MATH. SCAND. 73 (1993), 49–64

NEVANLINNA'S FIRST FUNDAMENTAL THEOREM FOR SUPERTEMPERATURES

N. A. WATSON

1. Introduction.

The principal result of this paper, Theorem 1, is the analogue for the heat equation of Nevanlinna's First Fundamental Theorem on superharmonic functions. The proof is necessarily very different from that given in [4], because the Green function for the appropriate domain is unknown, and the point at which we would want to evaluate it is on the boundary. The remainder of the paper is devoted to some of the consequences of Theorem 1. After a few simple ones have been deduced as corollaries, we estimate the upper and lower limits as $c \to 0+$ of a quotient of surface means $\mathscr{M}(u, p_0, c)/\mathscr{M}(v, p_0, c)$, where v is a supertemperature and u is the difference of two, and $v(p_0) = \infty$, in terms of the behaviour near p_0 of the measures associated with v and u by the Riesz decomposition theorem. A judicious choice of v then leads to a similar result in which $\mathscr{M}(v, p_0, c)$ is replaced by a given function of c. Then, in Theorem 4, the size of the (polar) sets on which $\limsup_{c \to 0+} (\mathscr{M}(w, p_0, c)/F(c))$ is infinite, for a given supertemperature w and function F, is estimated in terms of the parabolic Hausdorff measures described in [6]. Theorems 2–4 are analogous to results of Armitage ([1]), although Theorem 4 is considerably more general than a direct parallel. Finally, we establish analogues of two elementary results from the Nevanlinna theory of subharmonic and δ-subharmonic functions. In Theorem 5, we give necessary and sufficient conditions for a measure on $\mathsf{R}^n \times]-\infty, a[$ to be the Riesz measure of a supertemperature. In Theorem 6, we characterize those differences of two supertemperatures on $\mathsf{R}^n \times]-\infty, a[$ which can be expressed as a difference of two positive supertemperatures.

We work in R^{n+1}, a typical point of which is denoted by p or q, except that where necessary it is written as (x, t) or (y, s), where $x, y \in \mathsf{R}^n$ and $t, s \in \mathsf{R}$. An arbitrary open subset of R^{n+1} is denoted by D, its Green function by G_D, and its

Received July 20, 1992.

adjoint Green function by G_D^* (so that $G_D^*(p,q) = G_D(q,p)$). The Green function for $\mathbb{R}^{n+1}$ is denoted by G, so that

$$G((x,t),(y,s)) = (4\pi(t-s))^{-n/2}\exp(-\|x-y\|^2/(4(t-s)))$$

if $t > s$, and $G((x,t),(y,s)) = 0$ if $t \leqq s$. All our measures are locally finite Borel measures, and we use the expression "signed measure" to signify a set function which can be written as the difference of two positive measures (which are not necessarily finite). Give a positive measure μ on D, its Green potential is defined by

$$G_D\mu(p) = \int_D G_D(p,q)\,d\mu(q)$$

for all $p \in D$; if it is finite on a dense subset of D, then it is called a potential. The adjoint potential $G_D^*\mu$ is defined analogously. A temperature (or parabolic function) is a solution of the heat equation, and a supertemperature (or superparabolic function) is the corresponding analogue of a superharmonic function. If w is a supertemperature on D, then the Riesz decomposition theorem associates with w a positive measure on D, which is equal to $-\theta w$ in the sense of distributions (where $\theta = \sum_{i=1}^n D_i^2 - D_t$ is the heat operator); that measure is called the Riesz measure of w. If it exists, the greatest thermic (or parabolic) minorant of w on D is denoted by $GM_D w$. If $A \subseteq D$, the smoothed reduction of w over A is written $\hat{R}_w^A$. See [2] for details.

For all $c > 0$, we put $\tau(c) = (4\pi c)^{-n/2}$. If $p_0 \in \mathbb{R}^{n+1}$ and $c > 0$, we put $\Omega(p_0,c) = \{p \in \mathbb{R}^{n+1} : G(p_0,p) > \tau(c)\}$, $\partial\Omega'(p_0,c) = \partial\Omega(p_0,c)\backslash\{p_0\}$, and $\bar{\Omega}'(p_0,c) = \bar{\Omega}(p_0,c)\backslash \{p_0\}$. If, in addition, $d > c$, then we write

$$A(p_0,c,d) = \Omega(p_0,d)\backslash\bar{\Omega}(p_0,c),$$

and put

$$A(p_0,c,\infty) = \bigcup_{d>c} A(p_0,c,d).$$

The p_0 may be omitted from these notations, if no confusion will be caused. If $p = (x,t)$ and $p_0 = (x_0,t_0)$, we define $K(p_0,p)$ whenever $t < t_0$ by

$$K(p_0,p) = \|x_0 - x\|^2(4\|x_0 - x\|^2(t_0 - t)^2 + (\|x_0 - x\|^2 - 2n(t_0 - t))^2)^{-1/2}.$$

If w is a function on $\partial\Omega(p_0,c)$ such that the integral exists, we put

$$\mathcal{M}(w,p_0,c) = \tau(c)\int_{\partial\Omega(p_0,c)} K(p_0,p)w(p)\,d\sigma(p),$$

where σ denotes surface area measure. These means were used to define subtemperatures in [7], and were studied further in [8, 10].

The terms "increasing" and "positive" are used here in the wide sense.

Let v and w be supertemperatures on D, and let $u = v - w$ whenever the difference is defined, hence q.e. on D (that is, on D less a polar set). Such a function u is called a *δ-subtemperature on D.* Since $\mathscr{M}(w, p_0, c)$ and $\mathscr{M}(v, p_0, c)$ are finite whenever $\bar{\Omega}(p_0, c) \subseteq D$, the mean of u is also defined and finite. If μ and ν are the Riesz measures of w and v respectively, then $\nu - \mu$ will be called the Riesz measure of u. As in the case of δ-subharmonic functions ([3], p. 507), the Riesz measure of u is uniquely determined.

2. The fundamental theorem.

THEOREM 1. *Suppose that w is a supertemperature on D, that μ is its Riesz measure, that $\bar{\Omega}(p_0, d) \subseteq D$, and that $0 < c \leqq d$. Then*

$$\mathscr{M}(w, p_0, c) = \mathscr{M}(w, p_0, d) - \int_c^d \tau'(\gamma)\mu(\bar{\Omega}'(p_0, \gamma))\,d\gamma, \tag{1}$$

and

$$w(p_0) = \mathscr{M}(w, p_0, d) - \int_0^d \tau'(\gamma)\mu(\bar{\Omega}'(p_0, \gamma))\,d\gamma, \tag{2}$$

PROOF. Since the assertion is local, we can assume that w is positive and that μ is finite. If v is any positive supertemperature on D, with Riesz measure ν, then $v = G_{\Omega(c)}\nu + GM_{\Omega(c)}v$ on $\Omega(c)$. Since the function which is equal to v on $D \backslash \Omega(c)$, and to $GM_{\Omega(c)}v$ on $\Omega(c)$, can be expressed as the limit of a decreasing sequence of supertemperatures on D, its lower semicontinuous smoothing is a supertemperature on D. By Theorem 6 of [11], there is only one supertemperature on D which is equal to v on $D \backslash \bar{\Omega}(c)$ and is a temperature on $\Omega(c)$, so that this supertemperature is the smoothed reduction of v over $D \backslash \Omega(c)$, and therefore coincides on $\Omega(c)$ with the PWB solution of the Dirichlet problem thereon with boundary function v. It therefore follows from Lemma 4 of [11] that $GM_{\Omega(c)}v$ has a fine limit $\mathscr{M}(v, p_0, c)$ at p_0.

Since μ is finite, $G\mu$ is a potential on R^{n+1}. On $\Omega(c)$ we have

$$w = G_{\Omega(c)}\mu + GM_{\Omega(c)}w$$

and

$$G\mu = G_{\Omega(c)}\mu + GM_{\Omega(c)}G\mu,$$

so that

$$w = G\mu - GM_{\Omega(c)}G\mu + GM_{\Omega(c)}w.$$

Taking fine limits at p_0, we deduce that

$$w(p_0) = G\mu(p_0) - \mathscr{M}(G\mu, p_0, c) + \mathscr{M}(w, p_0, c), \tag{3}$$

since the means are finite by Theorem 2 of [8]. Supposing, temporarily, that $w(p_0) < \infty$, so that $G\mu(p_0) < \infty$ also, we obtain from (3) and the corresponding formula with d in place of c, the identity

$$\mathscr{M}(w, p_0, c) = \mathscr{M}(w, p_0, d) + \mathscr{M}(G\mu, p_0, c) - \mathscr{M}(G\mu, p_0, d). \tag{4}$$

By the example in [8],

$$\mathscr{M}(G(\cdot, q), p_0, c) = \tau(c) \wedge G(p_0, q)$$

whenever $q \in \mathsf{R}^{n+1}$. Therefore

$$\mathscr{M}(G\mu, p_0, c) = \int_{\mathsf{R}^{n+1}} (\tau(c) \wedge G(p_0, q))\, d\mu(q),$$

so that

$$\mathscr{M}(G\mu, p_0, c) - \mathscr{M}(G\mu, p_0, d) = (\tau(c) - \tau(d)) \int_{\bar{\Omega}'(c)} d\mu(q) + \int_{A(c,d)} (G(p_0, q) - \tau(d))\, d\mu(q).$$

If we put $\lambda(\gamma) = \mu(\bar{\Omega}'(\gamma))$ for all $\gamma \in]0, d]$, then

$$\begin{aligned} \int_{A(c,d)} (G(p_0, q) - \tau(d))\, d\mu(q) &= \int_c^d (\tau(\gamma) - \tau(d))\, d\lambda(\gamma) \\ &= (\tau(d) - \tau(c))\lambda(c) - \int_c^d \tau'(\gamma)\lambda(\gamma)\, d\gamma, \end{aligned}$$

so that

$$\mathscr{M}(G\mu, p_0, c) - \mathscr{M}(G\mu, p_0, d) = -\int_c^d \tau'(\gamma)\lambda(\gamma)\, d\gamma.$$

This, together with (4), yields (1) in the case where $w(p_0) < \infty$.

Now suppose that $w(p_0) = \infty$, and let $w_0 = \hat{R}_w^{D \backslash \Omega(c)}$. Then w_0 is a supertemperature on D and a temperature on $\Omega(c)$, so that

$$w_0(p_0) = \mathscr{M}(w_0, p_0, c) < \infty,$$

by Theorems 4 and 2 in [8]. Therefore, if μ_0 is the Riesz measure of w_0,

$$\mathscr{M}(w_0, p_0, c) = \mathscr{M}(w_0, p_0, d) - \int_c^d \tau'(\gamma)\mu_0(\bar{\Omega}'(\gamma))\, d\gamma. \tag{5}$$

Since there is only one supertemperature on D which is a temperature on $\Omega(c)$ and equal to w on $D\backslash\bar{\Omega}(c)$ (by Theorem 6 of [11]), and that supertemperature is necessarily equal to w on $\partial\Omega'(c)$ also (by Theorem 5 of [8]), we see that $w_0 = w$ on $D\backslash(\Omega(c) \cup \{p_0\})$. Therefore we can replace w_0 by w in (5), so that it remains to prove that we can also replace μ_0 by μ. Let $\gamma \in]c,d[$, and let ν, ν_0 denote the restrictions of μ, μ_0 to $\bar{\Omega}'(\gamma)$. Then $G_{\Omega(d)}\nu = G_{\Omega(d)}\nu_0$ on $A(c,d)$, by the following reasoning. Let $\{c_j\}$ be an increasing sequence in $]c,d[$, with limit d. On $\Omega(d)$, the decreasing sequence of positive supertemperatures $\{\hat{R}_w^{D\backslash\Omega(c_j)}\}$ converges to $GM_{\Omega(d)}w$. A similar statement is true for w_0, so that $GM_{\Omega(d)}w = GM_{\Omega(d)}w_0$ because $w = w_0$ on $D\backslash\bar{\Omega}(c)$. Therefore, for the same reason, $G_{\Omega(d)}\mu = G_{\Omega(d)}\mu_0$ on $A(c,d)$. It follows that, in particular, the restrictions of μ and μ_0 to $A(\gamma, d)$ are equal, so that their potentials on $\Omega(d)$ are equal, and hence $G_{\Omega(d)}\nu = G_{\Omega(d)}\nu_0$ on $A(c,d)$. If $\rho \in]\gamma, d[$, then the function

$$\left(\frac{G(p_0,\cdot) - \tau(d)}{\tau(\rho) - \tau(d)}\right) \wedge 1$$

is a potential relative to the adjoint heat equation on $\Omega(d)$, and is identically equal to 1 on $\Omega(\rho)$. Since it is also a solution of that equation on $A(\rho, d)$, it can be written as $G^*_{\Omega(d)}\alpha$ for some positive measure α supported by $\partial\Omega'(\rho)$. Hence

$$\begin{aligned} \nu(\bar{\Omega}'(\gamma)) &= \int_{\bar{\Omega}'(\gamma)} G^*_{\Omega(d)}\alpha \, d\nu = \int_{\partial\Omega'(\rho)} G_{\Omega(d)}\nu \, d\alpha \\ &= \int_{\partial\Omega'(\rho)} G_{\Omega(d)}\nu_0 \, d\alpha = \int_{\bar{\Omega}'(\gamma)} G^*_{\Omega(d)}\alpha \, d\nu_0 = \nu_0(\bar{\Omega}'(\gamma)), \end{aligned}$$

so that $\mu(\bar{\Omega}'(\gamma)) = \mu_0(\bar{\Omega}'(\gamma))$ whenever $\gamma \in]c,d[$. Thus (5) is the same as (1). Making $c \to 0+$ in (1), we obtain (2).

REMARKS. The identity (2) is analogous to Nevanlinna's First Fundamental Theorem for superharmonic functions ([4], p. 127). The formula (1) is analogous to an extension of that theorem, which is proved, and described as "essentially well-known", in [1]. If we put

$$N(p_0, c) = -\int_0^c \tau'(\gamma)\mu(\bar{\Omega}'(p_0,\gamma))\,d\gamma,$$

then $N(p_0,\cdot)$ is obviously increasing, and a standard technique ([4], p. 127) shows that there is a convex function ϕ such that $N(p_0,\cdot) = \phi \circ \tau$.

We now give three corollaries of Theorem 1, all of which are analogues of results on superharmonic functions given by Kuran in [5]. The proofs of the first two follow Armitage [1], while that of the third is adapted from Kuran's argument. Theorem 3 (iii) of [8] implies that the surface means can be replaced by the appropriate volume means in the first two corollaries.

COROLLARY 1. *If* w *is a supertemperature on* D, *and* $p_0 \in D$, *then* $\mathscr{M}(w, p_0, c) = o(\tau(c))$ *as* $c \to 0+$.

PROOF. If $\bar{\Omega}(p_0, d) \subseteq D$, then (1) yields

$$\mathscr{M}(w, p_0, c) = -\int_c^d \tau'(\gamma)\mu(\bar{\Omega}'(\gamma))\,d\gamma + 0(1) \tag{6}$$

as $c \to 0+$. Since $\mu(\bar{\Omega}'(\cdot))$ is increasing on $]0, d[$, and $\mu(\bar{\Omega}'(0+)) = 0$, it is an elementary exercise to prove that the integral in (6) is $o(\tau(c))$ as $c \to 0+$.

REMARK. A more elementary proof of Corollary 1 was outlined in [9] (p. 255).

COROLLARY 2. *If* w *is a positive supertemperature on an open superset of* $(\mathsf{R}^n \times]-\infty, t_0[) \cup \{p_0\}$, *where* $p_0 = (x_0, t_0)$, *and* μ *is the Riesz measure of* w, *then*

$$\tau(c)\mu(\bar{\Omega}'(p_0, c)) \leqq \mathscr{M}(w, p_0, c)$$

for all $c > 0$.

PROOF. By (1), if $0 < c < d$ then

$$\mathscr{M}(w, p_0, c) \geqq -\int_c^d \tau'(\gamma)\mu(\bar{\Omega}'(\gamma))\,d\gamma \geqq \mu(\bar{\Omega}'(c))\int_c^d -\tau'(\gamma)\,d\gamma$$
$$= (\tau(c) - \tau(d))\mu(\bar{\Omega}'(c)) \to \tau(c)\mu(\bar{\Omega}'(c))$$

as $d \to \infty$.

COROLLARY 3. *Let* μ *be the Riesz measure of a positive supertemperature* w *on* $H_a = \mathsf{R}^n \times]-\infty, a[$. *Then*

$$\lim_{c \to \infty} \tau(c)\mu(\bar{\Omega}'(p, c)) = 0$$

for all $p \in H_a$.

PROOF. By Theorem 3 of [10], the function u, defined for all $p \in H_a$ by

$$u(p) = \lim_{c \to \infty} \mathscr{M}(w, p, c),$$

is $GM_{H_a}w$. Since μ is also the Riesz measure of $w - u$, it follows from Corollary 2 that

$$\tau(c)\mu(\bar{\Omega}'(p, c)) \leqq \mathscr{M}(w - u, p, c) = \mathscr{M}(w, p, c) - u(p) \to 0$$

as $c \to \infty$.

We shall return to the topic of Corollary 3 in Section 5.

3. The behaviour of the means of small c.

Let u be a δ-subtemperature and v a supertemperature on D, with Riesz measures μ and ν respectively. We shall compare the behaviour, for small c, of the quotients $\mathscr{M}(u,p_0,c)/\mathscr{M}(v,p_0,c)$ and $\mu(\bar{\Omega}'(p_0,c))/\nu(\bar{\Omega}'(p_0,c))$. This leads to a similar result comparing $\mathscr{M}(u,p_0,c)/F(c)$ with $\mu(\bar{\Omega}'(p_0,c))/f(c)$ under suitable conditions on f, where F is an integral of f with weight function τ'. An application is given in Section 4. The results of this section were suggested by the work of Armitage [1], and the proof of Theorem 2 follows his argument; but the proof of Theorem 3 is necessarily different.

THEOREM 2. *Let u be a δ-subtemperature and v a supertemperature on D, with Riesz measures μ and ν respectively. If $v(p_0) = \infty$, then*

$$\limsup_{c\to 0+} \frac{\mathscr{M}(u,p_0,c)}{\mathscr{M}(v,p_0,c)} \leqq \limsup_{c\to 0+} \frac{\mu(\bar{\Omega}'(p_0,c))}{\nu(\bar{\Omega}'(p_0,c))}, \tag{7}$$

and the reverse inequality holds for lower limits.

PROOF. Let λ denote the right-hand side of (7). If $\lambda = \infty$ there is nothing to prove, so suppose otherwise. Given $\varepsilon > 0$, choose $d > 0$ such that

$$\mu(\bar{\Omega}'(p_0,c)) < (\lambda + \varepsilon)\nu(\bar{\Omega}'(p_0,c))$$

whenever $c \in]0,d]$. As $c \to 0+$, it follows from Theorem 1 that

$$\begin{aligned}\mathscr{M}(u,p_0,c) &= -\int_c^d \tau'(\gamma)\mu(\bar{\Omega}'(p_0,\gamma))d\gamma + O(1)\\ &\leqq -(\lambda+\varepsilon)\int_c^d \tau'(\gamma)\nu(\bar{\Omega}'(p_0,\gamma))d\gamma + O(1) = (\lambda+\varepsilon)\mathscr{M}(v,p_0,c) + O(1).\end{aligned}$$

Since $\mathscr{M}(v,p_0,0+) = v(p_0) = \infty$, we deduce that

$$\limsup_{c\to 0+} \frac{\mathscr{M}(u,p_0,c)}{\mathscr{M}(v,p_0,c)} \leqq \lambda + \varepsilon,$$

which proves (7). The inequality for lower limits now follows if we replace u by $-u$.

THEOREM 3. *Let $\alpha \in]0,\infty[$, and let f be a positive, continuous, increasing function on $[0,\alpha]$ such that f is differentiable on $]0,\alpha[$ and*

$$-\int_0^\alpha \tau'(\gamma)f(\gamma)d\gamma = \infty. \tag{8}$$

Let

$$F(c) = -\int_c^\alpha \tau'(\gamma)f(\gamma)\,d\gamma$$

for all $c \in]0,\alpha[$. *If* u *is a* δ*-subtemperature on* D *with Riesz measure* μ, *and* $p_0 \in D$, *then*

$$\limsup_{c\to 0+} \frac{\mathscr{M}(u,p_0,c)}{F(c)} \leqq \limsup_{c\to 0+} \frac{\mu(\bar{\Omega}'(p_0,c))}{f(c)},$$

and the reverse inequality holds for lower limits.

Proof. If $f(0) \neq 0$, then $\mu(\bar{\Omega}'(p_0,c)) = o(f(c))$ as $c \to 0+$, so we must prove that $\mathscr{M}(u,p_0,c) = o(F(c))$. Now,

$$F(c) \geqq f(0)\int_c^\alpha -\tau'(\gamma)\,d\gamma = f(0)(\tau(c) - \tau(\alpha)),$$

so that $\tau(c)/F(c)$ is bounded as $c \to 0+$, and hence

$$\frac{\mathscr{M}(u,p_0,c)}{F(c)} = \frac{\mathscr{M}(u,p_0,c)}{\tau(c)}\cdot\frac{\tau(c)}{F(c)} \to 0,$$

by Theorem 1, Corollary 1.

Now suppose that $f(0) = 0$, and choose $d \leqq \alpha$ such that $\bar{\Omega}(p_0,d) \subseteq D$. Let $p_0 = (x_0,t_0)$, put $g(\gamma) = \gamma\tau(\gamma)f'(\gamma)$ for all $\gamma \in]0,d[$, and define a measure ν on D by putting

$$d\nu(x,t) = g\left((t_0 - t)\exp\left(\frac{\|x_0 - x\|^2}{2n(t_0 - t)}\right)\right)\frac{\|x_0 - x\|^2}{2n(t_0 - t)^2}\,\chi(x,t)\,dx\,dt,$$

where χ is the characteristic function of $\Omega(p_0,d)$. Then, if $c \in]0,d[$, it follows from Lemma 3 in [7] that

$$\begin{aligned}\nu(\bar{\Omega}'(p_0,c)) &= \int_{\Omega(c)} G(p_0,(x,t))\frac{\|x_0 - x\|^2}{2n(t_0 - t)}\exp\left(\frac{\|x_0 - x\|^2}{2n(t_0 - t)}\right)\\ &\quad\times f'\left((t_0 - t)\exp\left(\frac{\|x_0 - x\|^2}{2n(t_0 - t)}\right)\right)dx\,dt\\ &= \int_0^c d\gamma\int_{\partial\Omega(\gamma)} \tau(\gamma)K(p_0,(x,t))f'(\gamma)\,d\sigma = \int_0^c f'(\gamma)\mathscr{M}(1,p_0,\gamma)\,d\gamma = f(c).\end{aligned}$$

It therefore follows from Theorem 1 that, as $c \to 0+$,

$$\mathscr{M}(G_D\nu,p_0,c) = -\int_c^d \tau'(\gamma)\nu(\bar{\Omega}'(\gamma))\,d\gamma + O(1) = -\int_c^d \tau'(\gamma)f(\gamma)\,d\gamma + O(1) = F(c) + O(1).$$

Since Theorem 1 shows that

$$G_D v(p_0) = \mathscr{M}(G_D v, p_0, d) - \int_0^d \tau'(\gamma) f(\gamma) d\gamma = \infty$$

by (8), the result now follows from Theorem 2.

COROLLARY. *If u is a δ-subtemperature on D with Riesz measure μ, if $p_0 \in D$ and if $\kappa_n = (4\pi)^{-n/2}(n/2)$, then*

$$\limsup_{c \to 0+} \frac{\mathscr{M}(u, p_0, c)}{\log(1/c)} \leqq \kappa_n \limsup_{c \to 0+} c^{-n/2} \mu(\bar{\Omega}'(p_0, c)),$$

and

$$\limsup_{c \to 0+} \left(\frac{n-\beta}{2}\right) c^{(n-\beta)/2} \mathscr{M}(u, p_0, c) \leqq \kappa_n \limsup_{c \to 0+} c^{-\beta/2} \mu(\bar{\Omega}'(p_0, c))$$

if $0 \leqq \beta < n$.

PROOF. Take $\alpha = 1$ and $f(\gamma) = \gamma^{\beta/2}$ in Theorem 3, where $0 \leqq \beta \leqq n$. If $\beta = n$, we have

$$F(c) = \kappa_n \int_c^1 \gamma^{-1} d\gamma = \kappa_n \log(1/c),$$

and if $0 \leqq \beta < n$ we have

$$F(c) = \kappa_n \int_c^1 \gamma^{((\beta-n)/2)-1} d\gamma = \kappa_n \left(\frac{c^{(\beta-n)/2} - 1}{(n-\beta)/2}\right).$$

4. Parabolic Hausdorff measures of certain polar sets.

We use Theorem 3 to study the size of the set of points p where $\mathscr{M}(w, p, c)/F(c)$ is unbounded as $c \to 0+$, for a given function F and supertemperature w. The size is estimated in terms of the parabolic Hausdorff measures discussed in [6], which have the appropriate mixed homogeneity.

We recall the relevant definitions. Let h be an increasing function on $]0, \infty[$ such that $h(0+) = 0$. Let $\mathscr{P}$ denote the class of all sets of the form

$$\left(\prod_{i=1}^{n} [a_i, a_i + r]\right) \times [a, a + r^2].$$

The set of this form which is centred at p is denoted by $P(p, r)$. For an arbitrary set E, the outer parabolic h-measure of E is defined by

$$\mathscr{P} - h - m_*(E) = \lim_{\delta \to 0+} \inf\left\{\sum_{i=1}^{\infty} h(\operatorname{diam} P_i) : P_i \in \mathscr{P}, E \subseteq \bigcup_{i=1}^{\infty} P_i, \operatorname{diam} P_i < \delta\right\}.$$

The associated measure, defined on a σ-field that contains the Borel sets, is denoted by $\mathscr{P} - h - m$. When $h(s) = s^\alpha$ for some $\alpha > 0$, we write $\mathscr{P} - \Lambda^\alpha - m$ for $\mathscr{P} - h - m$. The parabolic dimension is defined by

$$\mathscr{P} - \dim E = \inf\{\alpha > 0 : \mathscr{P} - \Lambda^\alpha - m_*(E) = 0\}.$$

Theorem 4 below was suggested by Theorem 3 of [1]. However, it is much more general than the direct analogue of Armitage's theorem, which is given as a corollary.

We require the following slight modification of a result in [6].

LEMMA. *Let h be an increasing function on $]0, \infty[$ with $h(0+) = 0$, to which there corresponds a constant λ such that $h(2s) \leqq \lambda h(s)$ for all $s > 0$. If μ is a positive measure on R^{n+1}, and*

$$Z = \left\{p : \limsup_{r \to 0+} \frac{\mu(P(p,r))}{h(r)} = \infty\right\},$$

then $\mathscr{P} - h - m(Z) = 0$.

PROOF. We can suppose that μ is finite. For each $k \in \mathsf{N}$, put

$$Z_k = \left\{p : \limsup_{r \to 0+} \frac{\mu(P(p,r))}{h(r)} \geqq k\right\}.$$

Then, by a simple modification of the proof of Lemma 3 in [6], there is a constant λ_n such that $\mathscr{P} - h - m(Z_k) \leqq \lambda_n \mu(\mathsf{R}^{n+1})/k$. Since $Z \subseteq Z_k$ for all k, the result follows.

THEOREM 4. *Let $\alpha \in]0, \infty[$, and let h be an increasing, continuous function on $]0, \infty[$ with $h(0+) = 0$, which is differentiable on $]0, \sqrt{\alpha}[$, to which there corresponds a constant λ such that $h(2s) \leqq \lambda h(s)$ for all $s > 0$, and which satisfies*

$$\int_0^{\sqrt{\alpha}} s^{-n-1} h(s)\, ds = \infty. \tag{9}$$

Let

$$F(c) = -\int_c^\alpha \tau'(\gamma) h(\sqrt{\gamma})\, d\gamma$$

for all $c \in]0, \alpha[$. If w is a supertemperature on D, then the set

$$\left\{p \in D : \limsup_{c \to 0+} \frac{\mathscr{M}(w, p, c)}{F(c)} = \infty\right\} \tag{10}$$

has $\mathscr{P} - h$-measure zero.

PROOF. It follows from (9) that

$$\int_0^\alpha \gamma^{-(n+2)/2} h(\sqrt{\gamma})\,d\gamma = 2\int_0^{\sqrt{\alpha}} s^{-n-1} h(s)\,ds = \infty,$$

so that we can take $f(\gamma) = h(\sqrt{\gamma})$ in Theorem 3, and deduce that the set (10) is a subset of

$$\left\{p \in D : \limsup_{c \to 0+} \frac{\mu(\tilde{\Omega}'(p,c))}{h(\sqrt{c})} = \infty\right\}, \tag{11}$$

where μ is the Riesz measure of w. Given $c > 0$, we can choose $r = (6nc/e)^{1/2}$, so that $\tilde{\Omega}'(p,c) \subseteq P(p,r)$. It follows that the set (11) is contained in

$$\left\{p \in D : \limsup_{r \to 0+} \frac{\mu(P(p.r))}{h(\delta r)} = \infty\right\}, \tag{12}$$

where $\delta = (e/6n)^{1/2} < 1$. If i is chosen such that $2^i\delta > 1$, then

$$h(\delta r) \geqq \lambda^{-i} h(2^i \delta r) \geqq \lambda^{-i} h(r)$$

by our conditions on h. It follows that the set (12), and hence the set (10), is a subset of the set Z in the above lemma (for the present choice of μ), which has $\mathscr{P} - h$- measure zero. The result follows.

COROLLARY. *Let w be a supertemperature on D.*

(i) *The set*

$$\left\{p \in D : \limsup_{c \to o+} \frac{\mathscr{M}(w,p,c)}{-\log c} = \infty\right\} \tag{13}$$

has $\mathscr{P} - \Lambda^n$-measure 0.

(ii) *If* $0 < \beta < n$, *then the set*

$$\left\{p \in D : \limsup_{c \to o+} c^{(n-\beta)/2} \mathscr{M}(w,p,c) = \infty\right\} \tag{14}$$

has $\mathscr{P} - \Lambda^\beta$-measure 0, *so that the set*

$$\left\{p \in D : \limsup_{c \to o+} c^{(n-\beta)/2} \mathscr{M}(w,p,c) > 0\right\} \tag{15}$$

has $\mathscr{P}$-dimension at most β.

PROOF. The assertions about the sets (13) and (14) follow from Theorem 4, with $\alpha = 1$ and $h(s) = s^\beta$ for $\beta \in]0, n]$. The last part then follows, because if $\beta < \gamma < n$ then the set in (15) is contained in the set

$$\left\{p \in D : \limsup_{c \to o+} c^{(n-\gamma)/2} \mathscr{M}(w, p, c) = \infty\right\},$$

which has $\mathscr{P} - \Lambda^{\gamma}$-measure 0.

REMARKS. It is already known that the set (13) has $\mathscr{P}$-dimension at most n, because it is clearly polar and a set with larger $\mathscr{P}$-dimension cannot be ([6], Theorem 3). Part (i) of the corollary is worthwile only because there exist polar sets with strictly positive $\mathscr{P} - \Lambda^{n}$-measure ([6], Example 3).

5. The Riesz measures of potentials on lower half-spaces.

Let $H_a = \mathsf{R}^n \times]-\infty, a[$, where $-\infty < a \leqq \infty$, let $p \in H_a$, and let μ be a positive measure on H_a. We put $\lambda(p, \gamma) = \mu(\bar{\Omega}'(p, \gamma))$ for all $\gamma > 0$. If $p = (x, t)$, then $\Lambda(p)$ denotes H_t. The following theorem is analogous to the first part of Theorem 3.20 in [4]; no exact analogue of the second part is possible due to the existence of non-constant positive temperatures on H_a. Recall that G_{H_a} is the restriction of G to $H_a \times H_a$.

THEOREM 5. *Let μ be a positive measure on H_a.*

(i) *If μ is the Riesz measure of a potential on H_a, then*

$$-\int_1^{\infty} \tau'(\gamma)\lambda(p, \gamma)d\gamma < \infty \tag{16}$$

for all $p \in H_a$.

(ii) *Conversely, if there is $p \in H_a$ such that* (16) *holds, then the Green potential of μ is a supertemperature on $\Lambda(p)$. If, in addition,*

$$-\int_0^{1} \tau'(\gamma)\lambda(p, \gamma)d\gamma < \infty,$$

then $G\mu(p) < \infty$.

PROOF. (i) If $G\mu$ is a supertemperature on H_a, and $p \in H_a$, then since $G\mu \geqq 0$ it follows from (1) that

$$-\int_1^{c} \tau'(\gamma)\lambda(p, \gamma)d\gamma \leqq \mathscr{M}(G\mu, p, 1)$$

whenever $c > 1$. Making $c \to \infty$, we obtain (16).

(ii) Suppose that (16) holds when $p = p_0$. If $u = G\mu$ on H_a, and $p \in H_a$, then

$$u(p) = \int_{\bar{\Omega}'(p_0, 1)} G(p, q)d\mu(q) + \int_{A(p_0, 1, \infty)} G(p, q)d\mu(q) = v_1(p) + v_2(p),$$

say. The function v_1 is a supertemperature on H_a. For each integer $k > 1$, let u_k denote the supertemperature given by

$$u_k(p) = \int_{A(p_0,1,k)} G(p,q)\,d\mu(q)$$

for all $p \in H_a$. Then $\{u_k\}$ is increasing, so that its limit v_2 is a supertemperature on $\Lambda(p_0)$ if it is finite at p_0 ([7]). If $\lambda = \lambda(p_0,\cdot)$, then

$$u_k(p_0) = \int_1^k \tau(\gamma)\,d\lambda(\gamma) = [\tau(\gamma)\lambda(\gamma)]_1^k - \int_1^k \tau'(\gamma)\lambda(\gamma)\,d\gamma.$$

Since (16) holds when $p = p_0$, given $\varepsilon > 0$ we can find c_0 such that

$$\lambda(c)\tau(c) = \lambda(c)\int_c^\infty -\tau'(\gamma)\,d\gamma \leqq -\int_c^\infty \tau'(\gamma)\lambda(\gamma)\,d\gamma < \varepsilon$$

whenever $c > c_0$, so that $\lambda(c)\tau(c) \to 0$ as $c \to \infty$. Hence

$$v_2(p_0) = \lim_{k\to\infty} u_k(p_0) = -\tau(1)\lambda(1) - \int_1^\infty \tau'(\gamma)\lambda(\gamma)\,d\gamma < \infty,$$

so that v_2, and therefore u, is a supertemperature on $\Lambda(p_0)$. For the last part, we have

$$\begin{aligned} u(p_0) &= \lim_{c\to 0+} \int_{A(p_0,c,\infty)} G(p_0,q)\,d\mu(q) = \lim_{c\to 0+} \int_c^\infty \tau(\gamma)\,d\lambda(\gamma) \\ &= \lim_{c\to 0+}\left(-\tau(c)\lambda(c) - \int_c^\infty \tau'(\gamma)\lambda(\gamma)\,d\gamma\right) \leqq -\int_0^\infty \tau'(\gamma)\lambda(\gamma)\,d\gamma < \infty. \end{aligned}$$

6. Differences of positive supertemperatures on lower half-spaces.

Let u be a δ-subtemperature on D. If μ is the Riesz measure of u, then μ can be written minimally as a difference $\mu^+ - \mu^-$ of two positive measures on D, as in the subharmonic case ([3], pp. 505–7). Whenever $\bar{\Omega}(p,c) \subseteq D$, we put

$$\lambda^+(p,c) = \mu^+(\bar{\Omega}'(p,c)), \quad N^+(p,c) = -\int_0^c \tau'(\gamma)\lambda^+(p,\gamma)\,d\gamma,$$

and similarly for μ^-. We say that $u(p_0)$ is *finite* if $N^+(p_0,\cdot)$ and $N^-(p_0,\cdot)$ are both finite, in which case it follows from (2) that u is the difference of two supertemperatures which are finite at p_0. If $u(p_0)$ is finite, we define the *characteristic* T of u at p_0 by

$$T(u,p_0,c) = \mathscr{M}(u^+,p_0,c) + N^+(p_0,c) - u(p_0)$$

whenever $\bar{\Omega}(p_0, c) \subseteq D$. This is directly analogous to the definition for δ-subharmonic functions given in [3], p. 508. We can use T to characterize those δ-subtemperatures on $H_a = \mathsf{R}^n \times]-\infty, a[$, $-\infty < a \leqq \infty$, which can be written as a difference of two positive supertemperatures. This result is analogous to Theorem 7.42 in [3], which deals with δ-subharmonic functions on a disc; its proof is an adaption of the proof given in [3].

THEOREM 6. *Let u be a δ-subtemperature on H_a.*

(i) *If $u = u_1 - u_2$ is the difference of two positive supertemperatures on H_a, and $u(p_0)$ is finite, then $T(u, p_0, \cdot)$ is an increasing function such that $0 \leqq T(u, p_0, \cdot) \leqq u_2(p_0)$ on $]0, \infty[$, and there is a convex function ϕ such that $T(u, p_0, \cdot) = \phi \circ \tau$.*

(ii) *Conversely, if there is a sequence $\{p_j\}$ in H_a such that $H_a = \bigcup_{j=1}^{\infty} \Lambda(p_j)$, $u(p_j)$ is finite for all j, and $T(u, p_j, \cdot)$ is bounded above on $]0, \infty[$ for all j, then u is the difference of two positive supertemperatures on H_a.*

PROOF. (i) For $i \in \{1, 2\}$, let μ_i be the Riesz measure of u_i, and put

$$\lambda_i(p_0, c) = \mu_i(\bar{\Omega}'(p_0, c)), \quad N_i(p_0, c) = -\int_0^c \tau'(\gamma)\lambda_i(p_0, \gamma)d\gamma$$

for all $c > 0$. Since $u_1 \geqq 0$, it follows from (2) that

$$0 = \mathscr{M}(u_1^-, p_0, c) = \mathscr{M}(u_1, p_0, c) + N_1(p_0, c) - u_1(p_0).$$

Since μ_1 and μ_2 are positive and $\mu = \mu_1 - \mu_2$, we have $\mu^+ \leqq \mu_1$ and $\mu^- \leqq \mu_2$, so that

$$N^+(p_0, c) \leqq N_1(p_0, c) = u_1(p_0) - \mathscr{M}(u_1, p_0, c).$$

Furthermore, $u_1 \geqq u^+$ so that $\mathscr{M}(u^+, p_0, c) \leqq \mathscr{M}(u_1, p_0, c)$. Hence

$$T(u, p_0, c) \leqq \mathscr{M}(u_1, p_0, c) + (u_1(p_0) - \mathscr{M}(u_1, p_0, c)) - u_2(p_0).$$

Now let v_1 and v_2 be the potentials on H_a of μ^+ and μ^- respectively. Applying (2) to each v_i and subtracting, we obtain

$$u(p_0) = \mathscr{M}(u, p_0, c) + N^+(p_0, c) - N^-(p_0, c),$$

so that

$$\begin{aligned} T(u, p_0, c) &= \mathscr{M}(u^-, p_0, c) + N^-(p_0, c) \\ &= \mathscr{M}(u^-, p_0, c) + v_2(p_0) - \mathscr{M}(v_2, p_0, c) = v_2(p_0) - \mathscr{M}(v_1 \wedge v_2, p_0, c). \end{aligned}$$

Therefore, as $v_1 \wedge v_2$ is a supertemperature, the characteristic $T(u, p_0, \cdot)$ is increasing ([10]), there is a convex function ϕ such that $T(u, p_0, \cdot) = \phi \circ \tau$ ([8,10]), and

$$T(u, p_0, 0+) = v_2(p_0) - (v_1 \wedge v_2)(p_0) \geqq 0.$$

(ii) Let $u = w_1 - w_2$ for some supertemperatures w_i on H_a. Applying (2) to each w_i and subtracting, we obtain

$$T(u, p_j, c) = \mathscr{M}(u^-, p_j, c) + N^-(p_j, c) \tag{17}$$

for all j and all $c > 0$, as above. Hence $N^-(p_j, \cdot) \leqq T(u, p_j, \cdot)$, so that $N^-(p_j, \cdot)$ is bounded. Thus

$$-\int_0^\infty \tau'(\gamma)\lambda^-(p_j, \gamma)d\gamma < \infty$$

for all j, so that Theorem 5 (ii) shows that the Green potential v_2 of μ^- is a supertemperature on $\Lambda(p_j)$ for all j, and hence on H_a. Furthermore,

$$N^+(p_j, c) = T(u, p_j, c) - \mathscr{M}(u^+, p_j, c) + u(p_j) \leqq T(u, p_j, c) + u(p_j)$$

for all j and all $c > 0$, so that each $N^+(p_j, \cdot)$ is bounded, and hence the Green potential v_1 of μ^+ is a supertemperature on H_a and finite at every p_j. It follows that the function h, defined q.e. on H_a by $h = u + v_2 - v_1$, can be extended to a temperature h on H_a. Furthermore, because v_1 and v_2 are positive,

$$\begin{aligned} \mathscr{M}(h^-, p_j, \cdot) &\leqq \mathscr{M}(u^-, p_j, \cdot) + \mathscr{M}(v_1, p_j, \cdot) \\ &= T(u, p_j, \cdot) - N^-(p_j, \cdot) + \mathscr{M}(v_1, p_j, \cdot) \end{aligned}$$

by (17), so that

$$\mathscr{M}(h^-, p_j, \cdot) \leqq T(u, p_j, \cdot) + v_1(p_j),$$

and hence each $\mathscr{M}(h^-, p_j, \cdot)$ is bounded. By Theorem 3 in [10], the subtemperature h^- has a thermic majorant w on H_a. Therefore $h = (h + w) - w$ is a difference of two positive temperatures on H_a, and hence

$$u = h + v_1 - v_2 = (h + w + v_1) - (w + v_2)$$

is the difference of two positive supertemperatures.

REMARK. The representation formula for the difference of two positive supertemperatures on H_a, follows easily from the Riesz decomposition theorem and the representation theorem for positive temperatures given in [2], p. 294.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CANTERBURY
CHRISTCHURCH
NEW ZEALAND