such that the space $\mathrm{Spec}_R(A)$ is not quasi-compact.

EXAMPLE 2.5. Let A be an infinite boolean ring. Then all prime ideals are maximal and $\mathrm{Spec}(A)$ has a non quasi-compact subset B. Defining R from B following Remark 1.3 we have that $\mathrm{Spec}_R(A) = B$ is not quasi-compact.

3. Localizations.

Fix a radical operation R defined on a ring A. In this section we study localizations of rings in certain sets related to R. These localized rings will be used in §4 to define a sheaf of regular functions on $\mathrm{Spec}_R(A)$. To define regular functions we first define regular functions on the fundamental open sets X_f for any element $f \in A$. In the case of the usual spectrum the ring of regular functions on X_f is the localization of A in the multiplicatively closed set $S_f = \{1, f, f^2, \ldots\}$. When R is defined by polynomials we have the following natural generalization of S_f, introduced by Laksov ([4], page 5).

DEFINITION 3.1. Let R be a polynomial operation on k-algebras. For any element f in a k-algebra A let

$$S_R(f) = \{p(f, a_1, \ldots, a_n) \mid p \in R,\ a_1, \ldots, a_n \in A\}$$

Laksov defined a subset S of a ring A to be *semi-multiplicatively closed* if for each pair of elements $s, t \in S$ there is an elelement $a \in A$ such that $ast \in S$ ([4], page 8). Furthermore, Laksov showed that the fractions of A with denominators in a semi-multiplicatively closed subset S has a natural ring structure.

It turns out that the sets $S_R(f)$ are semi-multiplicatively closed exactly when R is radical i.e. satisfying the first axiom in Definition 1.1. Laksov and Westin ([6], §3, page 180) proved the first implication of the following equivalence.

PROPOSITION 3.2. *Let R be a polynomial operation on the ring A. Suppose that $RR(I) = R(I)$ for all ideals of A. Then R is a radical polynomial operation if and only if $S_R(a)$ is a semi-multiplicatively closed set for each element a in A.*

PROOF. To prove the second implication suppose that there is an element $a \in A$ such that $S_R(a)$ is not semi-multiplicatively closed. Then there are elements $r = p(a, a_1, \ldots, a_n)$ and $s = q(a, b_1, \ldots, b_m)$ in $S_R(a)$ such that $crs \notin S_R(a)$ for all $c \in A$. Hence $a \in R(r)$ and $a \in R(s)$ but $a \notin R(rs)$. However if R were a radical operation we would have $R(r) \cap R(s) = R(rs)$ ([4], Proposition 15).

The following definition introduces sets $C_R(f)$ which are closely related to the sets $S_R(f)$. The former sets have the advantage that they do not make use of polynomials and hence apply to any radical operation.

DEFINITION 3.3. For a radical operation R defined on a ring A and an element $f \in A$ we write

$$C_R(f) = \bigcap_{x \in X_f} P_x^c = \bigcap_{\substack{P \not\ni f \\ P \text{ an } R\text{-prime ideal}}} P^c$$

When R is defined by polynomials, $C_R(f)$ is nothing but the saturation of the semi-multipicatively closed set $S_R(f)$. It is a well-known fact that in this case $S_R(f)^{-1} A$ and $C_R(f)^{-1} A$ are naturally isomorphic.

In the following example we shall show that in general the inclusion $S_R(f) \subseteq C_R(f)$ can be strict even if $S_R(f)$ is multiplicatively closed.

EXAMPLE 3.4. Let R denote the usual radical and consider the element $x(x+1)$ in the polynomial ring $k[x]$, where k is an integral domain. We have that $x(x+1) \in (x) = R(x)$. Hence the element $x \in C_R(x(x+1))$. However x is not in the multiplicatively closed set

$$S_R(x(x+1)) = \{(x(x+1))^n \mid n \in \mathbb{N}\}.$$

4. The sheaf of regular functions.

Fix a radical operation R defined on a ring A. Set $X = \operatorname{Spec}_R(A)$. Motivated by the results of §3 we define a presheaf on the open basis $\{X_f\}_{f \in A}$ of X by defining the sections over X_f to be $C_R(f)^{-1} A$ for any $f \in A$. For any inclusion $X_f \subseteq X_g$ the restriction map is induced by the inclusion $C_R(g) \subseteq C_R(f)$.

DEFINITION 4.1. We define *the sheaf of regular functions* $\mathcal{O}_X$ to be the sheaf associated to the above presheaf.

For any $P \in X$ we denote by $\mathcal{O}_{X,P}$ the stalk of the sheaf $\mathcal{O}_X$ at P.

It is easy to see that the stalks can be identified with the corresponding localization of the ring.

PROPOSITION 4.2. *Let* $P \in X = \operatorname{Spec}_R(A)$. *Then there is a natural isomorphism*

$$\mathcal{O}_{X,P} \simeq A_{P_X}$$

In the case when R is the usual radical of an ideal an important fact is that the natural homomorphism

$$\rho_f : S_R(f)^{-1} A \to \mathcal{O}_X(X_f)$$

is an isomorphism for any $f \in A$. We will see in Example 4.5 that this is not true in general. When R is defined by polynomials we shall prove that ρ_f is injective for all $f \in A$.

LEMMA 4.3. *Suppose that* $X_f = \cup_{i=1}^N X_{f_i}$ *and given elements* $s_i \in C_R(f_i)$ *for* $i = 1, \ldots N$. *Then* $f \in R(J)$ *where* J *is the ideal generated by* $\{s_i\}_{i=1,\ldots,N}$.

PROOF. Let I be the ideal generated by the elements $\{f_i\}_{i=1,\ldots,N}$. Then $Z_R(f) = Z_R(I)$ and it follows that $f \in R(I)$. Since $s_i \in C_R(f_i)$ we have equivalently that $f_i \in R(s_i)$. Hence $f \in R(I) \subseteq R(J)$.

PROPOSITION 4.4. *Suppose that* R *is a radical polynomial operation on a ring* A. *Then, for any* $f \in A$, *the homomorphism* ρ_f *is injective.*

PROOF. Suppose that $a/s \in C_R(f)^{-1}A$ and $\rho_f(a/s) = 0$. Since X_f is quasi-compact (Proposition 2.4) it follows that there is a finite cover $X_f = \cup_{i=1}^N X_{f_i}$ such that $a/s = 0$ in $C_R(f_i)^{-1}A$ for $i = 1, \ldots, N$. Then there are elements s_i in $C_R(f_i)$ such that $s_i a = 0$ for $i = 1, \ldots, N$. It follows from Lemma 4.3 that $f \in R(J)$ where J is the ideal generated by $\{s_i\}_{i=1}^N$. Since R is a radical polynomial operation there is a polynomial p in R and elements $a_1, \ldots, a_m, b_1, \ldots, b_N$ in A such that

$$p(f, a_1, \ldots, a_m) = \sum_{i=1}^{N} b_i s_i.$$

We multiply the last equality by a and get

$$ap(f, a_1, \ldots, a_m) = \sum_{i=1}^{N} b_i a s_i = 0.$$

Since $C_R(f)$ is multiplicatively closed and the elements g and $p(f, a_1, \ldots, a_m)$ are in $C_R(f)$ we have that

$$\frac{a}{g} = \frac{ap(f, a_1, \ldots, a_m)}{gp(f, a_1, \ldots, a_m)} = 0$$

in $C_R(f)^{-1}A$. Hence the map ρ_f is injective.

The homomorphism ρ_f is not always surjective as we will see from the following example.

EXAMPLE 4.5. Let R denote the real numbers. Denote by R the R-radical and let A be the ring

$$\mathsf{R}[x, y]/((x^2 - 1)^2 + y^2).$$

Set $X = \mathrm{Spec}_R(A)$. It is easy to see that X is the disjoint union $X_{\bar{x}+1} \cup X_{\bar{x}-1}$. Hence any pair of functions on $X_{\bar{x}+1}$ and $X_{\bar{x}-1}$ glue together to a global section of $\mathcal{O}_X$. We shall construct such a section and show that it is not in the image of ρ_1. In fact let s be the global section of $\mathcal{O}_X$ defined by the zero-section over $X_{\bar{x}+1}$ and by the constant section 1 over $X_{\bar{x}-1}$. Using the fact that A is an integral domain it is clear that s is not in the image of ρ_1.

5. The category of *R*-schemes.

In this last section we introduce the notion of an R-scheme in analogy with the well-known notion of a scheme. We fix a functorial radical operation R defined on a category of commutative rings $\mathscr{C}$.

DEFINITION 5.1. Let A be a ring in $\mathscr{C}$. The *R-prime spectrum* of A is the space $X = \mathrm{Spec}_R(A)$ together with the sheaf $\mathscr{O}_X$.

The following results are generalizations of well known results for affine schemes.

PROPOSITION 5.2. *Let A and B be in $\mathscr{C}$. Set $X = \mathrm{Spec}_R(A)$ and $Y = \mathrm{Spec}_R(B)$.*

(1) *The pair $(X, \mathscr{O}_X)$ is a locally ringed space.*

(2) *If $\varphi : A \to B$ is a homomorphism in $\mathscr{C}$, then φ induces a natural morphism of locally ringed spaces*

$$\phi : (Y, \mathscr{O}_Y) \to (X, \mathscr{O}_X).$$

PROOF. (1) This follows from Proposition 4.2.

(2) We know already from Proposition 1.5 that we have an induced map ϕ on the level of topological spaces. Furthermore it is easy to see that for any $f \in A$ we have that φ maps $C_R(f)$ into $C_R(\varphi(f))$ so there is an induced morphism of presheaves. Of course the morphism of presheaves induces a morphism $\mathscr{O}_X \to \phi_* \mathscr{O}_Y$ of sheaves.

DEFINITION 5.3. (1) *An affine R-scheme* is a locally ringed space $(X, \mathscr{O}_X)$ which is isomorphic to the R-prime spectrum of a ring A in $\mathscr{C}$.

(2) *An R-scheme* is a locally ringed space $(X, \mathscr{O}_X)$ in which every point has an open neighborhood U such that the topological space U, together with the restricted sheaf $\mathscr{O}_{X|U}$, is an affine R-scheme. A morphism of R-schemes is a morphism of locally ringed spaces.

An important fact in the theory of schemes is that the functor Spec induces an equivalence of categories between the category of commutative rings and the category of affine schemes. In the case of the category of affine R-schemes one asks whether there is an equivalence with the full subcategory of $\mathscr{C}$ consisting of rings $A \in \mathscr{C}$ such that the canonical homomorphism $A \to C_R(1)^{-1}A$ is an isomorphism. However this is impossible since Example 4.7 shows that the ring of global sections of $\mathrm{Spec}_R(A)$ can be different from the ring $C_R(1)^{-1}A$. Because of this fact one would like to modify the theory developed here to something which is more well-behaved. One possible strategy would be to consider a Grothendieck-topology on a given R-scheme. It is beyond my knowledge how and whether this can be carried out.

REFERENCES

1. M. F. Atiyah and I. G. MacDonald I. G, *Introduction to Commutative Algebra*, Addison-Wesley, London. 1969.
2. R. Hartshorne, *Algebraic Geometry*, Springer Verlag, New York, 1977.
3. D. Laksov, *Generalized radicals of ideals in algebras*, Rend. Sem. Math. Univ. Politec. Torino Fascicoslo speciale 33 (1986), 77–89.
4. D. Laksov, *Radicals defined by universal polynomials*, Trita. Mat. 27 (1986).
5. D. Laksov, *Radicals and Hilbert Nullstellensatz for not necessarily algebraically closed fields*, Enseign. Math. 33 (1987), 323–338.
6. D. Laksov and K. Westin, *Nullstellensätze; conjectures and counterexamples*, Proceedings Trento, Lecture notes in Math. 1420, (1988), 178–190.

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