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STRONG BARRELLEDNESS PROPERTIES IN $L_\infty(\mu, X)$

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Throughout this paper (Ω, Σ, μ) will stand for a finite measure space, Σ being a σ-algebra of subsets of a set Ω, and X is a normed space. $L_\infty(\mu, X)$ will denote the space of all (equivalence classes of) X-valued μ-measurable functions defined on Ω that are essentially bounded. On the other hand, $S(\mu, X)$ will denote the subspace of $L_\infty(\mu, X)$ of all X-valued μ-simple functions on Ω and $S_c(\mu, X)$ will stand for the subspace of $L_\infty(\mu, X)$ formed by the functions that take at most a countable number of different values μ-almost everywhere, all these endowed with the norm

$$\|f\|_\infty = \operatorname{ess\,sup}\{\|f(\omega)\|: \omega \in \Omega\}.$$

The subspace $S_c(\mu, X)$ happens to be dense in $L_\infty(\mu, X)$ as a consequence of the Pettis measurability theorem. Finally, $B(\mu, X)$ will denote the closure of $S(\mu, X)$ in $L_\infty(\mu, X)$; it is clear that $S(\mu, X) \subset S_c(\mu, X) \subset L_\infty(\mu, X)$ and $S_c(\mu, X) \subset B(\mu, X)$ if and only if X is finite-dimensional.

When no measure is considered, in [7] it has been shown that the space $S(\Sigma, X)$ of Σ-simple X-valued functions on Ω is barrelled iff X is finite-dimensional while it is proven in [8] that the space $B(\Sigma, X)$ of all X-valued functions that are the uniform limit of X-valued Σ-simple functions is barrelled iff X is barrelled. On the other hand, in [2] it has been shown that if μ is atomless, $L_\infty(\mu, X)$ is barrelled, and if μ is atomic and σ-finite, $L_\infty(\mu, X)$ is barrelled iff X is barrelled. In this paper we will show that if X is barrelled of class s, then $S_c(\mu, X)$ and $B(\mu, X)$ are barrelled of class s and, since $S_c(\mu, X)$ is dense in $L_\infty(\mu, X)$, this is also true in $L_\infty(\mu, X)$.

Let us start by recalling that a (real or complex Hausdorff locally convex) space$E$ is Baire-like [9] if, given any increasing sequence of closed absolutely convex subsets of E covering E, there is one that is a neighbourhood of the origin. E is said to be db or suprabarrelled [10, 11] if, given any increasing sequence of subspaces of E covering E, there is one that is dense and barrelled. Given $s \in \mathbf{N}$, and considering as $\mathscr{C}_0$ the class of Baire-like spaces, a space E is said to be barrelled of class s [5], or briefly $E \in \mathscr{C}_s$, if given any increasing sequence of subspaces of E covering E, there is one that belongs to $\mathscr{C}_{s-1}$, and E is said to be

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barrelled of class $\varkappa_0$ if $E \in \mathscr{C}_s$ for every $s \in \mathsf{N}$. So $\mathscr{C}_1$ coincides with the class of suprabarrelled spaces and for every $s \in \mathsf{N}$ we have,

$$\text{Baire-like} \supset \mathscr{C}_{s-1} \supset \mathscr{C}_s \supset \text{barrelled of class } \varkappa_0.$$

The following definition, [4], will help us to obtain other useful characterization of barrelled spaces of class s.

Definition. Given a positive integer s, a countable family of subspaces $W = \{L_{m_1 \ldots m_p}: m_r \in \mathsf{N},\ 1 \leqq r \leqq p \leqq s\}$ of a linear space L is an *s-net* in L if the sequence $\{L_m: m \in \mathsf{N}\}$ is increasing, covers L and, for each $p \in \{2, \ldots, s\}$, $\{L_{m_1 \ldots m_{p-1} m}: m \in \mathsf{N}\}$ is increasing and covers $L_{m_1 \ldots m_{p-1}}$. The family $\{L_{m_1 \ldots m_s}: m_i \in \mathsf{N},\ 1 \leqq i \leqq s\}$ will be denoted by W_s.

Proposition 1. *Given $s \in \mathsf{N}$, a space E is barrelled of class s if and only if, given any s-net W in E, there is some $F \in W_s$ that is Baire-like (or barrelled and dense in E).*

Proof. For $s = 1$ the result is immediate since any dense barrelled subspace of a Baire-like space is Baire-like (see Prop. 1 of [3]).

Let us assume the proposition is true for some $s \in \mathsf{N}$ and suppose $E \in \mathscr{C}_{s+1}$. Let $W = \{E_{m_1 \ldots m_p}: m_r \in \mathsf{N},\ 1 \leqq r \leqq p \leqq s+1\}$ be an $(s+1)$-net in E, then there is some $m_1 \in \mathsf{N}$ such that $E_{m_1} \in \mathscr{C}_s$ and is dense in E. Fixing this m_1, $\{E_{m_1 \ldots m_p}: m_r \in \mathsf{N},\ 2 \leqq r \leqq p \leqq s+1\}$ is an s-net in E_{m_1} and, by the induction hypothesis, some $E_{m_1 \ldots m_{s+1}}$ is barrelled and dense in E_{m_1} and therefore in E. On the other hand, assume that given any $(s+1)$-net W in E there is some $F \in W_{s+1}$ that is barrelled and dense. Suppose that $E \notin \mathscr{C}_{s+1}$, then there is an increasing sequence $\{E_n: n \in \mathsf{N}\}$ of subspaces of E covering E such that no $E_n \in \mathscr{C}_s$. As $E \in \mathscr{C}_s \subset \mathscr{C}_0$, every E_n may be assumed to be dense in E. So, by the induction hypothesis, for each $n \in \mathsf{N}$ there will be an s-net $W^n = \{F^n_{m_1 \ldots m_p}: m_r \in \mathsf{N},\ 1 \leqq r \leqq p \leqq s\}$ in E_n such that no $F \in (W^n)_s$ is barrelled and dense in E_n. Setting $E_{n m_1 \ldots m_p} := F^n_{m_1 \ldots m_p}$ for each $n, m_r \in \mathsf{N}$, $1 \leqq r \leqq p \leqq s$, then $W := \{E_{m_1 \ldots m_p}: m_r \in \mathsf{N},\ 1 \leqq r \leqq p \leqq s+1\}$ is an $(s+1)$-net in E and no $F \in W_{s+1}$ is barrelled and dense in E, a contradiction.

In what follows, given $A \in \Sigma$, $e(A)$ will denote the indicator function on A, and by a μ-measurable function we shall mean a function from Ω into X that is the μ-almost everywhere limit of a sequence of μ-simple X-valued functions.

Lemma 1. *If $f \in S_c(\mu, X)$, then there is a countable partition $\{A_n: n \in I\}$ of Ω formed by nonempty elements of Σ such that f is essentially constant on each A_n and takes a different value.*

Proof. If $f \in S(\mu, X)$, then I is finite and the result is obvious. If $f \in S_c(\mu, X) \backslash S(\mu, X)$ let g be a canonical representation of f with countable range $\{x_n: n \in \mathsf{N}\}$. Since $g^{-1}(x_n) \in \Sigma$ for each $n \in \mathsf{N}$ (see for example [1, p. 167]), setting $A_n := g^{-1}(x_n)$, $n \in \mathsf{N}$, the sequence $\{A_n: n \in \mathsf{N}\}$ satisfies the lemma.

Hereafter, given $A \in \Sigma$, $S(\mu, A, X)$ and $S_c(\mu, A, X)$ will stand for the spaces $S(\mu/\Sigma \cap A, X)$, and $S_c(\mu/\Sigma \cap A, X)$, respectively. We identify these spaces with their natural embeddings into $L_\infty(\mu, X)$. Although the two following results can be found enunciated in [2], we give an independent proof of the first of them in order to get in touch with the methods of proof that we use afterwards. On the other hand, a proof of Theorem 2 with similar methods to the ones used in our Theorem 1 can be found in [6].

THEOREM 1. *If X is barrelled, then $B(\mu, X)$ is barrelled.*

PROOF. Suppose that X is barrelled but there is a barrel T in $B(\mu, X)$ which is not a neighbourhood of the origin in $B(\mu, X)$. Then T cannot absorb the unit sphere S_1 of $S(\mu, X)$ since if it did so it would also absorb the closed unit ball of $B(\mu, X)$. Hence there must be some $f_1 \in S_1$ such that $f_1 \notin 2T$.

According to Lemma 1, let $\{Q_1^1, Q_2^1, \ldots, Q_{k_1}^1\}$ be a partition of Ω formed by nonempty elements of Σ such that f_1 is essentially constant on each Q_i^1 and takes a different value.

Now given that $S(\mu, X)$ is the topological direct sum of the subspaces $\{S(\mu, Q_i^1, X)\colon 1 \leqq i \leqq k_1\}$, T cannot absorb the unit spheres of all of them, and there must be some $m_1 \in \{1, \ldots, k_1\}$ and $f_2 \in S_2$, the closed unit sphere of $S(\mu, Q_{m_1}^1, X)$, such that $f_2 \notin 4T$. Let $\{Q_1^2, Q_2^2, \ldots, Q_{k_2}^2\}$ be a partition of $Q_{m_1}^1$ formed by nonempty elemets of Σ such that f_2 is essentially constant on each Q_i^2 and takes a different value.

Going on by recurrence, we obtain a normalized sequence $\{f_n\colon n \in \mathsf{N}\}$ of μ-simple functions, a sequence $\{m_n\colon n \in \mathsf{N}\}$ of positive integers and a countable family $\{Q_{m_n}^n\colon n \in \mathsf{N}\}$ formed by nonempty elements of Σ such that for each $n \in \mathsf{N}$, f_n is essentially constant on $Q_{m_n}^n$ in such a way that, for each $n \in \mathsf{N}$,

(i) $\operatorname{supp} f_{n+1} \subset Q_{m_n}^n$.
(ii) f_n is essentially constant in $\operatorname{supp} f_m$ for every $m > n$.
(iii) $Q_{m_{n+1}}^{n+1} \subset Q_{m_n}^n$.
(iv) $f_n \notin 2nT$.

Set $Q := \cap \{Q_{m_n}^n\colon n \in \mathsf{N}\}$. If $\mu(Q) \neq 0$ then $e(Q)$ is not the identically null mapping and the mapping $x \to e(Q)x$ is an isometry of X onto its image. Therefore if x_n denotes the value taken by f_n on $Q_{m_n}^n$ then $\|x_n\| \leqq 1\ \forall n \in \mathsf{N}$, since $\{f_n\colon n \in \mathsf{N}\}$ is normalized, and there must be some $n_0 \in \mathsf{N}$ such that $x_n e(Q) \in n_0 T\ \forall n \in \mathsf{N}$. Hence $x_n e(Q) \in nT\ \forall n \geqq n_0$.

If for each $n \in \mathsf{N}$ we define $g_n := f_n - x_n e(Q) \notin nT$, then

$$\cap\{\operatorname{supp} g_n\colon n \geqq n_0\} \subset \cap\{Q_{m_n}^n \backslash Q\colon n \geqq n_0\} = \emptyset.$$

If $\mu(Q) = 0$, for each $n \in \mathsf{N}$ we define $g_n(\omega) = f_n(\omega)$ if $\omega \notin Q$ and $g_n(\omega) = 0$ if $\omega \in Q$. Taking $n_0 = 1$, then $g_n = f_n$ μ-a.e. $\forall n \geqq n_0$ and $\cap\{\operatorname{supp} g_n\colon n \geqq n_0\} = 0$.

In any of these two cases, the sequence $\{g_n\colon n \geqq n_0\}$ is bounded in $S(\mu, X)$.

Therefore if $\xi \in l_1$, $\sum_{n=n_0}^{\infty} \xi_n g_n$ converges in the completion of $B(\mu, X)$ and, essentially, takes at most a countable number of values in X. Indeed if $\omega \in Q$, $\sum_{n=n_0}^{\infty} \xi_n g_n(\omega) = 0$, and if $\omega \notin Q$, there exists some positive integer $m_0 \geqq n_0$ such that $\omega \notin Q_{m_n}^n$ for all $n > m_0$ and so $\sum_{n=n_0}^{\infty} \xi_n g_n(\omega) = \sum_{n=n_0}^{\infty} \xi_n f_n(\omega) = \sum_{n=n_0}^{m_0} \xi_n f_n(\omega) \in X$. Therefore, $\sum_{n=n_0}^{\infty} \xi_g g_n \in B(\mu, X)$.

Hence, denoting by B_{l_1} the closed unit ball of l_1, the Banach disk $D := \left\{ \sum_{n=n_0}^{\infty} \xi_n g_n \colon \xi \in B_{l_1} \right\}$ in the completion of $B(\mu, X)$ is contained in $B(\mu, X)$. Thus, by the Baire category theorem, there exists some integer $q \geqq n_0$ with $D \subset qT$ and hence $g_q \in qT$, a contradiction.

THEOREM 2. *If X is barrelled, then $S_c(\mu, X)$ is barrelled.*

In the following two results we suppose that s is any positive integer, $W = \{E_{m_1 \ldots m_p} \colon m_r \in \mathbf{N}, 1 \leqq r \leqq p \leqq s\}$ is an s-net in E formed by dense subspaces of $S_c(\mu, X)$ covering $S_c(\mu, X)$. For each $m_1, \ldots, m_s \in \mathbf{N}$, suppose $T_{m_1 \ldots m_s}$ is a barrel of $E_{m_1 \ldots m_s}$, $B_{m_1 \ldots m_s}$ is its closure in $S_c(\mu, X)$ and $L_{m_1 \ldots m_s} := \langle B_{m_1 \ldots m_s} \rangle$. By decreasing recurrence, for $p = s - 1, \ldots, 1$, define the subspaces $F_{m_1 \ldots m_{p+1}} := \cap \{L_{m_1 \ldots m_p m} \colon m \geqq m_{p+1}\}$, $L_{m_1 \ldots m_p} := \cup \{F_{m_1 \ldots m_p m} \colon m \in \mathbf{N}\}$, and $F_{m_1} := \cap \{L_m \colon m \geqq m_1\}$. Notice that $\{F_m \colon m \in \mathbf{N}\}$ and $\{F_{m_1 m_2 \ldots m_p m} \colon m \in \mathbf{N}\}$ are 1-nets in $S_c(\mu, X)$ and $L_{m_1 \ldots m_p}$, $\forall m_r \in \mathbf{N}$, $1 \leqq r \leqq p \leqq s - 1$, and $E_{m_1 \ldots m_p} \subset F_{m_1 \ldots m_p}$, $\forall m_r \in \mathbf{N}$, $1 \leqq r \leqq p \leqq s$.

LEMMA 2. *If $\{A_n \colon n \in \mathbf{N}\}$ is a sequence of nonempty pairwise disjoint elements of Σ, then there exists some $n_0 \in \mathbf{N}$ such that $S_c(\mu, \cup \{A_n \colon n \geqq n_0\}, X) \subset F_{n_0}$.*

PROOF. Assume the lemma is false and that for each $p \in \mathbf{N}$ there is some $f_p \in S_c(\mu, \cup \{A_n \colon n \geqq p\}, X) \backslash F_p$ so that $\|f_p\| = 1$. Then $\{f_n \colon n \in \mathbf{N}\}$ is bounded in $S_c(\mu, X)$ and if $\xi \in l_1$, $\sum_{n=1}^{\infty} \xi_n f_n$ converges in the completion $L_\infty(\mu, \hat{X})$ of $S_c(\mu, X)$.

Now $\sum_{n=1}^{\infty} \xi_n f_n$ is essentially countably valued in X since if $\omega \in \Omega \backslash \cup \{A_n \colon n \in \mathbf{N}\}$, then $\sum_{n=1}^{\infty} \xi_n f_n(\omega) = 0$ and if $\omega \in \cup \{A_n \colon n \in \mathbf{N}\}$ there is some $r \in \mathbf{N}$ such that $\omega \in A_r$, i.e. $\omega \notin \cup \{A_n \colon n > r\}$ and, since $\operatorname{supp} f_n \subset \cup \{A_i \colon i \geqq n\}$, $\sum_{n=1}^{\infty} \xi_n f_n(\omega) = \sum_{n=1}^{r} \xi_n f_n(\omega)$.

Moreover, the sequence $\left\{\sum_{n=1}^{m} \xi_n f_n,\ m \in \mathrm{N}\right\}$ of $S_c(\mu, X)$ converges to $\sum_{n=1}^{\infty} \xi_n f_n$ in the completion $L_\infty(\mu, \hat{X})$ of $S_c(\mu, X)$. Hence, $\sum_{n=1}^{\infty} \xi_n f_n \in S_c(\mu, X)$.

Therefore $D := \left\{\sum_{n=1}^{\infty} \xi_n f_n\colon \xi \in B_{l_1}\right\}$ is a Banach disk and, denoting by E_D the normed space $\langle D \rangle$ whose norm is the gauge of D, there is some $m'_1 \in \mathrm{N}$ such that $F_{m_1} \cap E_D$ is a dense Baire subspace of E_D $\forall m_1 \geqq m'_1$. By finite induction, suppose that we have found m'_1 and the functions $m'_i(m_1, \ldots, m_{i-1})$, $2 \leqq i \leqq p \leqq s-1$, such that for any positive integer $m_1 \geqq m'_1$, $m_i \geqq m'_i(m_1, \ldots, m_{i-1})$, $2 \leqq i \leqq p$, $F_{m_1 \ldots m_1} \cap E_D$ is a dense Baire subspace of E_D. Then, for any $m_1 \geqq m'_1, \ldots, m_p \geqq m'_p(m_1, \ldots, m_{p-1})$ given that $\{F_{m_1 \ldots m_p m}\colon m \in \mathrm{N}\}$ covers $F_{m_1 \ldots m_p}$, there is some $m'_{p+1}(m_1, \ldots, m_p) \in \mathrm{N}$ such that $F_{m_1 \ldots m_{p+1}} \cap E_D$ is a dense Baire subspace of E_D $\forall m_{p+1} \geqq m'_{p+1}(m_1, \ldots, m_p)$. Hence $D \subset L_{m_1 \ldots m_s}$ if $m_1 \geqq m'_1, \ldots, m_s \geqq m'_s(m_1, \ldots, m_{s-1})$, since $B_{m_1 \ldots m_s} \cap L_{m_1 \ldots m_s} \cap E_D$ is a barrel and consequently a neighbourhood of the origin in the Baire space $L_{m_1 \ldots m_s} \cap E_D$ for $m_1 \geqq m'_1, \ldots, m_s \geqq m'_s(m_1, \ldots, m_{s-1})$. It follows from this that $D \subset F_{m_1 \ldots m_s}$ for $m_1 \geqq m'_1, \ldots, m_s \geqq m'_s(m_1, \ldots, m_{s-1})$ and therefore $D \subset L_{m_1 \ldots m_{s-1}}$ if $m_1 \geqq m'_1, \ldots, m_{s-1} \geqq m'_{s-1}(m_1, \ldots, m_{s-2})$. This implies that $D \subset F_{m_1 \ldots m_{s-1}}$ for $m_1 \geqq m'_1, \ldots, m_{s-1} \geqq m'_{s-1}(m_1, \ldots, m_{s-2})$. Going on in this way, we obtain that $D \subset F_{m_1}$ for $m_1 \geqq m'_1$, and, consequently, $f_{m_1} \in F_{m_1}$, a contradiction.

LEMMA 3. *If X is barrelled of class s, then there exists some $q \in \mathrm{N}$ such that $S(\mu, X) \subset F_q$.*

PROOF. Suppose the lemma is false and there is some $f_1 \in S(\mu, X)$, $f_1 \notin F_1$ so that $\|f_1\| = 1$. Let $\{Q_1^1, Q_2^1, \ldots, Q_{k_1}^1\}$ be a partition of Ω formed by nonempty elements of Σ such that f_1 is essentially constant on each Q_i^1 and takes a different value.

Now given that $S(\mu, X)$: is the topological direct sum of the subspaces $\{S(\mu, Q_i^1, X)\colon 1 \leqq i \leqq k_1\}$, there must be some $m_1 \in \{1, \ldots, k_1\}$ such that $S(\mu, Q_{m_1}^1, X)$ is not contained in F_n for each $n \in \mathrm{N}$ and, consequently, there is some $f_2 \in S(\mu, Q_{m_1}^1, X)$, $f_2 \notin F_2$ so that $\|f_2\| = 1$. Let $\{Q_1^2, Q_2^2, \ldots, Q_{k_2}^2\}$ be a partition of $Q_{m_1}^1$ formed by nonempty elements of Σ such that f_2 is essentially constant on each Q_i^2 and takes a different value. Now there is some $m_2 \in \{1, \ldots, k_2\}$ such that $S(\mu, Q_{m_2}^2, X)$ is not contained in F_n for each n.

Assume that we have obtained by induction a sequence $\{f_n\colon n \in \mathrm{N}\}$ of μ-simple functions, a sequence of positive integers $\{k_n\colon n \in \mathrm{N}\}$, and a countable family $\{Q_i^n\ n \in \mathrm{N}, 1 \leqq i \leqq k_n\}$ formed by nonempty elements of Σ such that:

a) for each $n \in \mathrm{N}$, f_n is essentially constant on each Q_i^n and takes a different value, and

b) for each $n \in \mathbb{N}$, the following properties are satisfied
(i) $\|f_n\| = 1$.
(ii) $\operatorname{supp} f_{n+1} \subset Q^n_{m_n}$ for some $m_n \in \{1, \ldots, k_n\}$.
(iii) $Q^{n+1}_{m_{n+1}} \subset Q^n_{m_n}$.
(iv) $f_n \notin F_n$.

Set $Q := \cap \{Q^n_{m_n}: n \in \mathbb{N}\}$. In case $\mu(Q) \neq 0$ we define $g_n := f_n - x_n e(Q)$ for each $n \in \mathbb{N}$ where x_n denotes the value taken by f_n on $Q^n_{m_n}$. Then, since the mapping of X into $S_c(\mu, X)$ such that $x \to e(Q)x$ is an isometry and $X \in \mathscr{C}_s$, using Proposition 1 it is easy to find some $m_1 \in \mathbb{N}$ such that $e(Q)x_j \in F_n\ \forall j \in \mathbb{N}$ and for all $n \geqq m_1$. Thus, $g_n \notin F_n$ for each $n \geqq m_1$ and

$$\cap \{\operatorname{supp} g_n: n \in \mathbb{N}\} = \emptyset.$$

In case $\mu(Q) = 0$, we define $g_n(\omega) := f_n(\omega)$ for $\omega \notin Q$ and $g_n(\omega) := 0$ for $\omega \in Q$ for each $n \in \mathbb{N}$. Then $g_n = f_n$ μ-a.e. and $\cap \{\operatorname{supp} g_n: n \in \mathbb{N}\} = \emptyset$ as well.

As in Theorem 1, $D := \left\{ \sum_{n=1}^{\infty} \xi_n g_n: \xi \in B_{l_1} \right\}$ is a Banach disk in $L_\infty(\mu, \hat{X})$ which is contained in $S_c(\mu, X)$ and there must be some $p \geqq m_1$ such that $D \subset F_p$.

Hence $g_p \in F_p$, a contradiction.

THEOREM 3. *Given $s \in \mathbb{N}$, if X is barrelled of class s then $S_c(\mu, X)$ is barrelled of class s.*

PROOF. By Theorem 2, $S_c(\mu, X) \in \mathscr{C}_0$ since it is a metrizable barrelled space. Proceeding by recurrence, let $p \in \{1, \ldots, s\}$ and assume $S_c(\mu, X) \in \mathscr{C}_{p-1} \backslash \mathscr{C}_p$. Then, by Proposition 1, there is a p-net $W := \{E_{m_1 \ldots m_i}: m_r \in \mathbb{N},\ 1 \leqq r \leqq i \leqq p\}$ in $S_c(\mu, X)$ formed by dense subspaces such that no $E_{m_1 \ldots m_i} \in W_i$ is barrelled of class $i - 1$, $1 \leqq i \leqq p$. And as $S_c(\mu, X)$ is metrizable, no $E_{m_1 \ldots m_p}$ is barrelled. For each $m_1, \ldots, m_p \in \mathbb{N}$, suppose $T_{m_1 \ldots m_p}$ is a barrel of $E_{m_1 \ldots m_p}$ which is not a neighbourhood of the origin in $E_{m_1 \ldots m_p}$, let $B_{m_1 \ldots m_p}$ be the closure of $T_{m_1 \ldots m_p}$ in $S_c(\mu, X)$ and let $L_{m_1 \ldots m_p} := \langle B_{m_1 \ldots m_p} \rangle$. By decreasing recurrence, for $i = p - 1, \ldots, 1$, define the subspaces $F_{m_1 \ldots m_{i+1}} := \cap \{L_{m_1 \ldots m_i m}: m \geqq m_{i+1}\}$, $L_{m_1 \ldots m_i} := \cup \{F_{m_1 \ldots m_i m}: m \in \mathbb{N}\}$, and $F_{m_1} := \cap \{L_m: m \geqq m_1\}$. Then $\{F_m: m \in \mathbb{N}\}$ and $\{F_{m_1 \ldots m_i m}: m \in \mathbb{N}\}$ are respectively 1-nets in $S_c(\mu, X)$ and in $L_{m_1 \ldots m_i}$, $\forall m_r \in \mathbb{N}$, $1 \leqq r \leqq i \leqq p - 1$. Besides $E_{m_1 \ldots m_i} \subset F_{m_1 \ldots m_i}$, $\forall m_r \in \mathbb{N}$ with $1 \leqq r \leqq i \leqq p$. Now if there is a $m_1 \in \mathbb{N}$ such that $S_c(\mu, X)$ coincides with F_{m_1}, then $L_{m_1} \in \mathscr{C}_{p-1}$ and there must be some $m_2 \in \mathbb{N}$ such that $F_{m_1 m_2} \in \mathscr{C}_{p-2}$ and is dense in $S_c(\mu, X)$. Thus, $L_{m_1 m_2} \in \mathscr{C}_{p-2}$ and is dense in $S_c(\mu, X)$. Continuing in this way we would find some $F_{m_1 \ldots m_p} \in \mathscr{C}_0$. So $B_{m_1 \ldots m_p} \cap F_{m_1 \ldots m_p}$ would be a neighbourhood of zero in $F_{m_1,,,m_p}$ and $E_{m_1 \ldots m_p}$ would be barrelled, a contradiction.

Hence, no F_n may coincide with $S_c(\mu, X)$. Now by the previous Lemma we may assume that $S(\mu, X) \subset F_n\ \forall n \in \mathbb{N}$. Let $f_1 \in S_c(\mu, X) \backslash F_1$ be such that $\|f_1\| = 1$ and let $\{Q_i^1: i \in \mathrm{N}\}$ be a partition of Ω formed by nonempty elements of Σ determined by Lemma 1 and defined by the μ-measurable function f_1 so that f_1 is essentially constant on each Q_i^1 and takes a different value.

By Lemma 2 there is some positive integer $n_2 > n_1 = 1$ so that $S_c(\mu, \cup\{Q_i^1: i \geqq n_2\}, X) \subset F_{n_2}$. Thus, setting $\Omega_1 := \cup\{Q_i^1: 1 \leqq i \leqq n_2\}$, $S_c(\mu, \Omega_1, X)$ cannot be contained in any F_n, $n \geqq n_2$, and there must be some $f_2 \in S_c(\mu, \Omega_1, X) \backslash F_{n_2}$ so that $\|f_2\| = 1$. Let $\{Q_i^2: i \in \mathrm{N}\}$ be a partition of Ω_1 formed by nonempty elements of Σ determined by the μ-measurable function f_2 so that f_2 is constant on each Q_i^2 and takes a different value.

Continuing in this way, we obtain a sequence of positive integers $\{n_i: i \in \mathrm{N}\}$ and a sequence $\{f_n: n \in \mathrm{N}\}$ of μ-measurable functions of $S_c(\mu, X)$ which determine the sequence $\{Q_i^n: n, i \in \mathrm{N}\}$ formed by nonempty elements of Σ such that, for each $n \in \mathrm{N}$, f_n is essentially constant on each Q_i^n and takes a different value, in such a way that setting $\Omega_n := \cup\{Q_i^n: 1 \leqq i \leqq n_{i+1}\}$ $\forall n \in \mathrm{N}$, for each $i \in \mathrm{N}$ we have that,

(i) $\operatorname{supp} f_{i+1} \subset \Omega_i$.

(ii) $e(\Omega_i)f_i \in S(\mu, \Omega_i, X) \subset S(\mu, X)$.

(iii) $\Omega_{i+1} \subset \Omega_i$.

(iv) $f_i \notin F_{n_i}$.

Now let $g_i := f_i - e(\Omega_i)f_i$ for each $i \in \mathrm{N}$. Then $g_i \notin F_{n_i}$ for each $i \in \mathrm{N}$ and $\operatorname{supp} g_i \cap \operatorname{supp} g_j = \emptyset$ for $i \neq j$. Hence $\overline{\langle\{g_n: n \in \mathrm{N}\}\rangle}$, where the closure is in $L_\infty(\mu, \hat{X})$, is a copy of c_0 since $\{g_n/\|g_n\|: n \in \mathrm{N}\}$ is equivalent to the unit vector basis of c_0. As it is easy to see that $\overline{\langle\{g_n: n \in \mathrm{N}\}\rangle} \subset S_c(\mu, X)$, using the Baire category theorem as above, there must be some $q \in \mathrm{N}$ such that $\{g_n: n \in \mathrm{N}\} \subset F_k$ for each $k \geqq n_q$. Hence $g_q \in F_{n_q}$, a contradiction. Thus $S_c(\mu, X) \in \mathscr{C}_p$.

Hence $S_c(\mu, X) \in \mathscr{C}_s$ and the proof is over.

THEOREM 4. *If X is a barrelled space of class s (barrelled of class $\varkappa_0$), then both $L_\infty(\mu, X)$ and $B(\mu, X)$ are barrelled of class s (barrelled of class $\varkappa_0$).*

PROOF. The first affirmation is an obvious consequence of the previous theorem, since $S_c(\mu, X)$ is dense in $L_\infty(\mu, X)$. The argument to prove the second affirmation is analogous to the one given in theorems above, but working with $B(\mu, X)$ instead of $S_c(\mu, X)$, and using Theorem 1 instead of Theorem 2.

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