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ON THE REITERATION PROPERTY OF $\vec{X}_{\varphi,q}$ SPACES

NATAN JA. KRUGLJAK

Abstract.

In [6] Janson introduced the spaces $\vec{X}_{\varphi,q} = (X_0, X_1)_{\varphi,q}$ with φ be a positive concave function on $\mathsf{R}_+ = (0, +\infty)$, $q \in [1, \infty]$ and gave sufficient conditions on concave positive functions $\varphi_0, \varphi_1, \varphi$ that the reiteration formula

$$(\vec{X}_{\varphi_0, q_0}, \vec{X}_{\varphi_1, q_1})_{\varphi,q} = \vec{X}_{\varphi(\varphi_0,\varphi_1),q}$$

$$\left(\varphi(\varphi_0, \varphi_1)(s) = \varphi_0(s))\varphi\left(\frac{\varphi_1(s)}{\varphi_0(s)}\right)\right)$$

holds for all $q_0, q_1, q \in [1, \infty]$. Here we will give necessary and sufficient conditions on $\varphi_0. \varphi_1, \varphi$ that this reiteration formula holds for all $q_0, q_1, q \in [1, \infty]$ and all Banach couples $\vec{X}$.

The spaces considered here are described by using an idea of optimal blocks which originates from a recent paper by Krugljak, Maligranda and Persson [8].

0. Introduction.

Let $\vec{X} = (X_0, X_1)$ be a Banach couple and for $x \in \Sigma(\vec{X}) = X_0 + X_1$ Peetre's K-functional is defined by the formula

$$K(t, x, \vec{X}) = \inf_{x = x_0 + x_1} (\|x_i\|_{X_0} + t\,\|x_1\|_{X_1}) \quad (t > 0).$$

In the theory if interpolation we have very nice theory and a lot of applications (see [2]) of $\vec{X}_{\theta,q} = (X_0, X_1)_{\theta,q}$ ($\theta \in (0, 1)$, $q \in [1, +\infty]$) spaces which are defined by the norm

$$(0.1) \qquad \|x\|_{\theta,q} = \left(\int_0^\infty \left(\frac{K(t, x; \vec{X})}{t^\theta}\right)^q \frac{dt}{t}\right)^{1/q}$$

with usual changes for $q = +\infty$.

One of the most important theoretical results for $\vec{X}_{\theta,q}$ spaces is the so called theorem of reiteration, which claims that

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$$(0.2) \qquad (\vec{X}_{\theta_0,q_0}, \vec{X}_{\theta_1,q_1})_{\alpha,q} = \vec{X}_{(1-\alpha)\theta_0+\alpha\theta_1,q},$$

if $\theta_0 \neq \theta_1$.

It seems natural to extend this theorem for more general spaces, for example replacing t^θ in (0.1) by $\varphi(t)$ where φ is a positive concave function on $\mathrm{R}_+ = (0, +\infty)$.

After some years (see [5, 7. 9] and also [1, 4]) occurs that such approach needs several restrictions on φ, for example that φ belongs to the so called "quasi-power" class P^{+-}, which means that φ behave like a power function near 0 and $+\infty$.

In 1981 Janson [6] made another approach to such spaces. A very surprising element of his theory is that for $\varphi \notin P^{+-}$ the spaces $\vec{X}_{\varphi,q} = (X_0, X_1)_{\varphi,q}$ are not defined by the norm

$$(0.3) \qquad \left\{ \int_0^{+\infty} \left[\frac{K(t, x; \vec{X})}{\varphi(t)} \right]^q \frac{dt}{t} \right\}^{1/q}$$

which is analogous to (0.1) but by the norm

$$(0.4) \qquad \|x\|_{\varphi,q} = \left(\sum_{n\in\mathbf{Z}} \left[\frac{K(t_n, x, \vec{X})}{\varphi(t_n)} \right]^q \right)^{1/q}$$

where $\{t_n\}$ is a special sequence which depends on φ. To explain why such definition "theoretically" is more correct than (0.3) we should mention that for the $\vec{X}_{\theta,q}$-spaces the extreme (maximal and minimal) functors with the same characteristic function t^θ are $\vec{X}_{\theta,\infty}$ and $\vec{X}_{\theta,1}$ respectively. The analogous statement is true for the $\vec{X}_{\varphi,\infty}$, $\vec{X}_{\varphi,1}$-spaces defined by (0.4), but it is not so if we define $\vec{X}_{\varphi,1}$ by the norm (0.3) when $\varphi \notin P^{+-}$. In [6] Janson proved that if $\varphi(\varphi_0, \varphi_1) \in P^{+-}$ and $\varphi_1/\varphi_0 \in P^{+-}$ then analogously to (0.2) the formula

$$(0.5) \qquad (\vec{X}_{\varphi_0,q_0}, \vec{X}_{\varphi_1,q_1})_{\varphi,q} = \vec{X}_{\varphi(\varphi_0,\varphi_1),q}$$

holds for any $q_0, q_1, q \in [0, +\infty]$ and $\varphi(\varphi_0, \varphi_1)(t) = \varphi_0(t)\,\varphi\left(\frac{\varphi_1(t)}{\varphi_0(t)}\right)$. Here we give necessary and sufficient conditions on φ_0, φ_1, φ so that (0.5) is true for any choice of q_0, q_1, $q \in [1, +\infty]$. Our main tools in the proof are general results for K and J methods from [3] combined with direct calculations for couples $\vec{L}_\infty$, $\vec{l}_1$.

1. Definitions and results.

Let φ be a positive concave function on $\mathrm{R}_+ = (0, +\infty)$. Here and below we will assume that

$$\lim_{t\to+0}\varphi(t)=\lim_{t\to+\infty}\frac{\varphi(t)}{t}=0 \tag{1.1}$$

For fixed φ we will construct a special sequence $\{t_i\}$.

Let $r>1$ be fixed. For $s>0$ by Ω_s^φ we denote the interval

$$\Omega_s^\varphi=\{t>0\,|\,\varphi(t)\leqq r\,\varphi(s)\min(1,t/s)\} \tag{1.2}$$

It is easy to see from (1.1) that there exist two sides sequence (finite or infinite) of increasing positive numbers $\{t_{2i+1}\}$ such that

$$\bigcup_i \Omega^\varphi_{t_{2i+1}}=\mathsf{R}_+,$$

$$\mathring{\Omega}^\varphi_{t_{2i+1}}\cap\mathring{\Omega}^\varphi_{t_{2i+1}}=\varnothing,\quad i\neq j,$$

where $\mathring{\Omega}_s^\varphi$ is the interior of Ω_s^φ.

Denote the left endpoint of $\Omega^\varphi_{t_{2i+1}}$ by t_{2i} and right endpoint by t_{2i+2}. We obtain the sequence $\{t_i\}$ such that (see [3], p. 322–323 and [8]):

$$t_{i+1}\geqq r\,t_i, \tag{1.3}$$

$$\bigcup_i [t_{2i},t_{2i+2})=\mathsf{R}_+, \tag{1.4}$$

$$\varphi(t)\leqq r\,\varphi(t_{2i+1})\min(1,t/t_{2i+1})\quad(t\in[t_{2i},t_{2i+2})), \tag{1.5}$$

$$\varphi(t_{2j+1})\min(1,t/t_{2j+1})\leqq \tag{1.6}$$

$$\leqq\frac{r}{r^{|j-i|}}\varphi(t_{2i+1})\min(1,t/t_{2i+1})\quad(t\in[t_{2i},t_{2i+2}))$$

REMARK 1.1. It is possible that the endpoints of Ω_s^φ equal 0, or equal $+\infty$. Then by $[0,t)$ in (1.4), (1.5), (1.6) we mean $[0,t)\cap\mathsf{R}_+$. We denote by Ω_i the interval $\Omega_i=[t_{2i},t_{2i+2})$. By $K_{\varphi,q}$ (φ – as above, $q\in[1,+\infty]$) we denote the K-functor defined for the Banach couple $\vec{X}=(X_0,X_1)$ by the formula

$$\|x\|_{\varphi,q}=\left(\sum_i\left[\sup_{s\in\Omega_i}\frac{K(s,x;\vec{X})}{\varphi(s)}\right]^q\right)^{1/q} \tag{1.7}$$

with usual changes if $q=+\infty$. The idea to define such a K-method of interpolation with blocks originates from a recent paper of Krugljak, Maligranda and Persson [8].

REMARK 1.2. It is easy to see that the sequence $\{t_i\}$ for fixed φ is uniquely determined by r and one of its elements let say t_0. If we change r and t_0, then we obtain another sequence $\{t_i'\}$ and so we can define $\|\cdot\|'_{\varphi,q}$ on (1.7) by $\{t_i'\}$. But it is not difficult to prove that

$$\|\cdot\|_{\varphi,q} \approx \|\cdot\|'_{\varphi,q}$$

It follows from the fact that the intervals $u_{ij} = [t_{2i}, t_{2i+2}) \cap [t'_{2j}, t'_{2j+1})$ divide the intervals $[t_{2k}, t_{2k+2})$, $[t'_{2l}, t'_{2l+2})$ into finite (and not greater than some natural number N) subintervals.

REMARK 1.3. If $q = +\infty$, then (1.7) imply that

$$\|x\|_{\varphi,\infty} = \sup_{t>0} \frac{K(t, x, \vec{X})}{\varphi(t)}$$

and $K_{\varphi,\infty}$ is equal to $K_{L^{\varphi}_{\infty}}$.

If $q = 1$ and $\lim_{t\to+0} \varphi(t)/t = \lim_{t\to+\infty} \varphi(t) = +\infty$, then (see section 2 below, or [6])

$$K_{\varphi,1} = J_{L^{\varphi}_1},$$

where $J_{L^{\varphi}_1}$ is a J-method with parameter L^{φ}_1:

$$\|f\|_{L^{\varphi}_1} = \int_0^{+\infty} \frac{|f(s)|}{\varphi(s)} \frac{ds}{s}.$$

Since $K_{L^{\varphi}_{\infty}}$, $J_{L^{\varphi}_1}$ are maximal and minimal interpolation functors with the same characteristic function (see [3], p. 438–444) we conclude that $K_{\varphi,q}$(φ – fix, $q \in [1, +\infty]$) is a scale which connects the extreme functors $J_{L^{\varphi}_1}$ and $K_{L^{\varphi}_{\infty}}$.

REMARK 1.4. If φ belongs to the so called quasi-power class P^{+-} (it means (see [8, Lemma1]) that $\sup \frac{t_{2i+2}}{t_{2i}} < +\infty$, then is easy to check that the norm (1.7) is equivalent to

$$\left\{ \int_0^{+\infty} \left[\frac{K(t, x; \vec{X})}{\varphi(t)} \right]^q \frac{dt}{t} \right\}^{1/q}.$$

REMARK 1.5. (Connections with the Janson definition). From (1.5) it follows that φ on $\Omega_i = [t_{2i}, t_{2i+2})$ is equivalent to $\varphi(t_{2i+1}) \min(1, t/t_{2i+1})$ and so

$$\sup_{t\in\Omega_i} \frac{K(t, x; \vec{X})}{\varphi(t)} \approx \max\left(\frac{K(t_{2i}, x; \vec{X})}{\varphi(t_{2i})}, \frac{K(t_{2i+2}, x, \vec{X})}{\varphi(t_{2i+2})} \right)$$

with constants of equivalence not depending on i. Therefore from (1.7) follows that

$$\|x\|_{\varphi,q} \approx \left(\sum_i \left(\frac{K(t_{2i}, x; \vec{X})}{\varphi(t_{2i})}\right)^q\right)^{1/q}$$

which is the original definition of $\vec{X}_{\varphi,q}$ spaces by Janson [6] (in fact, Janson chose the numbers t_{2i} in a slightly different way).

To formulate our main result let us denote by $\varphi(\varphi_0, \varphi_1)$ concave function which is defined by the formula

$$\varphi(\varphi_0, \varphi_1)(t) = \varphi(\varphi_0(t), \varphi_1(t)) = \varphi_0(t)\,\varphi(\varphi_1(t)/\varphi_0(t)),$$

where $\varphi_0, \varphi_1, \varphi$ are positive concave functions on R_+, which satisfies (1.1). It is easy to see that $\varphi(\varphi_0, \varphi_1)$ is again a positive concave function and satisfies (1.1). By $\{\tilde{t}_i\}$ we will define the sequence constructed above for $\varphi(\varphi_0, \varphi_1)$.
Here and everywhere below we shall use the function

$$S(t) = \varphi_1(t)/\varphi_0(t)$$

Our main result is

THEOREM 1.1. *Let φ_0, φ_1, φ be positive concave functions on R_+ which satisfies (1.1). Then, for any q_0, q_1, $q \in [1, +\infty]$, we have*

$$K_{\varphi,q}(K_{\varphi_0,q_0}, K_{\varphi_1,q_1}) = K_{\varphi(\varphi_0,\varphi_1),q} \tag{1.8}$$

if and only if there exists a natural number N such that for any i the set

$$U_i = S^{-1}([t_{2i}, t_{2i+2})) \tag{1.9}$$

intersect with not more than N numbers of intervals $[\tilde{t}_{2j}, \tilde{t}_{2j+2})$.

We leave the proof of the Theorem 1.1 to section 4, but here we will give only one corollary of it.

THEOREM 1.2. *If φ_0, $\varphi_1 \in P^{+-}$ and the map $S: s \to \varphi_1(s)/\varphi_0(s)$ maps R_+ onto R_+, then (1.8) holds for all q_0, q_1, $q \in [1, +\infty]$ if and only if*

a) $\varphi \in P^{+-}$

and

b) $\sup\limits_{n \in \mathsf{Z}} \operatorname{Card}\{k \in \mathsf{Z} \mid 2^n \leq S(2^k) < 2^{n+1}\} < +\infty$

hold.

PROOF. We should like to remind that $\varphi \in P^{+-}$ *iff* $\sup\limits_{2} \dfrac{t_{2i+2}}{t_{2i}} < +\infty$ (see [8, Lemma 1]). It is easy to see that it follows from definition (1.2) that

$$\Omega_s^{\varphi(\varphi_0,\varphi_1)} \subset \Omega_s^{\varphi_0} \cup \Omega_s^{\varphi_1}$$

Hence, since $\varphi_0, \varphi_1 \in P^{+-}$ it follows that $\varphi(\varphi_0, \varphi_1) \in P^{+-}$ and so $\sup_i \frac{\tilde{t}_{2i+2}}{\tilde{t}_{2i}} < +\infty$. Moreover $\tilde{t}_{2i+2} \geqq r^2 \tilde{t}_{2i}$ (see (1.3)) which implies that the intervals $[\tilde{t}_{2i}, \tilde{t}_{2i+2})$ in Theorem 1.1 can be replaced by the intervals $[2^m, 2^{m+1})$.
If $2^m \leqq s < 2^{m+1}$, then

$$\tfrac{1}{2} \leqq \frac{S(s)}{S(2^m)} \leqq 2. \tag{1.10}$$

From this it follows that S maps the interval $[2^m, 2^{m+1})$ in an interval $[T_m, 4T_m)$, $T_m = \frac{1}{2} S(2^m)$. Therefore if $\varphi \notin P^{+-}$, then $\sup \frac{t_{2i+2}}{t_{2i}} = +\infty$ and it is impossible for all i to cover the interval $[t_{2i}, t_{2i+2})$ with N numbers of intervals of the form $[T_m, 4\,T_m)$. Therefore φ must belong to P^{+-} and as above we could replace the intervals $[t_{2i}, t_{2i+2})$ by intervals of the form $[2^n, 2^{n+1})$. By using (1.10) it is easy to show that the conditions in Theorem 1.1 now is equivalent to the condition b).

2. Theorem of equivalence.

We need some extension of Janson's theorem on equivalence for $\vec{X}_{\varphi,q}$ spaces. By a general equivalence theorem for K-functors (see [3, Theorem 3.5.9]) $K_{\varphi,q}$ can be expressed in terms of the so called J functors:

$$K_{\varphi,q} = J_\phi + \Delta^c \quad (\phi = K_{\varphi,q}(\vec{l}_1)), \tag{2.1}$$

where $\vec{l}_1 = (l_1, l_1(\{2^{-n}\}))$ is a Banach couple consisting of two sides infinite sequences $\{\phi_n\}$ with the norms

$$\|\{a_n\}_{l_1} = \sum_{n=-\infty}^{+\infty} |a_n|, \quad \|\{a_n\}\|_{l_1(\{\frac{1}{2^n}\})} = \sum_{n=-\infty}^{+\infty} \frac{|a_n|}{2^n}$$

and $\Delta^{c'}$ is a Gagliardo completion of $\Delta(\vec{X}) = X_0 \cap X_1$ in $\Sigma(\vec{X})$. We will compute the parameter of the J-method $\phi = K_{\varphi,q}(\vec{l}_1)$ in (2.1).

Let us remind that $\Omega_i = [t_{2i}, t_{2i+2})$. Everywhere below the symbol $\approx$ means that finiteness of one part imply finiteness of the other and the constants of equivalence are absolute.

LEMMA 2.1. *Let $\phi = K_{\varphi,q}(\vec{l}_1)$ then*

$$\|\{a_n\}\|_\phi \approx \left(\sum_i \left(\sum_{2^k \in \Omega_i} \frac{|a_k|}{\varphi(2^k)}\right)^q\right)^{1/q}. \tag{2.2}$$

PROOF. Since

$$K(t, \{a_n\}; \vec{l}_1) \approx \sum_n |a_n| \min(1, t/2^n)$$

it follows that

$$\|\{a_n\}\|_\Phi \approx \left(\sum_i \left[\sup_{t\in\Omega_i} \frac{\sum_n |a_n| \min(1, \frac{t}{2^n})}{\varphi(t)}\right]^q\right)^{1/q}. \tag{2.3}$$

Let us denote

$$\alpha_i = \sup_{t\in\Omega_i} \frac{\sum_n \frac{|a_n|}{\varphi(2^n)} \varphi(2^n) \min(1, t/2^n)}{\varphi(t)}$$

Then

$$(\sum \alpha_i^q)^{1/q} \approx \|\{a_n\}\|_\Phi$$

and for $t = t_{2i+2}$ we obtain

$$\alpha_i \geqq \frac{1}{r} \sum_{2^n\in[t_{2i+1}, t_{2i+2})} \frac{|a_n|}{\varphi(2^n)}.$$

Analogously for $t = t_{2i}$ we obtain

$$\alpha_i \geqq \frac{1}{r} \sum_{2^n\in[t_{2i}, t_{2i+2})} \frac{|a_n|}{\varphi(2^n)}$$

and so

$$\sum_{2^n\in\Omega_i} \frac{|a_n|}{\varphi(2^n)} \leqq 2r\,\alpha_i.$$

Therefore the right hand side in (2.2) is not greater than

$$2r\left(\sum_i \alpha_i^q\right)^{1/q}$$

To prove the inverse estimate let us denote by β_i the sum

$$\beta_i = \sum_{2^n\in\Omega_i} \frac{|a_n|}{\varphi(2^n)}.$$

Then, the right hand side in (2.2) is finite means that

$$\left(\sum_i \beta_i^q\right)^{1/q} < +\infty$$

and so

$$\sum_{2^n\in\Omega_j} \frac{|a_n|}{\varphi(2^n)} \varphi(2^n)\min(1, t/2^n) \leqq \beta_j r\, \varphi(t_{2j+1})\min(1, t/t_{2j}).$$

Since on $\Omega_i = [t_{2i}, t_{2i+2})$ we have, according to (1.6),

$$\varphi(t_{2j+1})\min(1, t/t_{2j+1}) \leqq \frac{r}{r^{|j-i|}} \varphi(t_{2i+1})\min(1, t/t_{2i+1})$$

it follows that on $[t_{2i}, t_{2i+2})$ we have

$$\sum_n \frac{|a_n|}{\varphi(2^n)} \varphi(2^n)\min\left(1, \frac{t}{2^n}\right) \leqq r^2\left(\sum_j \frac{\beta_j}{r^{|j-i|}}\right)\varphi(t_{2i+1})\min\left(1, \frac{t}{t_{2i+1}}\right).$$

Therefore

$$\sup_{t\in\Omega_i} \frac{\dfrac{|a_n|}{\varphi(2^n)}\varphi(2^n)\min\left(1, \dfrac{t}{2^n}\right)}{\varphi(t)} \leqq r^2 \sum_j \frac{\beta_j}{r^{|j-i|}}.$$

By substitutions this estimate in the right hand side of (2.3) we obtain

$$\|\{a_n\}\|_\Phi \leqq C\left(\sum_i \left(\sum_j \frac{\beta_j}{r^{|j-i|}}\right)^q\right)^{1/q}.$$

In view of this inequality and the general Minkowski inequality for $q \geqq 1$ and direct estimates

$$\left(\sum_j \frac{\beta_j}{r^{|j-i|}}\right)^q \leqq \sum_j \frac{\beta_j^q}{r^{q|j-i|}}$$

for the case $q < 1$ we obtain

$$\|\{a_n\}\|_\Phi \leqq C\left(\sum_j \beta_j^q\right)^{1/q}.$$

This completes the proof.

3. Descriptions of the functors $F = K_{\varphi,q}(K_{\varphi_0,\infty}, K_{\varphi_1,\infty})$, $G = K_{\varphi,q}(K_{\varphi_0,1}, K_{\varphi_1,1})$.

We will need several lemmas.

A). Let $\vec{L}_\infty = (L_\infty, L_\infty(t^{-1}))$ be a Banach couple of functions on $\mathbf{R}_+ = (0, +\infty)$ with the norms

$$\|f\|_{L_\infty} = \sup_{t\in\mathbf{R}_+} |f(t)|, \quad \|f\|_{L_\infty(\frac{1}{t})} = \sup_{t\in\mathbf{R}_+}\left|\frac{f(t)}{t}\right|.$$

In our first lemma we are interested in the functor

$$F = K_{\varphi,q}(K_{\varphi_0,\infty}, K_{\varphi_1,\infty}) \quad (1 \leqq q \leqq \infty). \tag{3.1}$$

By a general reiteration theorem $F = K_\Psi$ where $\Psi = F(\vec{L}_\infty)$ (see [3] p. 345). In Lemma 3.1 we try to determine Ψ. Let $\{t_i\}$ be a sequence constructed by φ (cf. section 1) and let

$$U_i = S^{-1}([t_{2i}, t_{2i+2})) \tag{3.2}$$

LEMMA 3.1. *If $\Psi = F(\vec{L}_\infty)$, then*

$$\|f\|_\Psi \approx \left(\sum_i \left[\sup_{S \in U_i} \frac{f(s)}{\varphi(\varphi_0(s), \varphi_1(s))}\right]^q\right)^{1/q}. \tag{3.3}$$

PROOF. Since

$$K(f; \vec{L}_\infty) = |\hat{f}|,$$

where $|\hat{f}|$ is a least concave majorant of $|f|$, than according to the concavity of φ_i, we have

$$\|f\|_{\Psi_i} = \sup_{t>0} \frac{|\hat{f}|(t)}{\varphi_i(t)} = \sup_{t>0} \frac{|f(t)|}{\varphi_i(t)} \quad (\Psi_i = K_{\varphi_i,\infty}(\vec{L}_\infty)).$$

Since

$$K\left(t, f; L_\infty\left(\frac{1}{\varphi_0}\right), L_\infty\left(\frac{1}{\varphi_1}\right)\right) \approx \sup_{S>0}\left(\frac{1}{\varphi_0(s)}, \frac{t}{\varphi_1(s)}\right)|f(s)|$$

it follows that

$$\|f\|_\Psi \approx \left(\sum_i \left[\sup_{t \in \Omega_i} \frac{\sup_{S>0}\left(\frac{1}{\varphi_0(s)}, \frac{t}{\varphi_1(s)}\right)|f(s)|}{\varphi(t)}\right]^q\right)^{1/q}$$

Denote

$$\alpha_i = \sup_{t \in \Omega_i} \frac{\sup_{S>0}\left(\frac{1}{\varphi_0(s)}, \frac{t}{\varphi_1(s)}\right)|f(s)|}{\varphi(t)} \tag{3.4}$$

Then

$$\|f\|_\Psi \approx \left(\sum_i \alpha_i^q\right)^{1/q}.$$

If $s \in U_i$, then $\dfrac{\varphi_1(s)}{\varphi_0(s)} \in \Omega_i = [t_{2i}, t_{2i+2})$. Take t in (3.4) to be equal to $\varphi_1(s)/\varphi_0(s)$. Then we obtain

$$\frac{|f(s)|}{\varphi_0(s)\varphi\left(\dfrac{\varphi_1(s)}{\varphi_0(s)}\right)} = \frac{|f(s)|}{\varphi(\varphi_0(s), \varphi_1(s))} \leqq \alpha_i$$

for all $s \in U_i$. We conclude that the right side in (3.3) is not greater than $\left(\sum_i \alpha_i^q\right)^{1/q} \approx \|f\|_\Psi$.

To prove (3.3) in the reversed direction let us denote

$$\beta_i = \sup_{s \in U_i} \frac{|f(s)|}{\varphi(\varphi_0(s), \varphi_1(s))}. \tag{3.5}$$

Then, since the right side in (3.3) is finite, we have that $\left(\sum_i \beta_i^q\right)^{1/q} < +\infty$. Let us consider the series $\sum f \chi_{U_i}$ (χ_{U_i} is the characteristic function of U_i). We will show that this sum belongs to $J_\phi\left(L_\infty\left(\frac{1}{\varphi_0}\right), L_\infty\left(\frac{1}{\varphi_1}\right)\right)$, $\phi = K_{\varphi,q}(\vec{l}_1)$ and has norm not greater than $c(\sum \beta_i^q)^{1/q}$. From this it follows that $\sum f \chi_{U_i} = f$ and due to (2.1)

$$\|f\|_\Psi \leqq c(\sum \beta_i^q)^{1/q}.$$

Therefore we only need to prove that

$$\|\sum f \chi_{U_i}\|_{J_\varphi(L_\infty(\frac{1}{\varphi_0}),(L_\infty(\frac{1}{\varphi_1}))} \leqq c\left(\sum_i \beta_i^q\right)^{1/q}. \tag{3.6}$$

According to (3.5) we have

$$J\left(t_{2i+1}, f\chi_{U_i}; L_\infty\left(\frac{1}{\varphi_0}\right), L_\infty\left(\frac{1}{\varphi_1}\right)\right) =$$

$$= \max\left[\sup_{S \in U_i} \frac{|f(s)|}{\varphi_0(s)}, \sup_{S \in U_i}\left(t_{2i+1}\frac{|f(s)|}{\varphi_1(s)}\right)\right] \leqq r\,\beta_i\,\varphi(t_{2i+1})$$

and so

$$\left(\sum_i\left[\frac{J(t_{2i+1}, f\chi_{U_i}; L_\infty(\frac{1}{\varphi_0}), L_\infty(\frac{1}{\varphi_1}))}{\varphi(t_{2i+1})}\right]^q\right)^{1/q} \leqq r\left(\sum_i \beta_i^q\right)^{1/q}.$$

If we replace t_{2i+1} by $2^{k_i} \approx t_{2i+1}$, then we will have

$$\left(\sum_i\left[\frac{J(2^{k_i}, f\chi_{U_i}; L_\infty(\frac{1}{\varphi_0}), L_\infty(\frac{1}{\varphi_1}))}{\varphi(2^{k_i})}\right]^q\right)^{1/q} \leqq c\left(\sum_i \beta_i^q\right)^{1/q}.$$

Since in every set $\Omega_i = [t_{2i}, t_{2i+2})$ it lies only finite (and not more than some natural N) numbers of 2^{k_j} we can use Lemma 2.1. to obtain the required result.

B). Now, we will consider the interpolation functor

$$G = K_{\varphi,q}(K_{\varphi_0,1}, K_{\varphi_1,1}).$$

By a general equivalence theorem (see [3]) we obtain that G coincides with the interpolation functor J_ϕ, where $\phi = G(\vec{l}_1)$ on all relatively complete couples. Here we will determine ϕ.

LEMMA 3.2. *If $\phi = G(\vec{l}_1)$, then*

$$\|\{a_n\}\|_\phi \approx \left(\sum_i \left[\sum_{2^n \in U_i} \frac{|a_n|}{\varphi(\varphi_0(2^n), \varphi_1(2^n))}\right]^q\right)^{1/q}. \tag{3.7}$$

PROOF. From Lemma 2.1 we have

$$\|\{a_n\}\|_{K_{\varphi_i,1}(\vec{l}_1)} \approx \sum_n \frac{|a_n|}{\varphi_i(2^n)}$$

and so

$$K(t, \{a_n\}, K_{\varphi_0,1}(\vec{l}_1), K_{\varphi_1,1}(\vec{l}_1)) \approx \sum_n |a_n| \min\left(\frac{1}{\varphi_0(2^n)}, \frac{t}{\varphi_1(2^n)}\right).$$

Therefore,

$$\|\{a_n\}\|_\phi \approx \left(\sum_i \left[\sup_{t \in \Omega_i} \frac{\sum_n |a_n| \min\left(\frac{1}{\varphi_0(2^n)}, \frac{t}{\varphi_1(2^n)}\right)}{\varphi(t)}\right]^q\right)^{1/q}. \tag{3.8}$$

If we denote

$$\alpha_i = \sup_{t \in \Omega_i} \frac{\sum_n |a_n| \min\left(\frac{1}{\varphi_0(2^n)}, \frac{t}{\varphi_1(2^n)}\right)}{\varphi(t)},$$

then

$$\|\{a_n\}\|_\phi \approx \left(\sum_i \alpha_i^q\right)^{1/q}$$

Since, for $t = t_{2i+2}$, we have

$$\alpha_i \geqq \sum_{S(2^n) \in [t_{2i+1}, t_{2i+2})} \frac{|a_n|}{\varphi(\varphi_0(2^n), \varphi_1(2^n))} \cdot \frac{\varphi(\varphi_0(2^n), \varphi_1(2^n))}{\varphi(t_{2i+1})}.$$

$$\cdot \min\left(\frac{1}{\varphi_0(2^n)}, \frac{t_{2i+2}}{\varphi_1(2^n)}\right) \geqq \frac{1}{r} \sum_{S(2^n) \in [t_{2i+1}, t_{2i+2})} \frac{|a_n|}{\varphi(\varphi_0(2^n), \varphi_1(2^n))}$$

and, analogously, for $t = t_{2i}$:

$$\alpha_i \geqq \sum_{S(2^n)\in[t_{2i},t_{2i+1})} \frac{|a_n|}{\varphi(\varphi_0(2^n),\varphi_1(2^n))}\cdot\frac{\varphi(\varphi_0(2^n),\varphi_1(2^n))}{\varphi(t_{2i})}\cdot$$

$$\cdot\min\left(\frac{1}{\varphi_0(2^n)},\frac{t_{2i}}{\varphi_1(2^n)}\right) \geqq \frac{1}{r}\sum_{S(2^n)\in[t_{2i},t_{2i+1})}\frac{|a_n|}{\varphi(\varphi_0(2^n),\varphi_1(2^n))}$$

it follows that

$$\sum_{2^n\in U_n}\frac{|a_n|}{\varphi(\varphi_0(2^n),\varphi_1(2^n))} \leqq 2r\,\alpha_i.$$

Therefore, the right hand side in (3.7) is not greater than

$$2r(\textstyle\sum \alpha_i^q)^{1/q} \approx \|\{a_n\}\|_\phi.$$

Assume now that the right side in (3.7) is finite. If we denote

$$\beta_i = \sum_{2^n\in U_i}\frac{|a_n|}{\varphi(\varphi_0(2^n),\varphi_1(2^n))},$$

then

$$\left(\sum_i \beta_i^q\right)^{1/q} < +\infty.$$

Since

$$\sum_{2^n\in U_j}|a_n|\min\left(\frac{1}{\varphi_0(2^n)},\frac{t}{\varphi_1(2^n)}\right) \leqq \sum_{2^n\in U_j}\frac{|a_n|}{\varphi(\varphi_0(2^n),\varphi_1(2^n))}\cdot$$

$$\cdot\,\varphi(\varphi_0(2^n),\varphi_1(2^n))\cdot\min\left(\frac{1}{\varphi_0(2^n)},\frac{t}{\varphi_1(2^n)}\right) \leqq r\sum_{2^n\in U_j}\frac{|a_n|}{\varphi(\varphi_0(2^n),\varphi_1(2^n))}\cdot$$

$$\cdot\,\varphi(t_{2j+1})\min\left(1,\frac{t}{t_{2j+1}}\right) = r\,\beta_j\,\varphi(t_{2j+1})\min\left(1,\frac{t}{t_{2j+1}}\right)$$

it follows that

$$\sup_{t\in\Omega_i}\frac{\sum_n|a_n|\min\left(\frac{1}{\varphi_0(2^n)},\frac{t}{\varphi_1(2^n)}\right)}{\varphi(t)} \leqq \sup\frac{\sum_j r\beta_j\varphi(t_{2j+1})\min\left(1,\frac{t}{t_{2j+1}}\right)}{\varphi(t_{2i+1})\min\left(1,\frac{t}{t_{2i+1}}\right)} \leqq$$

$$\leqq r^2\sum_j\frac{\beta_j}{r^{|j-i|}}$$

Thus, by (3.8) we have

$$\|\{a_n\}\|_\Phi \leqq c\left(\sum_i\left(\sum_j \frac{\beta_j}{r^{|j-i|}}\right)^q\right)^{1/q}$$

and by applying a general Minkowski inequality for $q \geqq 1$ or direct estimates for $q < 1$ we obtain

$$\|\{a_n\}\|_\Phi \leqq c\left(\sum_j \beta_j^q\right)^{1/q}.$$

This completes the proof.

C). Let $\{\tilde{t}_i\}$ be a sequence constructed for the concave function $\varphi(\varphi_0, \varphi_1)$ and $\tilde{\Omega}_i = [\tilde{t}_{2i}, \tilde{t}_{2i+2})$ (see section 1). Take $j = j(i)$ such that

$$t_{2(j-1)+1} \leqq \frac{\varphi_1(\tilde{t}_{2i+1})}{\varphi_0(\tilde{t}_{2i+1})} \leqq t_{2j+1}. \tag{3.9}$$

LEMMA 3.3. If $j = j(i)$ is defined by (3.9), then

$$\tilde{\Omega}_i \subset \bigcup_{|k-j|\leqq 2} U_k \tag{3.10}$$

PROOF. If $s \in [\tilde{t}_{2i+1}, \tilde{t}_{2i+2})$, then

$$\varphi(\varphi_0(s), \varphi_1(s)) \leqq r\, \varphi(\varphi_0(\tilde{t}_{2i+1}), \varphi_1(\tilde{t}_{2i+1})),$$

which we can rewrite as

$$\varphi\left(\frac{\varphi_1(s)}{\varphi_0(s)}\right) \leqq r\, \frac{\varphi_0(\tilde{t}_{2i+1})}{\varphi_0(s)} \cdot \varphi\left(\frac{\varphi_1(\tilde{t}_{2i+1})}{\varphi_0(\tilde{t}_{2i+1})}\right) \leqq r\, \varphi(t_{2j+1}) \tag{3.11}$$

or

$$\frac{\varphi(\varphi_1(s)/\varphi_0(s))}{\varphi_1(s)/\varphi_0(s)} \leqq r\frac{\varphi_0(\tilde{t}_{2i+1})}{\varphi_1(s)} \cdot \varphi\left(\frac{\varphi_1(\tilde{t}_{2i+1})}{\varphi_0(\tilde{t}_{2i+1})}\right) \leqq r\frac{\varphi(t_{2(j-1)+1})}{t_{2(j-1)+1}}. \tag{3.12}$$

The estimates (3.11) and (3.12) imply that

$$t_{2(j-1)} \leqq \frac{\varphi_1(s)}{\varphi_0(s)} \leqq t_{2j+2}$$

which, in its turn, implies (3.10).
In an analogous way we can treat the case $s \in [\tilde{t}_{2i}, \tilde{t}_{2i+1})$.

COROLLARY 3.1. *Let*

$$V_k = \bigcup_{\tilde{t}_{2j+1} \in U_k} \tilde{\Omega}_j \tag{3.13}$$

If $V_k \cap U_l \neq \emptyset$, *then* $|k - l| \leqq 5$.

PROOF. If $s \in V_k \cap U_l$, then for some i we have $s \in \tilde{\Omega_i}$ and $\tilde{t}_{2i+1} \in U_k$. So $\tilde{\Omega_i} \cap U_k \neq \emptyset$ and $\tilde{\Omega_i} \cap U_l \neq \emptyset$. From Lemma 3.3 we obtain now that $|k-l| \leqq 5$. This completes the proof.

The next theorem gives the description of the functors F and G.

THEOREM 3.1. *If* $F = K_{\varphi,q}(K_{\varphi_0,\infty}, K_{\varphi_1,\infty})$ *and* $G = K_{\varphi,q}(K_{\varphi_0,1}, K_{\varphi_1,1})$, *then*

$$\|f\|_{F(\vec{L}_\infty)} \approx \left(\sum_i \left[\sup_{s \in V_i} \frac{|f(s)|}{\varphi(\varphi_0(s), \varphi_1(s))}\right]^q\right)^{1/q}$$

and

$$\|\{a_n\}\|_{G(\vec{l}_1)} \approx \left(\sum_i \left[\sum_{2^n \in V_i} \frac{|a_n|}{\varphi(\varphi_0(2^n), \varphi_1(2^n))}\right]^q\right)^{1/q}.$$

PROOF. From Lemma 3.1 and Corollary 3.1 we have

$$\|f\|_{F(\vec{L}_\infty)} \approx \left(\sum_i \left[\sup_{s \in U_i} \frac{|f(s)|}{\varphi(\varphi_0(s), \varphi_1(s))}\right]^q\right)^{1/q} \approx$$

$$\approx \left(\sum_{i,j} \left[\sup_{s \in U_i \cap V_j} \frac{|f(s)|}{\varphi(\varphi_0(s), \varphi_1(s))}\right]^q\right)^{1/q} \approx \left(\sum_j \left[\sup_{s \in V_j} \frac{|f(s)|}{\varphi(\varphi_0(s), \varphi_1(s))}\right]^q\right)^{1/q}.$$

In a quite analogous way we obtain the second statement.

4. Proof of the theorem 1.1.

A). *necessity*. If, for any Banach couple $\vec{X}$ and any $q_0, q_1, q \in [1, +\infty]$. we have

$$(\vec{X}_{\varphi_0,q_0}, \vec{X}_{\varphi_1,q_1})_{\varphi,q} = \vec{X}_{\varphi(\varphi_0,\varphi_1),q},$$

then it holds for couple $\vec{X} = \vec{l}_1$ and $q_0 = q_1 = 1, q = +\infty$. From Theorem 3.1 it follows that the norm of left hand side is equivalent to

$$\|\{a_n\}\|_{G(\vec{l}_1)} \approx \sup_i \sum_{2^n \in V_i} \frac{|a_n|}{\varphi(\varphi_0(2^n), \varphi_1(2^n))}. \tag{4.1}$$

Moreover, according to Lemma 2.1 we see that the norm of the right hand side is equivalent to

$$\|a_n\|_{K_{\varphi(\varphi_0,\varphi_1),q}(\vec{l}_1)} \approx \sup_i \sum_{2^n \in \tilde{\Omega_i}} \frac{|a_n|}{\varphi(\varphi_0(2^n), \varphi_1(2^n))}. \tag{4.2}$$

Since V_i is a union of some $\tilde{\Omega_j}$ and $\{a_k\}$ is arbitrary it is easy to see that (4.1) is equivalent to (4.2) if and only if there exists a natural number N such that V_i is a union of not more than N numbers of $\tilde{\Omega_j}$.

Now, from Corollary 3.1 it follows that for any i $U_i \cap \tilde{\Omega_j} \neq \emptyset$ not more than $11N$ numbers of j.

B). *sufficiency*. If there exists a natural N such that for any i, the numbers of j for which $\Omega_j^{\sim} \cap U_i \neq \emptyset$ is not more than N, then from Corollary 3.1 it follows that for any k the set V_k consists of not more than $11\,N$ numbers of $\Omega_j^{\sim}$. Thus, by Theorem 3.1 for

$$G = K_{\varphi,q}(K_{\varphi_0,1}, K_{\varphi_1,1})$$

we have

$$\|\{a_n\}\|_{G(\vec{l}_1)} \approx \left(\sum_i \left(\sum_{2^n \in V_i} \frac{|a_n|}{\varphi(\varphi_0(2^n), \varphi_1(2^n))}\right)^q\right)^{1/q}$$
$$\approx \left(\sum_i \left(\sum_{2^n \in \Omega_i^{\sim}} \frac{|a_n|}{\varphi(\varphi_0(2^n), \varphi_1(2^n))}\right)^q\right)^{1/q}.$$

Compare this with Lemma 2.1 and, in view of the minimal property of J functor, we have

$$J_{K_{\varphi(\varphi_0,\varphi_1),q}}(\vec{l}_1) \hookrightarrow K_{\varphi,q}(K_{\varphi_0,1}, K_{\varphi_1,1}) \hookrightarrow K_{\varphi,q}(K_{\varphi_0 q_0}, K_{\varphi_1 q_1}).$$

Since on the right hand side we have a K-functor, it follows that Δ^c is also contained in it. Thus

$$K_{\varphi(\varphi_0,\varphi_1),q} = J_{K_{\varphi(\varphi_0,\varphi_1),q}}(\vec{l}_1) + \Delta^c \hookrightarrow K_{\varphi,q}(K_{\varphi_0,q_0}, K_{\varphi_1,q_1}).$$

To prove the reversed imbedding, let us consider the functor

$$F = K_{\varphi,q}(K_{\varphi_0,\infty}, K_{\varphi_1,\infty})$$

on the couple $\vec{L}_\infty$.

By Theorem 3.1 we have

$$\|f\|_{F(\vec{L}_\infty)} \approx \left(\sum_i \left[\sup_{s \in V_i} \frac{|f(s)|}{\varphi(\varphi_0(s), \varphi_1(s))}\right]^q\right)^{1/q} \approx$$
$$\approx \left(\sum_i \left[\sup_{s \in \Omega_i^{\sim}} \frac{|f(s)|}{\varphi(\varphi_0(s), \varphi_1(s))}\right]^q\right)^{1/q}.$$

Since F is a K-functor, it follows from the maximal property of the K-functor that

$$\|f\|_{F(\vec{L}_\infty)} = \|K(f; \vec{L}_\infty)\|_{F(\vec{L}_\infty)} \approx \left(\sum_i \left[\sup_{s \in \Omega_i^{\sim}} \frac{K(s, f; \vec{L}_\infty)}{\varphi(\varphi_0(s), \varphi_1(s))}\right]^q\right)^{1/q} =$$
$$= \|f\|_{K_{\varphi(\varphi_0,\varphi_1),q}}(\vec{L}_\infty).$$

Thus

$$K_{\varphi,q}(K_{\varphi_0,q_0}, K_{\varphi_1,q_1}) \hookrightarrow K_{\varphi,q}(K_{\varphi_0,\infty}, K_{\varphi_1,\infty}) = K_{\varphi(\varphi_0,\varphi_1),q}$$

and the proof is complete.

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REFERENCES

1. S. V. Astashkin, *On stability of real interpolation method*, Dokl. Akad. Nauk SSSR (1983) 10–11 (Russian).
2. J. Bergh and J. Löfstrom, *Interpolation Spaces*, Springer Verlag, 1976.
3. Yu. A. Brudyi and N. Ja. Krugljak, *Interpolation Functors and Interpolation Spaces* I, North Holland, 1991.
4. C. E. Finol, L. Maligranda and L.-E. Persson, *Reiteration for and exact relations between some real interpolation spaces*, in: *Function Spaces* (Proc. Int. Conf. on Function Spaces, Poznan, Aug. 28–Sep. 2, 1989), Teubner Texte zur Math., 1991, 238–247.
5. J. Gustavsson, *A function parameter in connection with interpolation of Banach Spaces*, Math. Scand. 42 (1978), 289–305.
6. S. Janson, *Minimal and maximal methods of interpolation*, J. Funct. Anal. 44(1981), 50–73.
7. T. F. Kalugina, *Interpolation of Banach spaces with a functional parameter*, Vestnik Moskov. Univ., Ser. 1, Math., Mech. (1975), 68–77 (Russian).
8. N. Ja. Krugljak, L. Maligranda and L.-E. Persson, *A Carlson type inequality with blocks and interpolation*, Studia Math. 104 (1993), 161–180.
9. L.-E. Persson, *Interpolation with a parameter function*, Math. Scand. 59 (1986), 199–222.

DEPARTMENT OF MATHEMATICS
YAROSLAVL STATE UNIVERSITY
SOVETSKAYA 14
150000 YAROSLAVL
RUSSIA