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L-THEORY AND DIHEDRAL HOMOLOGY

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Abstract.

Let k be a commutative ring with $1/2 \in k$, A an involutive k-algebra, $\varepsilon = \pm 1$. We construct a Chern class map, ch′ from $L^{\varepsilon}_{*}(A)$ to the dihedral homology $\mathrm{HD}_{*}(A)$, in such a way that, if ch is Karoubi's Chern class, y is Loday's involution and $H :=$ the hyperbolic functor, the following is a commutative diagram

$$\begin{array}{ccc} \mathrm{ch}'_q : L_{*}(A) & \longrightarrow & \mathrm{HD}_{*+2q}(A) \\ {\scriptstyle H}\uparrow & & \uparrow{\scriptstyle 1+y} \\ \mathrm{ch}_q : K_{*}(A) & \longrightarrow & \mathrm{HC}_{*+2q}(A) \\ \uparrow & & \uparrow \\ \mathrm{ch}'_q : L_{*}(A) & \longrightarrow & \mathrm{HD}_{*+2q}(A) \end{array}$$

§0. Introduction.

Let k be a commutative ring, $1/2 \in k$, A an involutive (or hermitian) k-algebra with an identity. M. Karoubi has defined Chern classes ([Ka-I])

$$\mathrm{ch}^n_q \colon K_n(A) \longrightarrow \mathrm{HC}_{n+2q}(A) \quad (n, q \geqq 0)$$

where HC stands for cyclic homology. Since A is involutive, for $\varepsilon = \pm 1$ we can as well consider its L^{ε}-theory (in the sense of Karoubi [Ka-II]), that is related to K-theory by means of the "forget" functor. In particular, we can compose ch^n_q with the forgetful map ${}_{\varepsilon}L_{*}(A) \to K_{*}(A)$ and study its image. When $1/2 \in k$, there is a splitting due to Loday [Lo] $\mathrm{HC}_{*}(A) = {}_{+1}\mathrm{HC}_{*}(A) \oplus {}_{-1}\mathrm{HC}_{*}(A)$; ${}_{+1}\mathrm{HC}_{*}(A)$ is the dihedral homology of A, and it is often denoted by $\mathrm{HD}_{*}(A)$. In this paper we prove:

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THEOREM (§3.3). *The image of* $\mathrm{ch}_q^*: L_q^\varepsilon(A) \to \mathrm{HC}_{*+2q}(A)$ *lies in the subgroup*

$$ {}_{+1}\mathrm{HC}_{*+2q}(A).$$

In order to explain the organization of this paper, we give a sketch of the proof of this theorem. First recall that Karoubi's ch_q^n has a factorization

$$\begin{array}{ccc} L_n^\varepsilon(A) = \pi_n(B({}_\varepsilon O(A))^+) & \longrightarrow & K_n(A) = \pi_n(\mathrm{BG}l(A)^+) \\ \downarrow & & \downarrow \\ H_n({}_\varepsilon O(A), k) & \longrightarrow & H_n(\mathrm{G}l(A) \\ \downarrow & & \downarrow \\ \mathrm{HN}_n(k[{}_\varepsilon O(A)]) & \longrightarrow & \mathrm{HN}_n(k[\mathrm{G}l(A)]) \\ & \searrow \qquad \swarrow & \\ & \operatorname{colim}_m \mathrm{HN}_n(M_m(A)) & \\ & \mathrm{Tr}\downarrow & \\ & \mathrm{HN}_n(A) & \\ & \downarrow & \\ & \mathrm{HP}_n(A) & \\ & \downarrow & \\ & \mathrm{HC}_{n+2q}(A) & \end{array}$$

Where α is Goodwillie's map ([G, II.3.1]), Tr is the trace map, and the composite of the right column is the Chern class ch_q^n. In §1 we give appropriate definitions for the ${}_\varepsilon\mathrm{HP}_*$, ${}_\varepsilon\mathrm{HN}_*$ of an involutive algebra, and we provide $M_{2m}(A)$ with an involution ${}^\varepsilon$,(ex. 1.2) which coincides with the standard involution on ${}_\varepsilon O(A)$. Thus we only need to prove

i) The image of $\alpha: H_n({}_\varepsilon O(A), k) \to \mathrm{HN}_n(k[{}_\varepsilon O(A)])$ lies in ${}_{-1}\mathrm{HN}_n(k[{}_\varepsilon O(A)])$. This is proved for any group G (instead of ${}_\varepsilon O(A)$) in §2

ii) Tr: $\mathrm{HN}_*(M_{2m}(A)) \to \mathrm{HN}_*(A)$ sends ${}_{+1}\mathrm{HN}_*(M_{2m}(A))$ to ${}_{+1}\mathrm{HN}(A)$. This proved in §3

§4 is devoted to the study of the compatibility between our Chern classes and the hyperbolic functor H. We prove the

THEOREM (4.6). *If* $p\colon \mathrm{HC}_*(A) \to {}_{+1}\mathrm{HC}_*(A)$ *denotes the projection, then* $\mathrm{ch}'_q \circ H = 2p \circ \mathrm{ch}_q$.

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§1. Definitions and Notations.

1.0. Let k be a commutative *with* $1/2 \in k$, G any fixed group; $R := k[G]$, the *group algebra*, A any unital associative k-algebra. We consider the following chain complexes ($\otimes := \otimes_k$, $\bar{A} := A/k$, $\bar{A}^n := A^{\otimes n}$)

– *The bar resolution of k as a left R-module* ([C-E])

$$(X_*, \partial) \quad X_n := R \otimes \bar{R}^n \ (n \geqq 0)$$

$$\partial(g[g_1,\dots,g_n]) := g(g_1[g_2,\dots,g_n] + \sum_{i=1}^{n-1} (-1)^i [g_1,\dots,g_i g_{i+1},\dots,g_n] +$$

$$+ (-1)^n [g_1,\dots,g_{n-1}])$$

– *The bar resolution of k as a right* R^{op}*-module*

$$(X^{\mathrm{op}}_*, \partial^{\mathrm{op}}) \quad X^{\mathrm{op}}_n := \bar{R}^n \otimes R^{\mathrm{op}} \ (n \geqq 0)$$

$$\partial^{\mathrm{op}}([g_1,\dots,g_n]g) := ([g_2,\dots,g_n] + b \sum_{i=1}^{n-1} (-1)^{\varepsilon} [g_1,\dots,g_i g_{i+1},\dots,g_n] +$$

$$+ (-1)^n [g_1,\dots,g_{n-1}] g_n) \mathrm{g}$$

– The *bar complex of* G([C-E])

$$(\bar{X}_*, \partial) = k \otimes_R (X_*, \partial) \equiv (X^{\mathrm{op}}_*, \partial^{\mathrm{op}}) \otimes_R k.$$

– The *Hochschild bar resolution of* A([C-E]) as a left $A^e := A \otimes A^{\mathrm{op}}$-module

$$(U_*(A), b^1) \quad U_n(A) = A \otimes \bar{A}^n \otimes A^{\mathrm{op}} \ (n \geqq 0)$$

$$b'(a_0 \otimes \dots \otimes a_{n+1}) = \sum_{i=0}^{n} (-1)^i a_0 \otimes \dots \otimes a_i a_{i+1} \otimes \dots \otimes a_{n+1}$$

– The *Hochschild bar complex*

$$(\bar{U}_*(A), b) = A \otimes_{A^e} (U_*(A), b')$$

We use the following notations for homology

– The (k-) group homology of G, $H_n(G) = H_n(\tilde{X}_*, \partial)$

– The Hochchild homology of A, $\mathrm{HH}_n(A) = H_n(\tilde{U}_*(A), b)$

The following classical complex maps will be considered (see [Ka-I])

$$\tau\colon \tilde{X}_* \to \tilde{U}_*(R)\colon \tau([g_i,\dots,g_n]) = (g_1\dots g_n)^{-1}\otimes g_1\otimes\dots\otimes g_n$$

$$\pi\colon \tilde{U}_*(R)\to \tilde{X}_x\colon \pi(g_0\otimes\dots\otimes g_n) = \begin{cases} [g_1,\dots,g_n] & \text{if } \prod_{i=1}^{n} g_i = 1\\ 0 & \text{otherwise}\end{cases}$$

Following [G], we provide $(\tilde{U}_*(A), b)$ with a cyclic module structure and consider its *cyclic periodic* and *negative cyclic* homologies, that we denote respectively by

$$\mathrm{HC}_*(A),\ \mathrm{HP}_*(A),\ \mathrm{HN}_*(A).$$

Namely, let $B\colon \tilde{U}_*(A)\to\tilde{U}_{*+1}(A)$ as in [L-Q] and put

$$V_{p,q}(A) := \tilde{U}_{q-p} := \tilde{U}_{q-p}(A) = A\otimes \bar{A}^{q-p}q,\ p\in\mathbb{Z},\ q\geqq p.$$

Pictorially

$$\begin{array}{ccccccccccccc}
 & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots\\
 & & \downarrow b & & \downarrow b & & \downarrow b & & \downarrow b & & \downarrow b & & \downarrow b\\
\dots & \xleftarrow{B} & \tilde{U}_5 & \xleftarrow{B} & \tilde{U}_4 & \xleftarrow{B} & \tilde{U}_3 & \xleftarrow{B} & \tilde{U}_2 & \xleftarrow{B} & \tilde{U}_1 & \xleftarrow{B} & \tilde{U}_0\\
 & & \downarrow b & & \downarrow b & & \downarrow b & & \downarrow b & & \downarrow b & & \\
\dots & \xleftarrow{B} & \tilde{U}_4 & \xleftarrow{B} & \tilde{U}_3 & \xleftarrow{B} & \tilde{U}_2 & \xleftarrow{B} & \tilde{U}_1 & \xleftarrow{B} & \tilde{U}_0 & & \\
 & & \downarrow b & & \downarrow b & & \downarrow b & & \downarrow b & & & & \\
\dots & \xleftarrow{B} & \tilde{U}_3 & \xleftarrow{B} & \tilde{U}_2 & \xleftarrow{B} & \tilde{U}_1 & \xleftarrow{B} & \tilde{U}_0 & & & & \\
 & & \downarrow b & & \downarrow b & & \downarrow b & & & & & & \\
\dots & \xleftarrow{B} & \tilde{U}_2 & \xleftarrow{B} & \tilde{U}_1 & \xleftarrow{B} & \tilde{U}_0 & \dots & \dots & \dots & \dots & \dots & \\
\dots & \xleftarrow{B} & \tilde{U}_1 & \xleftarrow{B} & \tilde{U}_0 & & & & & & & & \\
\dots & \xleftarrow{B} & \tilde{U}_0 & & & & & & & & & & \\
\dots & & p=-2 & & p=-1 & & p=0 & & p=1 & & p=2 & & p=3
\end{array}$$

figure 1. The double complex V_{**}

then the periodic homology of A is the homology of $W_*(A) = \mathrm{Tot}(V_{**}(A), b, B)$. Similarly, if $V_{**}^{-}(A)$ (resp. $V_{**}^{+}(A)$) denotes the second (resp. first) quadrant truncation of $V_{**}(A)$, then the negative cyclic (resp. cyclic) homology of A is the homology of $W^{-}(A) = \mathrm{tot}_*(V^{-}(A))$ (resp. of $W^{+}(A) = \mathrm{tot}_*(V^{+}(A))$).

1.1. Let A be an involutive k-algebra with involution "$-$", i.e. $\bar{\lambda}=\lambda$, $\overline{ab} = \bar{b}\,\bar{a}$, $\lambda\in k$, $b\in A$) and $\varepsilon = \pm 1$. Following Loday [Lo] we consider the natural map of chain complexes

$$y\colon \tilde{U}_*(A)\to\tilde{U}_*(A)\colon y(a_0\otimes\dots\otimes a_n) = (-1)^{\frac{n(n+1)}{2}}(\overline{a_0}\otimes\overline{a_n}\otimes\overline{a_{n-1}}\otimes\dots\otimes\overline{a_1})$$

Let ${}_\varepsilon\tilde{U}_n = \{\alpha \in \tilde{U}_n : y\alpha = \varepsilon\alpha\}$, ${}_\varepsilon\mathrm{HH}_n(A) = \mathrm{H}_n({}_\varepsilon\tilde{U}_*(A))$, the ε-Hochschild homology; then $\mathrm{HH}_*(A) = {}_{+1}\mathrm{HH}_*(A) \oplus {}_{-1}\mathrm{HH}_*(A)$ ([Lo]). Also from [Lo] we learn that $B({}_\varepsilon\tilde{U}_*) \subseteq (-\varepsilon\tilde{U}_*)$, so that we can consider the complex

$$
\begin{array}{ccccccccccc}
 & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
 & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle b} \\
\cdots & \xleftarrow{B} & {}_\varepsilon\tilde{U}_4 & \xleftarrow{B} & {}_{-\varepsilon}\tilde{U}_3 & \xleftarrow{B} & {}_\varepsilon\tilde{U}_2 & \xleftarrow{B} & {}_{-\varepsilon}\tilde{U}_1 & \xleftarrow{B} & {}_\varepsilon\tilde{U}_0 \\
 & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle b} & & \\
\cdots & \xleftarrow{B} & {}_\varepsilon\tilde{U}_3 & \xleftarrow{B} & {}_{-\varepsilon}\tilde{U}_2 & \xleftarrow{B} & {}_\varepsilon\tilde{U}_1 & \xleftarrow{B} & {}_{-\varepsilon}\tilde{U}_0 & & \\
 & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle b} & & & & \\
\cdots & \xleftarrow{B} & {}_\varepsilon\tilde{U}_2 & \xleftarrow{B} & {}_{-\varepsilon}\tilde{U}_1 & \xleftarrow{B} & {}_\varepsilon\tilde{U}_0 & & & & \\
 & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle b} & & & & & & \\
\cdots & \xleftarrow{B} & {}_\varepsilon\tilde{U}_1 & \xleftarrow{B} & {}_{-\varepsilon}\tilde{U}_0 & & & & & & \\
 & & \downarrow{\scriptstyle b} & & & & & & & & \\
\cdots & \xleftarrow{B} & {}_\varepsilon\tilde{U}_0 & & & & & & & & \\
\cdots & & p=-2 & & p=-1 & & p=0 & & p=1 & & p=2\ \cdots
\end{array}
$$

figure 2. The double complex ${}_\varepsilon V_{**}$

Then we put ${}_\varepsilon W_* = \mathrm{tot}_*({}_\varepsilon V_{**})$, ${}_\varepsilon\mathrm{HP}_*(A) = H_*({}_\varepsilon W(A))$. We can as well define ${}_\varepsilon W^+$, ${}_\varepsilon W^-$, ${}_\varepsilon\mathrm{HN}(A)$, ${}_\varepsilon\mathrm{HC}(A)$ in the obvious way; ${}_{+1}\mathrm{HC}(A)$ is called the dihedral homology ([Lo]). In the rest of the paper, we will be interested in the following

1.2. EXAMPLES. i) Let $R = k[G]$, as above; we view R as a hermitian algebra with the involution $g \to g^{-1}$. For reasons that will be uncovered in the next §, the corresponding involution on $\tilde{U}_*(R)$ will be denoted by y_2.

ii) Let $n \geqq 1$, $M_{2n}(A)$ the matrix ring, $I \in M_n(A)$ the identity matrix, $\varepsilon = \pm 1$, put

$$h = {}_\varepsilon h := \begin{pmatrix} 0 & I \\ {}_\varepsilon I & 0 \end{pmatrix}$$

We consider two involutions on $M_{2n}(A)$;

$$a = (a_{ij}) \to {}^t\bar{a} := (\overline{a_{ji}})$$
$$a \to \overset{\varepsilon}{\check{a}} := h^{-1}({}^t\bar{a})h$$

The corresponding involutions on $\tilde{U}_*(M_{2n}(A))$ will be named y, y_ε respectively.

1.3. REMARK. With the notations in the example ii) above, let $G = {}_\varepsilon 0_{n,n}(A) = \{g \in M_{2n}(A) : {}^t\bar{g}({}_\varepsilon h)g = {}_\varepsilon h\}$; then the natural morphism $R \to (M_{2n}(A), \overset{\varepsilon}{\check{\ }})$ is a morphism of hermitian algebras, i.e. $\overset{\varepsilon}{\check{g}} = g^{-1}$ $(g \in G)$.

§2. Relation between $H_*(G)$ and ${}_{+1}\mathrm{HH}(k[G])$.

2.0. Following the notations in §1, we define k-linear maps

$$y_1'\colon X_n \to X_n^{\mathrm{op}}\colon y_1'(g[g_1,\dots,g_n]) := (-1)^{\frac{n(n+1)}{2}}[g_n^{-1},\dots,g_1^{-1}]g^{-1}$$

$$y_2'\colon U_n(R) \to U_n(R)\colon y_2'(g \otimes g_1 \otimes \dots \otimes g_h \otimes h) := h^{-1} \otimes g_n^{-1} \otimes \dots g_1^{-1} \otimes g^{-1}$$

$$\tau'\colon X_n \to U_n(R)\colon g[g_1,\dots,g_n] \to g \otimes g_1 \otimes \dots \otimes g_n \otimes (g_1 \dots g_n)^{-1} g^{-1}$$

$$\eta'\colon X_n^{\mathrm{op}} \to U_n(R)\colon [g_1,\dots,g_n]h \to h^{-1}(g_1 \dots g_n)^{-1} \otimes g_1 \otimes g_1 \otimes \dots \otimes g_n \otimes h$$

$$\Theta\colon X_n \to X_n^{\mathrm{op}}\colon \Theta(g[g_1,\dots,g_n]) = [g_1,\dots,g_n](gg_1 \dots g_n)^{-1}.$$

2.1 PROPOSITION. i) *All the above are chain complex maps and induce maps*

$$\tilde{y}_1' = y_1\colon (\tilde{X},\partial) \to (\tilde{X}^{\mathrm{op}},\partial^{\mathrm{op}})$$

$$y_2' = y_2\colon (\tilde{U},b) \to (\tilde{U},b)$$

$$\tilde{\tau}' = \tilde{\eta}' = (\tilde{X},\partial) \to (\tilde{U},b)$$

$$\tilde{\Theta} = \text{identity: } (\tilde{X},\partial) \to (\tilde{X},\partial).$$

where τ, y_2 are the same as those defined in §1.0.

ii) *The following diagram commutes*

$$\begin{array}{ccc} X_* & \xrightarrow{\tau'} & U_*(R) \\ {\scriptstyle y_1'}\big\downarrow & & \big\downarrow{\scriptstyle y_2'} \\ X_*^{\mathrm{op}} & \xrightarrow{\eta'} & U_*(R) \end{array}$$

iii) *There is a chain homotopy h such that*

$$\theta - y_1' = h\partial + \partial^{\mathrm{op}} h$$

$$h(gx) = h(x)g^{-1} \textit{ for all } g \in G,\ x \in X_*$$

PROOF. All the commutativity issues involved are straightforward. As to the induced maps, it is enough to observe that all $\tilde{X}$, $\tilde{U}$ are quotients of X, U respectively and then one checks easily that the above maps come down to those quotients. As to iii), we first need some notations; let M be an R-left module, N an R^{op}-right module,(*) a group morphism is called an R-skew morphism if $f(gm) = f(m)g^{-1}$ for all $m \in M$, $g \in G$. For example Θ, y_1' are R-skew morphisms; moreover, if $\varepsilon\colon X_* \to k$, $\varepsilon'\colon X_*^{\mathrm{op}} \to k$ denote the standard augmentations, then the square

(*) Or equivalently, an R-left module.

$$\begin{array}{ccc} (X_*, \partial) & \xrightarrow{\varepsilon} & k \\ {\scriptstyle \Theta - y_1'} \downarrow & & \downarrow {\scriptstyle 0} \\ (X_*^{\mathrm{op}}, \partial^{\mathrm{op}}) & \longrightarrow & k \end{array}$$

is commutative. Then we are done if we prove the following statement: Given a chain complex of free R-left modules (C_*, d) and an acyclic complex of R^{op}-right modules (D_*, d') and a skew morphism $f\colon C_0 \to D_0$ then there exists a chain skew morphism $\alpha_*\colon C_* \to D_*$ such that $\alpha_0 = f$ and α_* is unique up to skew-homotopy.

The above becomes a well known fact ([C-E]) if we replace "skew morphism" by morphism. We can as well mimic the proof in [C-E]; the only crucial point to prove is that if L is a left R-free module and $M \xrightarrow{\pi} N \to 0$ is an exact sequence of R^{op}-right modules then for any skew morphism $\alpha\colon L \to N$ there is a lifting $\bar{\alpha}$ (i.e. $\pi\bar{\alpha} = \alpha$). Now let $\{e_i\colon i \in \bar{\alpha}\}$ be a basis for L, $n_i = \alpha(e_i)$ and pick $m_i \in M$ such that $\pi(m_i) = n_i$; then $\bar{\alpha}(\sum_i g_i e_i) = \sum_i m_i g_i^{-1}$ does the job.

2.2. Corollary. *The maps*

$$y_2\tau \text{ and } \tau\colon (\tilde{X}_*, \partial) \to (\tilde{U}_*(R), b)$$

are chain homotopic.

Proof. From 2.1. i), iii) we get that y_1 is homotopic to the identity map $\tilde{\Theta}$. Then i) and ii) yield the desired result.

As a straightforward consequence of 2.2, we get the

2.3. Theorem. *Let* $\tau\colon H_*(G) \to \mathrm{HH}_*(k[G])$ *as in* 1.0; *then the image of* τ *lies in the subgroup* ${}_{+1}\mathrm{HH}_*(k[G])$

Proof. In view of 2.2., both τ and $\dfrac{\tau + y_2\tau}{2}$ are the same thing in homology.

2.4. In [G; II-3.1] it is shown that the chain map $\tau\colon (\tilde{X}_*, \partial) \to (\tilde{U}, b)$ has a natural lifting (unique up to natural homotopy)

$$\begin{array}{ccc} & & W_*^- \\ & {\scriptstyle \alpha} \nearrow & \downarrow {\scriptstyle \pi} \\ \tilde{X}_* & \xrightarrow{\tau} & \tilde{U}_* \end{array}$$

where W^- is as in §1 and π is the canonical projection. This result, together with 2.3 give the following

2.5. Corollary. *Let* $\alpha\colon H_n(G) \to \mathrm{HN}_n^-(k[G])$ *be Goodwillie's map (see* 2.4.*); then the image of* α *lands in* ${}_{+1}\mathrm{HN}_n(k[G])$

Proof. We will go through some steps. First, we prove the existence of a complex map $\bar{\alpha}\colon \tilde{X} \to {}_{+1}W$ (using the notations of §1.1) that makes the following into a commutative diagram

$$\begin{array}{ccc} & & {}_{+1}W_*^- \\ & \overset{\bar{\alpha}}{\nearrow} & \big\downarrow \bar{\pi} \\ \tilde{X}_* & \xrightarrow{p\tau} & {}_{+1}\tilde{U}_*(R) \end{array}$$

For that sake, we mimic Goodwillie's proof of [G, II.3.1]; the only crucial point is that, if F_p is the free group in p letters, then $H_n({}_{+1}W_*^-(F_p))$ must be zero for $n \geqq p-1$. But $H_n({}_{+1}W_*^-(F_p))$ is a direct summand of $H_n(W_*^-(F_p))$, which is zero for $n \geqq p-1$, done.

Next, we know from 2.2. that $\tau, y_2\tau$ are naturally chain homotopic through say, h; as a second step, we choose a natural lifting $\tilde{h}$, such that $\pi\tilde{h} = h$. The existence of such a lifting is clear from Goodwillie's arguments, i.e., we can choose a lifting for the free groups and then extend by naturally. Last, let $\Delta\colon W_*^- \to W_{*-1}^-$ be the boundary map, and put $\beta := \frac{1}{2}(\tilde{h}\partial + \Delta\tilde{h})$, $\alpha' := \bar{\alpha} + \beta$. Then

$$\pi\alpha' = p\tau + \frac{\pi\tilde{h}\partial + \pi\Delta\tilde{h}}{2} = p\tau + \frac{h\partial + bh}{2} = \frac{\tau + y_2\tau}{2} + \frac{\tau - y_2\tau}{2} = \tau$$

In view of the uniqueness of α (2.4), α' is chain homotopic to α, so that $H_*(\alpha) = H_*(\alpha') = H_*(\bar{\alpha})$, done.

§3. Orthogonal Chern Characters.

Let A be a hermitian k-algebra (§1.1), $\varepsilon = \pm 1$ the purpose of this § to define Chern characters

$$\mathrm{ch}_q'^{(n)}\colon L_n^\varepsilon(A) \to {}_{+1}\mathrm{HP}_n(A) \to {}_{+1}\mathrm{HC}_{n+2q}(A) \quad (n, q \geqq 0)$$

We first need to have a corresponding Dennis trace map; this is provided by the following

3.0. Theorem. *Let* $n \geqq 0$, *A a hermitian k-algebra, D the Dennis trace map. There is a commutative diagram*

$$\begin{array}{ccc} L_n^\varepsilon(A) & \xrightarrow{D'} & {}_{+1}\mathrm{HH}_n(A) \\ \big\downarrow & & \big\downarrow i \\ K_n(A) & \xrightarrow{D} & \mathrm{HH}_n(A) \end{array}$$

where the left column is the "forget" morphism, i is the natural split inclusion. In other words, the image of D lies in $_{+1}\mathrm{HH}_*(A)$.

In view of 2.0., it seems natural to state the following

3.1. DEFINITION. The map D' of 2.0. will be called the *orthogonal Dennis trace map.*

In order to prove 3.0., we need the following (refer to 1.1 for notations).

3.2. LEMMA i) *Let* $n \geqq 1$, $\mathrm{Tr} :=$ *the trace map; the diagram*

$$\begin{array}{ccc} \tilde{U}_*(M_n(A)) & \xrightarrow{\mathrm{Tr}} & \tilde{U}_*(A) \\ \downarrow{\scriptstyle y} & & \downarrow{\scriptstyle y} \\ \tilde{U}_*(M_n(A)) & \xrightarrow{\mathrm{Tr}} & \tilde{U}_*(A) \end{array}$$

is commutative.

ii) *Let* $P \in \mathrm{Gl}_n(A)$, *and consider the map*

$$f_P \colon U_*(M_n(A)) \longrightarrow U_*(M_n(A)):$$

$$f_P(a_0 \otimes \ldots \otimes a_n) := a_0 P^{-1} \otimes \ldots \otimes Pa_{n-1}P^{-1} \otimes Pa_n$$

then f_P *is an* A^e*-chain complex map, k-linearly chain homotopic to the identity.*

iii) *Let* $y_\varepsilon \colon \tilde{U}_*(M_n(A)) \to \tilde{U}_*(M_n(A))$ *be the chain complex involution corresponding to* $^{\varepsilon}$ (cf. 1.1); *the following diagram is commutative*

$$\begin{array}{ccc} \mathrm{HH}_*(M_{2n}(A)) & \xrightarrow{\mathrm{Tr}} & \mathrm{HH}_*(A) \\ \downarrow{\scriptstyle y_\varepsilon} & & \downarrow{\scriptstyle y} \\ \mathrm{HH}_*(M_{2n}(A)) & \xrightarrow{\mathrm{Tr}} & \mathrm{HH}_*(A) \end{array}$$

iv) *Let* $m \geqq 1$ $e \in M_{2m}(A)$, ${}_\varepsilon h \in M_{2m}(A)$ *as in* 1.1. *and suppose* $e^2 = e$ *and* $({}^t\bar{e})_\varepsilon h(1-e) = 0$
(that is, e is an ε*-orthogonal projector) then*

$$\overset{\varepsilon}{\check{e}} = e.$$

PROOF. Let $r \geqq 0$, $a_0 \otimes \ldots \otimes a_r \in \tilde{U}_r(M_n(A))$, $a_l^{i,j}$ $0 \leqq l \leqq r$ $1 \leqq i,j \leqq n$ will denote the (i,j)-th entry of a_l. Then

$$\mathrm{Tr}(y(a_0 \otimes \ldots \otimes a_n)) = (-1)^{\frac{n(n+1)}{2}} \sum_{i_0,\ldots,i_n} \bar{a}_0^{i_1,i_0} \otimes \bar{a}_n^{i_2,i_1} \otimes \ldots \otimes \bar{a}_1^{i_0,i_n} =$$

$$= (-1)^{\frac{n(n+1)}{2}} \sum_{i_0,\ldots,i_n} \bar{a}_0^{i_0,i_1} \otimes \bar{a}_n^{i_n,i_0} \otimes \ldots \otimes \bar{a}_y^{i_1,i_2} = y(\mathrm{Tr}(a_0 \otimes \ldots \otimes a_n))$$

This proves i).

ii) Since proving that f_P is a chain map is trivial, we only prove our second assertion. Let μ: $M_n(A^e) \to M_n(A)$ be the canonical augmentation of U_*; then

$$\begin{array}{ccc} U_*(M_n(A)) & \xrightarrow{\mu} & M_n(A) \\ {\scriptstyle f_P}\downarrow & & \| \\ U_*(M_n(A)) & \xrightarrow{\mu} & M_n(A) \end{array}$$

is commutative. The assertion is now clear since μ is a relative projective resolution.

iii) is a trivial consequence of i) and ii), since, for $P = h = {}_\varepsilon h$, if $\bar{f}_h$: $\tilde{U}_* \to \tilde{U}_*$ is the induced map, then $y_\varepsilon = \bar{f}_h \circ y$.

iv) Let $h = {}_\varepsilon h$; then

$$e - \overset{\varepsilon}{e} = e - h^{-1}\,{}^t\bar{e}h = e - h^{-1}({}^t\bar{e})he = (1 - h^{-1}({}^t\bar{e})h)\,e =$$

$$= h^{-1}(1 - ({}^t\bar{e}))\,he = h^{-1}\varepsilon\left({}^t\overline{(({}^t\bar{e})h(1-e))}\right) = 0$$

Proof of 3.1. First consider the case $n = 0$. An element in $L_0^\varepsilon(A)$ can be thought of as the equivalence class of a projector e as in 3.2iv). Then $D_0(e)$ is the homology class of $\mathrm{Tr}(e)$; but in view of 3.2 iii), iv)

$$y\mathrm{Tr}(e) \cong \mathrm{Tr}(y_\varepsilon(e)) = \mathrm{Tr}(\overset{\varepsilon}{e}) = \mathrm{Tr}(e).$$

As to the case $n \geqq 1$, 3.2 iii) and 2.4 yield the commutative diagram (recall 1.3.)

$$\begin{array}{ccc} L_n^\varepsilon(A) = \pi_n(B({}_\varepsilon O(A))^+) & \xrightarrow{\text{forget}} & \pi_n(\mathrm{BG}l(A)^+) = K_n(A) \\ {\scriptstyle \text{Hurewicz}}\downarrow & & {\scriptstyle \text{Hurewicz}}\downarrow \\ H_n(B({}_\varepsilon O(A))^+ k) & \longrightarrow & H_n(\mathrm{BG}l(A)^+, k) \\ \downarrow & & \downarrow \\ H_n({}_\varepsilon O(A)k) & \longrightarrow & H_n(\mathrm{Gl}(A), k) \\ {\scriptstyle \bar{\tau}}\downarrow & & {\scriptstyle \tau}\downarrow \\ {}_{(+1)}\mathrm{HH}_n(k[{}_\varepsilon O(A)]) & \longrightarrow & \mathrm{HH}_n(k(\mathrm{Gl}(A)]) \\ \downarrow & & \downarrow \\ \operatorname{colim}_m ({}_{+1}\mathrm{HH}_n(M_m(A)) & \longrightarrow & \operatorname{colim}_m \mathrm{HH}_n(M_m(A)) \\ {\scriptstyle \mathrm{Tr}}\downarrow & & {\scriptstyle \mathrm{Tr}}\downarrow \\ {}_{+1}\mathrm{HH}_n(A) & \longrightarrow & \mathrm{HH}_n(A) \end{array}$$

The ordinary Dennis map is the composite of the right side column in the above diagram; the theorem is done by calling $D' :=$ the composite of the left column.

Just as we have done in 2.4., we can extend 3.0. to cyclic homology.

3.3. THEOREM. *Let*

$$\mathrm{ch}_q^n : K_n(A) \longrightarrow \mathrm{HN}_n(A) \longrightarrow \mathrm{HP}_n(A) \longrightarrow \mathrm{HC}_{n+2q}(A)$$

be the Karoubi Chern character (*see* [Ka-I], [We]). *Then there is a commutative diagram*

$$\begin{array}{ccccccc}
\mathrm{ch}_q^n : K_n(A) & \longrightarrow & \mathrm{HN}_n(A) & \longrightarrow & \mathrm{HP}_n(A) & \longrightarrow & \mathrm{HC}_{n+2q}(A) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\mathrm{ch}'_q : L_n^\varepsilon(A) & \longrightarrow & {}_{+1}\mathrm{HN}_n(A) & \longrightarrow & {}_{+1}\mathrm{HP}_n(A) & \longrightarrow & {}_{+1}\mathrm{HC}_{n+2q}(A)
\end{array}$$

PROOF. Taking into account Karoubi's explicit computation for ch_q^0, the $n = 0$ case is trivial after 3.1. Let $n \geqq 1$; in view of 1.1., we only need to prove the commutativity of the left side square. Now that square can be decomposed as follows

$$\begin{array}{ccc}
L_n^\varepsilon(A) & \longrightarrow & K_n^\varepsilon(A) \\
\downarrow{\scriptstyle \text{Hurewicz}} & & \downarrow \\
H_n(B({}_\varepsilon O(A))^+, k) & \longrightarrow & H_n(\mathrm{BLl}(A)^+, k) \\
\downarrow & & \downarrow \\
H_n({}_\varepsilon O(A)) & \longrightarrow & H_n(\mathrm{Gl}(A), k) \\
\downarrow{\scriptstyle \bar\alpha} \quad \searrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \bar\alpha} \\
{}_1\mathrm{HC}_n^-(k[{}_\varepsilon O(A)]) \to \mathrm{HC}_n^-(k[{}_\varepsilon O(A)]) & \to & \mathrm{HC}_n^-(k[\mathrm{Gl}(A)]) \\
\downarrow & \searrow \quad \swarrow & \\
\operatorname{colim}_m {}_{+1}\mathrm{HC}_n^-(M_n(A)) & \longrightarrow & \operatorname{colim} \mathrm{HC}_n^-(M_n(A)) \\
\downarrow{\scriptstyle \mathrm{Tr}} & & \downarrow{\scriptstyle \mathrm{Tr}} \\
{}_{+1}\mathrm{HC}_n^-(A) & \longrightarrow & \mathrm{HC}_n^-(A)
\end{array}$$

where α, $\bar\alpha$ are as in 2.4. Moreover, 2.4., 3.1 prove that the above is a commutative diagram

3.4. DEFINITION. The map $\mathrm{ch}_q'^{(n)}: L_n^\varepsilon(A) \to {}_{+1}\mathrm{HC}_{n+2q}(A)$ above is the *orthogonal Chern class.*

§4. Compatibility with the hyperbolic map.

4.0. I am grateful to C. Weibel, who insisted that something like 4.1. should hold. Let A be a hermitian k-algebra, $1/2 \in k$; put

$\mathscr{P}(A) :=$ the category of all finitely-generated-projective-right-A-modules

and for $\varepsilon = \pm 1$ ([Ka-II])

${}_\varepsilon Q(A) :=$ the category of all right-A-ε-quadratic modules in the sense of Karoubi.

Recall that the hyperbolic functor is (see for example [Ka-II]).

$$H = {}_\varepsilon H : \mathscr{P}(A) \longrightarrow {}_\varepsilon Q(A)$$

$$H(P) = \left(P \oplus \bar{P}, \begin{bmatrix} 0 & 1 \\ \varepsilon & 0 \end{bmatrix} \right) \qquad (P \in \mathscr{P}(A))$$

$$H(\alpha) = \begin{bmatrix} \alpha & 0 \\ 0 & ({}^t\bar{\alpha})^{-1} \end{bmatrix} \qquad (\alpha \text{ an isomorphism in } \mathscr{P}(A)).$$

where

$$\bar{P} := \{ f: P \to A, k \text{ linear} / f(p\lambda) = \bar{\lambda} f(p), p \in P, \lambda \in A \}$$

then H induces maps from $K_*(A)$ to $L_*(A)$ and from $H_*(\mathrm{Gl}(A))$ to $H_*({}_\varepsilon O(A))$, all of which will be denoted by H. We have a commutative diagram

$$\begin{array}{ccc} K_*(A) & \longrightarrow & H_*(\mathrm{Gl}(A)) \\ {\scriptstyle H}\downarrow & & \downarrow{\scriptstyle H} \\ L_n^\varepsilon(A) & \longrightarrow & H_*({}_\varepsilon O(A)) \end{array}$$

In what follows, we study the composite map $\mathrm{ch}_q' \circ H$.

4.1. THEOREM. *With the notations of* 3.0, 3.1, *the following is a commutative diagram*

$$\begin{array}{ccc} K_n(A) & \xrightarrow{\mathrm{D}} & \mathrm{HH}_n(A) \\ {\scriptstyle H}\downarrow & & \downarrow{\scriptstyle 1+y} \\ L_n^\varepsilon(A) & \xrightarrow{\mathrm{D}'} & \mathrm{HH}_n(A) \end{array} \qquad (n \geq 0)$$

PROOF. We first do the case $n = 0$. Let π, $Q \in \mathscr{P}(A)$ such that $P \oplus Q = A^n$ ($n \geqq 1$), and consider the idempotent matrix $e \in M_n(A)$ of the projection to P with respect to the canonical basis. Then $e \oplus {}^t\bar{e}$ is the matrix corresponding to the orthogonal projection from $H(A^n)$ to $H(P)$. Now 3.2.i) yields $D'(H(P)) = \mathrm{Tr}(e \oplus {}^t\bar{e}) = \mathrm{Tr}(e \oplus ye) = \mathrm{Tr}(e) + y\mathrm{Tr}(e) = (1 + y)D(P)$. We now proceed with the case $n \geqq 1$. Consider the involution on the complex $\tilde{X}$ of 1.0. given by

$$(4.2) \quad y_3 : \tilde{X}(\mathrm{Gl}(A)) \to \tilde{X}(\mathrm{Gl}(A)) : y_3[g_1, \ldots, g_n] = (-1)^{\frac{n(n+1)}{2}}[{}^+\bar{g}_n, \ldots, {}^+\bar{g}_1]$$

It is easy to see that y_3 is a chain map; moreover

$$y_1 y_3[g_1, \ldots, g_n] = y_1\left((-1)^{\frac{n(n+1)}{2}}[{}^t\bar{g}_n, \ldots, {}^t\bar{g}_1]\right) = [({}^t\bar{g}_1)^{-1}, \ldots, ({}^t\bar{g}_n)^{-1}].$$

Consequently we get that $H = 1 + y_1 y_3$ as chain morphisms from $\tilde{X}(\mathrm{Gl}(A))$ to $\tilde{X}({}_\varepsilon O(A))$. We know from 2.1 iii) that y_1 is homotopic to the identity, so that, at the level of group homology, H is the same as $1 + y_3$. But if $g \in {}_\varepsilon O(A)$, then ${}^t\bar{g} = h(h^{-1}({}^t\bar{g})h)h^{-1} = hg^{-1}h^{-1}$, which in view of 3.2. ii) implies that the restriction of y_3 to $\tilde{X}(O(A))$ is homotopic to the identity. Summing up, we have a commutative diagram

$$(4.3) \quad \begin{array}{ccccc} & & H_*(\mathrm{Gl}(A)) & & \\ & \nearrow & \downarrow H & \searrow 1 + y_3 & \\ H_*({}_\varepsilon O(A)) & \xrightarrow{2} & H_*(O(A)) & \longrightarrow & H_*({}_\varepsilon O(A)) \end{array}$$

Next, if we let $y_4 : \tilde{U}(k[\mathrm{Gl}(A)]) \longrightarrow \tilde{U}(k[\mathrm{Gl}(A)])$ be the "y"-map corresponding to the involution $g \to {}^t\bar{g}$ (i.e. $y_4(g_0 \otimes \ldots \otimes g_n) = (-1)^{\frac{n(n+1)}{2}}({}^t\bar{g}_n \otimes \ldots \otimes {}^t\bar{g}_1)$). Then it is easy to see that $\tau y_4 = y_4 \tau$ (τ is defined in §1.0), and in view of 3.2.i), we can complete (4.3) to yield

$$\begin{array}{ccccccc} D\colon K_*(A) & \longrightarrow & H_*(\mathrm{Gl}(A)) & \xrightarrow{\tau} & \mathrm{HH}_*(k[\mathrm{Gl}(A)]) & \xrightarrow{\mathrm{Tr}} & \mathrm{HH}_n(A) \\ \downarrow H & & \downarrow H & & \downarrow Hy_4 & & \downarrow Hy \\ D'\colon L^\varepsilon_*(A) & \longrightarrow & H_*({}_\varepsilon O(A)) & \xrightarrow{\tau} & {}_t\mathrm{HH}_*(k[{}_\varepsilon\mathrm{HH}_*[{}_\varepsilon O(A)]) & \xrightarrow{\mathrm{Tr}} & {}_+\mathrm{HH}_n(A) \end{array}$$

where ${}_t\mathrm{HH}_*(k[{}_\varepsilon O(A)])$ is the "+"-summand in the decomposition corresponding to the involution $\alpha \to {}^t\bar{\alpha}$. (see 1.1.).

4.4. Consider the involution y_3 defined in (4.2), and let $\varepsilon = \pm 1$. In the spirit of 1.1., it is natural to define (t stands for transpose)

$${}_{\varepsilon t}\tilde{X}(\mathrm{Gl}(A)) = \{x \in \tilde{X}(\mathrm{Gl}(A)) \colon y_3(x) = \varepsilon x\}.$$

$${}_{\varepsilon t}H(\mathrm{Gl}(A), k) = {}_{\varepsilon t}H_*(\mathrm{Gl}(A)) = H_*({}_{\varepsilon t}\tilde{X}(\mathrm{Gl}(A)))$$

and, since we are assuming that $1/2 \in k$,

$$H_*(\mathrm{Gl}(A)) = {}_tH_*(\mathrm{Gl}(A)) \oplus {}_{-t}H_*(\mathrm{Gl}(A))$$

Now by choosing the diagram (4.3) we get the

4.5. COROLLARY. *If* $1/2 \in k$, *then* $H_*({}_\varepsilon O(A), k)$ *is a direct summand of* $H_*(\mathrm{Gl}(A), k)$. Moreover, with the notations of 3.8.,

$$H_*({}_\varepsilon O(A), k) = {}_tH_*(\mathrm{Gl}(A), k).$$

PROOF. See 4.4. above.

4.6. COROLLARY. *With the notations of* 3.3, 3.4., *the following diagram is commutative*

$$\begin{array}{ccc} \mathrm{ch}_q \colon K_*(A) & \longrightarrow & \mathrm{HC}_{*+2}(A) \\ {\scriptstyle H}\uparrow & & \uparrow{\scriptstyle 1+y} \\ \mathrm{ch}'_q \colon L^\varepsilon_*(A) & \longrightarrow & {}_{+1}\mathrm{HD}_{*+2q}(A) \end{array}$$

PROOF. The case $n = 0$ is derived from the arguments in the proof of 3.3, taking into account Karoubi's explicit computation of ch_q^0 ([Ka-I, 2.17]). Now we go to the case $n \geqq 1$. Consider the automorphism β: $\mathrm{Gl}(A) \to \mathrm{Gl}(A)$ $\beta(g) = ({}^t\bar{g})^{-1}$; then β induces chain automorphisms on both $\tilde{X}(\mathrm{Gl}(A))$, $\tilde{U}(k[\mathrm{Gl}(A)]$ that we denote also by β. Next, let α: $H_*(\mathrm{Gl}(A)) \to \mathrm{HN}_*(k[\mathrm{Gl}(A)]$ be Goodwillie's map (as in 2.4.); in the proof of 2.5, we showed that $y_2\alpha = \alpha y_1$; keeping in mind the naturality of α, and the fact that y_2, y_4 can both be extended to HN_* (see 1.1.), we get

$$\alpha y_3 = (\alpha y_3 y_1) y_1 = (\alpha\beta) y_1 = \beta\alpha y_1 = (y_4 y_2)\alpha y_1 = y_4 \alpha y_1^2 = y_4 \alpha.$$

Now this computation, together with 4.3 and 3.2 i) yield the desired result.

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