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TRANSITION PROBABILITIES AND TRACE FUNCTIONS FOR C*-ALGEBRAS

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Abstract.

Transition probabilities are used to show that certain trace functions are lower semi-continuous on the space of closed ideals of a C*-algebra, equipped with the τ_w-topology. This generalizes the well-known theorem of Dismier that the trace functions are lower semi-continuous on the spectrum of a C*-algebra. The result is then applied to characterize C*-algebras of Type I_0 (or Fell C*-algebras) in terms of the existence of a dense ideal of elements for which the trace functions are continuous. The points of continuity of transition probabilities are characterized, and a necessary and sufficient condition is given for the reduced C*-algebra of a second countable r-discrete principal groupoid to be of type I_0.

1. Introduction.

In this paper we use transition probabilities for pure states of a C*-algebra to study trace functions defined on the spectrum and on certain ideal spaces. This approach enables us to work with an arbitrary C*-algebra, and not just with liminal ones. We also extend the results of [6] on the continuity of transition probabilities and illustrate these results in the case of groupoid C*-algebras.

Let A be a C*-algebra with spectrum $\hat{A}$ (the space of equivalence classes of irreducible representations of A). Suppose that (π_α) is a convergent net in $\hat{A}$ and let L be the set of limits. Dixmier [12; 3.5.9] showed that if $\pi \in L$ then

$$\liminf \operatorname{Tr} \pi_\alpha(a) \geqq \operatorname{Tr} \pi(a) \quad (a \in A^+), \tag{1}$$

where A^+ is the set of positive elements of A and Tr is the usual trace for positive operators on a Hilbert space (note that values $+\infty$ may occur in (1) and in (2) below). In the special case where A is liminal (that is, when every irreducible representation of A consists of compact operators), Milicic [20] extended this result as follows:

$$\liminf \operatorname{Tr} \pi_\alpha(a) \geqq \sum_{\pi \in L} \operatorname{Tr} \pi(a) \quad (a \in A^+). \tag{2}$$

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We refer the reader to [25], [20], [19] for related convergence properties in uniformly liminal C*-algebras and for applications to semisimple Lie groups.

One of the results of this paper (Theorem 2.4) is that (2) is valid for an arbitrary C*-algebra. In fact, we obtain Theorem 2.4 as a corollary of a quite general lower semi-continuity result for trace functions on the ideal space of a C*-algebra (Theorem 2.3). Whilst the main interest of these results may lie in the case where A contains a non-zero postliminal ideal, we note that even an antiliminal C*-algebra may have a dense set of finite dimensional representations in $\hat{A}$ (for example, the full C*-algebra of the free group on two generators [10] and the rotation algebra [17], [2]). In the rest of Section 2 we use the generalized version of (2) to give alternative proofs of some results of Fell [14] on finite dimensional irreducible representations.

Once the lower semi-continuity of the trace functions has been established, it is natural, by analogy with continuous trace C*-algebras, to look for C*-algebras containing a dense ideal for which the trace functions are continuous, in some appropriate sense. In Section 3, we study three possible extensions of the class of continuous trace C*-algebras. We show that they all coincide, and that they are precisely equal to the class of C*-algebras of Type I_0 studied in [24; 6.1, 6.2]. Since these are precisely the C*-algebras whose spectra satisfy Fell's condition at each point we will refer to them as Fell C*-algebras. It is possible to obtain still larger classes of C*-algebras with dense ideals of elements of 'continuous trace' by introducing 'multiplicity' integers, see [14], [25], [20]. We hope to pursue this subject further, and one of the aims of this paper is to clarify the 'multiplicity one' case. In the rest of Section 3 we study first the relationship between Fell C*-algebras and C*-algebras of generalized continuous trace (GCT C*-algebras), and then the set of separated points in the spectrum of a Fell C*-algebra, when the spectrum is compact. We give an example of a Fell C*-algebra which does not have GCT, providing a new and simpler example of a separable, liminal C*-algebra for which the set of non-separated points is dense in the spectrum.

Transition probabilities for pure states of a C*-algebra play an important rôle in our approach to Section 2. They are also related to Fell's condition in [6]. In Section 4, we continue the study of their continuity properties. We recall from [29] the definition of the transition probability $\langle\phi,\psi\rangle$ for a pair (ϕ,ψ) in $P(A)\times P(A)$ (where $P(A)$ is the set of pure states of A). If the Gelfand-Naimark-Segal representations π_ϕ and π_ψ are inequivalent then $\langle\phi,\psi\rangle = 0$. On the other hand, if $\pi_\phi \simeq \pi_\psi$ then there exists an irreducible representations π of A and unit vectors ξ and η in the Hilbert space H_π such that $\phi = \langle\pi(.)\xi,\xi\rangle$ and $\psi = \langle\pi(.)\eta,\eta\rangle$. In this case $\langle\phi,\psi\rangle = |\langle\xi,\eta\rangle|^2$. Let $T\colon P(A)\times P(A)\to[0,1]$ be defined by

$$T(\phi,\psi) = \langle\phi,\psi\rangle \quad (\phi,\psi\in P(A)).$$

As observed in [6; p. 8, Remark 2], T is upper semi-continuous for the product w^*-topology on $P(A) \times P(A)$ and hence is continuous at points (ϕ, ψ) for which $\langle \phi, \psi \rangle = 0$. It is this fact which is the key to our results in Section 2. In Theorem 4.1 we give a complete description of the set of points of continuity for T.

In Theorem 4.2 we consider the restriction T_0 of T to $R(A)$, the subset of $P(A) \times P(A)$ consisting of those pairs (ϕ, ψ) such that π_ϕ and π_ψ are equivalent. It was shown in [6; Theorem 2.3] that T_0 is continuous (for the product w^*-topology) if and only if every π in $\hat{A}$ satisfies Fell's condition. Here we localize this result by showing that if $(\phi, \psi) \in R(A)$ then T_0 is continuous at (ϕ, ψ) if and only if either $\langle \phi, \psi \rangle = 0$ or π_ϕ satisfies Fell's condition.

In Section 5 we investigate a class of C*-algebras which illustrate the results of Sections 3 and 4, namely the reduced C*-algebras of separated topological equivalence relations. These are a special class of groupoid C*-algebras for which the theory can be developed topologically, without recourse to integration [26]. A separated topological equivalence relation R is equipped with two topologies, one finer than the other. We show that the Fell points in the spectrum of the reduced C*-algebra of R correspond exactly to those points of R where the two topologies coincide. We conclude, as a consequence, that the C*-algebra is a Fell C*-algebra if and only if these two topologies are equal. This extends the results of [22], [23] in a special case. (In [22], [23] a characterization is given of continuous trace C*-algebras in the class of principal groupoid C*-algebras with twists.) The results of Section 5 were obtained jointly with Mark Priest, and we are grateful for his permission to include them in this paper.

In the rest of this section we establish some more notation and prove some preliminary lemmas.

Let A be a C*-algebra with Banach dual A^* equipped with the w^*-topology. Let $S(A)$ denote the state space of A and for $\phi \in S(A)$ let $\{\pi_\phi, H_\phi, \xi_\phi\}$ be the usual GNS triple associated with ϕ (see Section 5). If π is an irreducible representation of A we shall adopt the common practice of using the same symbol to denote the corresponding equivalence class in $\hat{A}$. Thus $\pi_1 \simeq \pi_2$ (as irreducible representations) means $\pi_1 = \pi_2$ (in $\hat{A}$). We shall denote by $\theta : P(A) \to \hat{A}$ the continuous, open mapping given by $\phi \to \pi_\phi$ (see [12; 3.4.11]). Recall that pure states ϕ and ψ are said to be *equivalent* if π_ϕ and π_ψ are equivalent.

By an ideal of A we shall always mean a two-sided ideal. The set $\mathrm{Id}(A)$ of all closed ideals of A can be equipped with strong and weak topologies, τ_s and τ_w. The precise definition and origins of these topologies are described in [3; Section 2]. The main features are that a net (I_α) is τ_s-convergent to I in $\mathrm{Id}(A)$ if and only if $\|a + I_\alpha\| \to \|a + I\|$ for all $a \in A$, whilst a base for τ_w is given by the family of sets of the form

$$U(F) = \{I \in \mathrm{Id}(A) : J \nsubseteq I \text{ for all } J \in F\}$$

where F is a finite set (possibly empty) of closed ideals of A. The method of proof of [3; 3.5b)] shows that the functions $I \to \|a + I\|$ $(a \in A)$ are lower semi-continuous on $(\mathrm{Id}(A), \tau_w)$. As a matter of fact it is easy to show that the τ_w-topology is the weakest topology with respect to which all these functions are lower semi-continuous. The space $(\mathrm{Id}(A), \tau_s)$ is compact and Hausdorff whilst $(\mathrm{Id}(A), \tau_w)$ is compact but not usually Hausdorff. The restriction of τ_w to $\mathrm{Prim}(A)$ (the set of primitive ideals of A) coincides with the Jacobson topology. This will be the default topology on $\mathrm{Prim}(A)$. The only topology which we shall use on $\hat{A}$ will be the Jacobson topology. If L is a non-empty subset of $\mathrm{Prim}(A)$ we will let $\ker L$ be equal to $\bigcap_{P \in L} P$.

A closed ideal I of A is said to be *primal* if whenever $n \geqq 2$ and $J_1, J_2, \ldots, J_n$ are closed ideals of A with zero product then $J_i \subseteq I$ for ar least one value of i. Primal ideals have been found to arise naturally in the theory of limits of factorial states [5]. We denote by Primal(A) (respectively Primal$'(A)$) (respectively MinPrimal(A)) the set of all primal (respectively proper primal) (respectively minimal primal) ideals of A. Let Sub(A) denote the τ_s-closure of MinPrimal(A) in Primal$'(A)$. Sub(A) is a natural base-space for a C*-bundle representation of A (see [3], [7] for investigations in the case when Sub(A) is equal to MinPrimal(A)).

We shall use $\tilde{A}$ to denote A itself (if A is unital) or $A + \mathbf{C}1$ (if A is non-unital and 1 is an adjoined identity).

For a Hilbert space H we denote by $L(H)$ (respectively LC(H)) the C*-algebra of all bounded (respectively compact) linear operators on H. If ξ is a unit vector in H, the associated vector state ω_ξ is defined by

$$\omega_\xi(T) = \langle T\xi, \xi \rangle \quad (T \in L(H)).$$

We now prove some lemmas on convergence in the τ_w and τ_s-topologies. The first extends [3; Proposition 3.2].

LEMMA 1.1. *Let A be a C*-algebra and let (I_α) be a net in* Id(A) *which is τ_w-convergent to a proper ideal in* Id(A). *Let L be the set of τ_w-limits of (I_α) in* Prim(A) *and set $I = \ker L$. For $J \in \mathrm{Id}(A)$ the following are equivalent:*

(i) $I_\alpha \to J\ (\tau_w)$,

(ii) $J \supseteq I$.

PROOF. (i) $\Rightarrow$ (ii) If $I_\alpha \to J(\tau_w)$ then $I_\alpha \to P(\tau_w)$ for each $P \in \mathrm{Prim}(A/J)$, so $\mathrm{Prim}(A/J) \subseteq L$. Hence $J \supseteq I$.

(ii) $\Rightarrow$ (i) It is sufficient to show that $I_\alpha \to I\ (\tau_w)$. To do this it is sufficient to show that whenever $K \in \mathrm{Id}(A)$ with $I \not\supseteq K$ then I_α is eventually in the set

$$X = \{R \in \mathrm{Id}(A) : R \not\supseteq K\}$$

(since sets of this type form a sub-base for the τ_w-topology). If $I \not\supseteq K$ then there is a $P \in L$ such that $P \not\supseteq K$, so X is a τ_w-open neighbourhood of P. Since $I_\alpha \to P(\tau_w)$, I_α is eventually in X, as required.

LEMMA 1.2. *Let A be a C^*-algebra and let (I_α) and (J_α) be nets in* $\mathrm{Id}(A)$ *with $I_\alpha \subseteq J_\alpha$ for each α. Let $I, J \in \mathrm{Id}(A)$ and suppose that $I_\alpha \to I(\tau_s)$ and $J_\alpha \to J(\tau_w)$. Then $I \subseteq J$.*

PROOF. If $a \in A$ then

$$\|a + J\| \leqq \liminf \|a + J_\alpha\| \leqq \lim \|a + I_\alpha\| = \|a + I\|.$$

Hence $I \subseteq J$.

COROLLARY 1.3. *Let A be a C^*-algebra and let (I_α) be a net in* $\mathrm{Id}(A)$ *which is τ_s-convergent to a proper ideal J in* $\mathrm{Id}(A)$. *Let L be the set of τ_w-limits of (I_α) in* $\mathrm{Prim}(A)$ *and set $I = \ker L$. Then $J = I$.*

Proof. Since $I_\alpha \to J(\tau_w)$, $J \supseteq I$ by Lemma 1.1. Since $I_\alpha \to I(\tau_w)$ (by Lemma 1.1), $I \supseteq J$ by Lemma 1.2.

Recall that a point in a topological space is a *cluster point* of a net if it is a limit of a subnet of the original net. The following result may be regarded as a generalization of [14; Theorem 2.1].

LEMMA 1.4. *Let A be a C^*-algebra and let (I_α) be a net in* $\mathrm{Id}(A)$ *which is τ_w-convergent to a proper ideal of A. The following conditions are equivalent:*

(i) *(I_α) is τ_s-convergent,*

(ii) *every τ_w-cluster point of (I_α) is a τ_w-limit,*

(iii) *every primitive τ_w-cluster point of (I_α) is a τ_w-limit.*

PROOF. Let L be the set of primitive τ_w-limits of (I_α) and set $I = \ker L$.

(i) $\Rightarrow$ (ii) Assuming (i), $I_\alpha \to I(\tau_s)$ by Corollary 1.3. Suppose that (I_β) is any subnet of (I_α) and that (I_β) is τ_w-convergent to an ideal $J \in \mathrm{Id}(A)$. Since $I_\beta \to I(\tau_s)$, it follows from Lemma 1.2 (applied to (I_β)) that $I \subseteq J$. By Lemma 1.1, $I_\alpha \to J(\tau_w)$.

(ii) $\Rightarrow$ (iii) This is immediate.

(iii) $\Rightarrow$ (i) Let (I_β) be any subnet of (I_α). The τ_s-compactness of $\mathrm{Id}(A)$ implies that (I_β) has a subnet (I_γ) which is τ_s-convergent to some $J \in \mathrm{Id}(A)$. The set of primitive τ_w-limits of (I_γ) is exactly L, by assumption, so by applying Corollary 1.3 to (I_γ) we obtain that $J = I$. It follows that $I_\alpha \to I(\tau_s)$.

2. The lower semi-continuity of the trace on Id(A).

In this section we show that the trace-evaluation functions are lower semi-continuous on $(\mathrm{Id}(A), \tau_w)$, Theorem 2.3, and derive some consequences, among them the generalization of Milicic's result (see Introduction) to arbitrary C*-algebras, Theorem 2.4.

We need the following well-known lemma, which may be proved by a routine induction on M (we omit the details). Alternatively it is possible to give an operator-theoretic proof by considering the matrix $(\langle \xi_i, \xi_j \rangle)$, as was pointed out to us by J. Spielberg. This latter argument is given in [9] with a reference to an unpublished paper of Haagerup.

LEMMA 2.1. *Given a positive integer M and $\delta_0 > 0$, there exists $\delta_1 > 0$ (depending only on M and δ_0) such that whenever $1 \leqq m \leqq M$ and $\xi_1, \xi_2, \ldots, \xi_m$ are unit vectors in a Hilbert space H satisfying*

$$|\langle \xi_i, \xi_j \rangle| < \delta_1 \quad (i \neq j)$$

then there exists an orthonormal system $\{\eta_1, \eta_2, \ldots, \eta_m\}$ in H such that

$$\|\eta_i - \xi_i\| < \delta_0 \quad (1 \leqq i \leqq m).$$

DEFINITION. If A is a C*-algebra, J is a closed ideal in A and $a \in A^+$, define $\mathrm{Tr}_J(a + J)$ to be $\mathrm{Tr}\, \pi_J(a + J)$ where π_J is the reduced atomic representation of A/J [24; 4.3.7]. Equivalently, $\mathrm{Tr}_J(a + J) = \sum_{\sigma \in (A/J)^\wedge} \mathrm{Tr}\, \sigma(a + J)$. Define $f_a : \mathrm{Id}(A) \to [0, \infty]$ by

$$f_a(J) = \mathrm{Tr}_J(a + J) \quad (J \neq A)$$

and let $f_a(A) = 0$.

LEMMA 2.2. *Let A be a C*-algebra, and let $a \in A^+$. If $P \in \mathrm{Prim}(A)$ and $\pi \in \hat{A}$ with $P = \ker \pi$ then*

$$f_a(P) = \mathrm{Tr}\, \pi(a).$$

PROOF. Since $f_a(P) \geqq \mathrm{Tr}\, \pi(a)$, we may suppose that $\mathrm{Tr}\, \pi(a) < \infty$. In this case $\pi(a)$ is compact. Let σ be an irreducible representation of A/P and let Φ be the canonical isomorphism of $\pi(A)$ onto A/P. Then either $\sigma \circ \Phi$ annihilates $\mathrm{LC}(H_\pi)$ or else $\sigma \circ \Phi$ is equivalent to the identity representation of $\pi(A)$ [24; 6.1.4]. Thus

$$f_a(P) = \sum_{\sigma \in (A/P)^\wedge} \mathrm{Tr}\, \sigma(a + P) = \sum_{\sigma \in (A/P)^\wedge} \mathrm{Tr}(\sigma \circ \Phi)(\pi(a)) = \mathrm{Tr}\, \pi(a).$$

It follows from Lemma 2.2 that if $I \in \mathrm{Id}(A)$ and $I \neq A$ then

$$f_a(I) = \sum_{P \in \mathrm{Prim}(A/I)} f_a(P).$$

We shall use this fact often in the sequel, without further mention.
We now prove the main theorem of this section.

THEOREM 2.3. *Let A be a C*-algebra. For each $a \in A^+$ the function f_a is lower semi-continuous on* $(\mathrm{Id}(A), \tau_w)$.

PROOF. If $a = 0$ then $f_a = 0$. We may suppose, therefore, that $a \neq 0$. Since $f_a \geqq 0$ and $f_a(A) = 0$, f_a is lower semi-continuous at A.

Suppose that I is a proper, closed ideal of A, that $\alpha \in \mathsf{R}$, and that $f_a(I) > \alpha$. We seek a τ_w-neighbourhood W of I in $\mathrm{Id}(A)$ such that

$$(1) \qquad f_a(J) > \alpha \quad (J \in W).$$

There is a finite set $\{\pi_1, \dots, \pi_n\}$ of inequivalent irreducible representations of A/I such that

$$\sum_{i=1}^{n} \mathrm{Tr}\, \pi_i(a) > \alpha \quad (1 \leqq i \leqq n).$$

For each $i = 1, \dots, n$ there exists an orthonormal set $\{\xi_k^{(i)} : 1 \leqq k \leqq m_i\}$ in H_{π_i} such that

$$(2) \qquad \sum_{i=1}^{n} \sum_{k=1}^{m_i} \langle \pi_i(a)\xi_k^{(i)}, \xi_k^{(i)} \rangle = \alpha + \varepsilon$$

for some $\varepsilon > 0$. For $1 \leqq i \leqq n$ and $1 \leqq k \leqq m_i$ let

$$\phi_k^{(i)} = \langle \pi_i(\,.\,)\xi_k^{(i)}, \xi_k^{(i)} \rangle \in P(A)$$

and let $\lambda_k^{(i)} = \phi_k^{(i)}(a)$. Let β and ε_1 be positive numbers such that

$$(3) \qquad \beta + \varepsilon_1 M = \frac{\varepsilon}{n}$$

where $M = m_1 + m_2 + \dots + m_n$. Let $\delta_0 = \beta(2M\|a\|)^{-1}$. From Lemma 2.1 we can obtain δ_1 corresponding to M and δ_0.

Whenever $(k,i) \neq (p,j)$ the transition probability $\langle \phi_k^{(i)}, \phi_p^{(j)} \rangle$ is zero and so $(\phi_k^{(i)}, \phi_p^{(j)})$ is a point of continuity for T, see Section 1. So there exists an open neighbourhood N of 0 in A^* such that whenever $(k,i) \neq (p,j)$ and

$$\phi \in (\phi_k^{(i)} + N) \cap P(A), \quad \psi \in (\phi_p^{(j)} + N) \cap P(A)$$

then $\langle \phi, \psi \rangle^{1/2} < \delta_1$.

Let $N_1 = N \cap \{\rho \in A^* : |\rho(a)| < \varepsilon_1\}$ and for $1 \leqq i \leqq n$ and $1 \leqq k \leqq m_i$ let

$$U_{k,i} = (\phi_k^{(i)} + N_1) \cap P(A).$$

Then $V_i = \cap_{k=1}^{m_i} \theta(U_{k,i})$ is an open neighbourhood of π_i in $\hat{A}$ $(1 \leqq i \leqq n)$. Since V_i is open, there exists a closed ideal J_i of A such that $V_i = \hat{J}_i (1 \leqq i \leqq n)$. Let

$$W = \{J \in \mathrm{Id}(A) : J \not\supseteq J_i (1 \leqq i \leqq n)\},$$

a τ_w-open neighbourhood of I in $\mathrm{Id}(A)$.

Let $J \in W$. Since $J \not\supseteq J_i$ there exists $\sigma_i \in V_i$ such that $\sigma_i(J) = \{0\}$ or, equivalently, $\sigma_i \in (A/J)^\wedge$ $(1 \leqq i \leqq n)$. Let $\pi \in \{\sigma_1, \sigma_2, \ldots, \sigma_n\}$. By re-ordering if necessary, we may suppose that $\pi = \sigma_1 \simeq \sigma_2 \simeq \ldots \simeq \sigma_r$ and $\pi \not\simeq \sigma_s$ for $s > r$. To establish (1) it suffices, by (2), to show that

$$\text{(4)} \qquad \mathrm{Tr}(\pi(a)) > \left(\sum_{i=1}^{r} \sum_{k=1}^{m_i} \lambda_k^{(i)}\right) - \frac{\varepsilon}{n}$$

(note that $\{\sigma_1, \sigma_2, \ldots, \sigma_n\}$ partitions into at most n blocks of equivalent representations and that each block contributes a summand like $\mathrm{Tr}(\pi(a))$ the the expression $f_a(J)$).

Since $\sigma_i \in V_i (1 \leqq i \leqq r)$ there exist in H_π unit vectors $\xi_{k,i} (1 \leqq k \leqq m_i)$ such that if $\phi_{k,i} = \langle \pi(.)\xi_{k,i}, \xi_{k,i} \rangle$ then $\phi_{k,i} \in U_{k,i}$. Hence if $(k,i) \neq (p,j)$ then $\langle \phi_{k,i}. \phi_{p,j} \rangle^{1/2} < \delta_1$ and so $|\langle \xi_{k,i}, \xi_{p,j} \rangle| < \delta_1$. Since $m_1 + \ldots + m_r \leqq M$, it follows from Lemma 2.1 that there exist orthonormal vectors $\eta_{k,i}$ in H_π $(1 \leqq i \leqq r, 1 \leqq k \leqq m_i)$ such that $\|\eta_{k,i} - \xi_{k,i}\| < \delta_0$. Hence

$$\text{(5)} \qquad \left|\sum_{i=1}^{r} \sum_{k=1}^{m_i} (\langle \pi(a)\eta_{k,i}, \eta_{k,i} \rangle - \phi_{k,i}(a))\right| \leqq 2\delta_0 \|a\| M = \beta.$$

Since $\phi_{k,i} - \phi_k^{(i)} \in N_1$,

$$\text{(6)} \qquad \left|\sum_{i=1}^{r} \sum_{k=1}^{m_i} (\phi_{k,i}(a) - \lambda_k^{(i)})\right| < \varepsilon_1 M.$$

It follows from (5), (6) and (3) that

$$\begin{aligned} \mathrm{Tr}\,\pi(a) &\geqq \sum_{i=1}^{r} \sum_{k=1}^{m_i} \langle \pi(a)\eta_{k,i}, \eta_{k,i} \rangle \\ &> \left(\sum_{i=1}^{r} \sum_{k=1}^{m_i} \lambda_k^{(i)}\right) - \beta - \varepsilon_1 M \\ &= \left(\sum_{i=1}^{r} \sum_{k=1}^{m_i} \lambda_k^{(i)}\right) - \frac{\varepsilon}{n} \end{aligned}$$

as required by (4).

It is easy to show that the τ_w-topology is actually the weakest topology with respect to which all the functions f_a are lower semi-continuous.

The next theorem generalizes [20; Lemma 8] from the liminal to the general case.

THEOREM 2.4. *Let A be a C^*-algebra and let (π_α) be a convergent net in $\hat{A}$. Let L be the set of limits of (π_α). Then, in the extended interval $[0, \infty]$,*

$$\liminf \operatorname{Tr} \pi_\alpha(a) \geqq \sum_{\pi \in L} \operatorname{Tr} \pi(a) \quad (a \in A^+).$$

PROOF. Let $a \in A^+$. Since L is closed there exists an ideal J such that $L = (A/J)^\wedge$. Hence

$$f_a(J) = \sum_{\pi \in L} \operatorname{Tr} \pi(a). \tag{1}$$

Since $\pi_\alpha \to \pi$ for all $\pi \in L$ we have $\ker \pi_\alpha \to \ker \pi\ (\tau_w)$ for all $\pi \in L$ and so $\ker \pi_\alpha \to J$ (τ_w), by Lemma 1.1. By Theorem 2.3 and equation (1):

$$\liminf f_a(\ker \pi_\alpha) \geqq \sum_{\pi \in L} \operatorname{Tr} \pi(a).$$

By Lemma 2.2

$$f_a(\ker \pi_\alpha) = \operatorname{Tr}(\pi_\alpha(a)).$$

The proof of Theorem 2.4 given above does not use Milicic's special case. Indeed, the whole thrust of our approach has been to show how transition probabilities allow one to work in complete generality. Nevertheless, it is possible to give an alternative proof of Theorem 2.4 by using Milicic's result. This requires some care since A may not have a non-zero liminal ideal (even if the equivalence classes of finite dimensional, irreducible representations are dense in $\hat{A}$). We outline the argument.

Fix $a \in A^+$ and suppose that the inequality in Theorem 2.4 fails. Then, writing $l = \liminf \operatorname{Tr} \pi_\alpha(a)$, we have $0 < l < \infty$ (note that if $l = 0$ then $\operatorname{Tr} \pi(a) = 0$ for all $\pi \in L$ by [12; 3.5.9]). There exists a subnet $(\pi_\beta)_{\beta \in \Gamma}$ of (π_α) such that

(i) $\lim \operatorname{Tr} \pi_\beta(a) = l$,

(ii) $0 < \operatorname{Tr} \pi_\beta(a) \leqq l + 1$ for all $\beta \in \Gamma$.

Let L_0 be the set of limits of (π_β) and let J be the closed ideal of A such that $(A/J)^\wedge$ is the closure of $\{\pi_\beta : \beta \in \Gamma\}$. Then $(A/J)^\wedge \supseteq L_0 \supseteq L$. For $\pi \in (A/J)^\wedge$ we have that $\operatorname{Tr} \pi(a) \leqq l + 1$ by (ii) and [12; 3.5.9]. Thus $a + J \in I$ where I is the liminal ideal of A/J defined by

$$I = \{x \in A/J : \pi(x) \in LC(H_\pi) \text{ for all } \pi \in (A/J)^\wedge\}.$$

Working in $(A/J)^\wedge$, we have that $\pi_\beta \in \hat{I}$ for all β, and that $\hat{I} \cap L_0$ is non-empty (for otherwise $\pi(a) = 0$ for all $\pi \in L$). Writing $b = a + J$ and applying Milicic's result to the net (π_β) in $\hat{I}$, we obtain

$$l = \lim \operatorname{Tr} \pi_\beta(b) \geqq \sum_{\pi \in \bar{I} \cap L_0} \operatorname{Tr} \pi(b)$$

$$= \sum_{\pi \in L_0} \operatorname{Tr} \pi(b) \geqq \sum_{\pi \in L} \operatorname{Tr} \pi(a) > l,$$

which is a contradiction.

The next result is essentially a combination of Lemma 2.4 and Corollary 1 from [14; Section 10]. The original proofs involve polynomial identities, universal subnets, and spectral theory.

COROLLARY 2.5. *Let A be a C^*-algebra and let n be a positive integer. Let (π_α) be a convergent net in $\hat{A}$ such that* $\dim \pi_\alpha \leqq n$ *for all α, and let L be the set of limits of (π_α). Then*

(i) *L is a finite set with at most n elements,*
(ii) *for each $\pi \in L$,* $\dim \pi \leqq n$,
(iii) $\sum_{\pi \in L} \dim \pi \leqq n$.

PROOF. It suffices to prove (iii). This follows from Corollary 2.4 by putting $a = 1$ (if A is non-unital we adjoin an identity and note that the canonical homeomorphism of $\hat{A}$ onto an open subset of $(\tilde{A})^\wedge$ preserves dimension).

The next result is a slight extension of Corollary 3 of [14; Section 10].

COROLLARY 2.6. *Let A be a C^*-algebra and let n be a positive integer. Let (π_α) be a convergent net in $\hat{A}$ and suppose that* $\dim \pi_\alpha = n$ *for all α. Let L be the (necessarily finite) sets of limits of (π_α).*

If A is unital the following are equivalent:
(i) $\sum_{\pi \in L} \dim \pi = n$,
(ii) $\operatorname{Tr} \pi_\alpha(a) \to \sum_{\pi \in L} \operatorname{Tr} \pi(a) \quad (a \in A)$.

If A is non-unital let σ denote the element of $(\tilde{A})^\wedge$ which annihilates A. Then the following conditions are equivalent:
(i)′ $\sum_{\pi \in L} \dim \pi = n$,
(ii)′ $\operatorname{Tr} \pi_\alpha(a) \to \sum_{\pi \in L} \operatorname{Tr} \pi(a)\ (a \in A)$ *and* $\pi_\alpha \not\to \sigma$ *in* $(\tilde{A})^\wedge$.

PROOF. Suppose that A is unital. Suppose also that (i) holds and that $x \in A$ with $0 \leqq x \leqq 1$. By applying Theorem 2.4 to x and $1 - x$ we obtain that

$$\limsup \operatorname{Tr} \pi_\alpha(x) \leqq \sum_{\pi \in L} \operatorname{Tr} \pi(x) \leqq \liminf \operatorname{Tr} \pi_\alpha(x).$$

Then (ii) follows by linearity. Conversely, if (ii) holds we can obtain (i) by putting $a = 1$.

Suppose that A is non-unital. Given that (i)′ holds it follows, by applying

Corollary 2.5 (iii) to (π_α) (regarded as a net in $(\tilde{A})^\wedge$), that $\pi_\alpha \not\to \sigma$. The rest of (ii)′ is obtained as before. Conversely, if (ii)′ holds there is an open neighbourhood U of σ in $(\tilde{A})^\wedge$ such that $\pi_\alpha \notin U$ frequently. There exists a closed ideal J of $\tilde{A}$ such that $U = \{\sigma' \in (\tilde{A})^\wedge : \sigma'(J) \neq \{0\}\}$. Since $J \nsubseteq A$ there exists $x \in A$ such that $1 - x \in J$. If $\pi \in L$ then, since $\pi_\alpha \to \pi$, $\pi \notin U$ and so $\pi(x) = \pi(1)$. Also, $\pi_\alpha(x) = \pi_\alpha(1)$ frequently. Hence, using (ii)′,

$$\sum_{\pi \in L} \dim \pi = \sum_{\pi \in L} \operatorname{Tr} \pi(x) = \lim \operatorname{Tr} \pi_\alpha(x) = n.$$

The next result is similar to [20; Lema 11]. It explains why we will restrict attention to τ_s-convergent nets in the next section.

COROLLARY 2.7. *Let A be a C*-algebra and let (I_α) be a net in* Id (A) *converging* (τ_w) *to $I \in$* Id (A). *Suppose that there exists a dense subset S of A^+ such that*

$$\lim f_a(I_\alpha) = f_a(I) < \infty \quad (a \in S).$$

Then $I_\alpha \to I$ (τ_s).

PROOF. Let (I_β) be any subnet of (I_α). By τ_s-compactness, (I_β) has a subnet (I_γ) which is τ_s-convergent to some $J \in \operatorname{Id}(A)$. Since $I_\gamma \to I$ (τ_w), $I \supseteq J$ by Lemma 1.2. For $a \in S$

$$f_a(I) = \lim f_a(I_\gamma) \geqq f_a(J) \quad \text{(by Theorem 2.3)}.$$

Hence $f_a(I) = f_a(J)$. If $P \in \operatorname{Prim}(A/J)$ there exists $a \in S$ with $f_a(P) > 0$. Hence $P \supseteq I$ and so $I = J$. It follows that $I_\alpha \to I(\tau_s)$.

REMARK. If it is further assumed that S above is the positive part of a self-adjoint subalgebra then the method of [20; Lemma 11] can be used to show that the assumption that $I_\alpha \to I(\tau_w)$ is redundant.

3. Fell C*-algebras.

In this section we study Fell C*-algebras (C*-algebras of Type I_0), viewing them as a natural generalization of the class of continuous trace C*-algebras. We show that they can be characterized in terms of an ideal of 'continuous trace'; we give an example of a Fell C*-algebra which is not a GCT-algebra, and we study the set of separated points in their spectra.

We begin with some definitions:

A positive element x in a C*-algebra A is said to be *abelian* if rank $\pi(x) \leqq 1$ for all $\pi \in \hat{A}$. If A is generated, as a C*-algebra, by its abelian elements then it is said to be a C*-algebra of *Type* I_0.

C*-algebras of Type I_0 were studied in [24; 6.1, 6.2] where it was shown that they are liminal, and that the class is closed under passage to quotients and to hereditary subalgebras. Continuous trace C*-algebras are of Type I_0; in fact a C*-algebra has continuous trace if and only if it is of Type I_0 and has Hausdorff spectrum, see below. Part of the interest of C*-algebras of Type I_0 is that each postliminal C*-algebra has a canonical composition series of Type I_0, whereas its continuous trace composition series are obtained using Zorn's Lemma [24; 6.2.12]. C*-algebras of Type I_0 also arise in the study of continuity of transition probabilities [6]. We shall look at this in more detail in the next section.

Let A be a C*-algebra. A point $\pi_0 \in \hat{A}$ *satisfies the Fell condition* if there exists $a \in A^+$ such that $\pi(a)$ is a 1-dimensional projection for all π in some neighbourhood of π_0 in $\hat{A}$. We will call such points *Fell points*. It is easy to see that if J is an ideal in A and $\pi \in \hat{J} \subseteq \hat{A}$ then π is a Fell point in $\hat{J}$ if and only if π is a Fell point in $\hat{A}$. Elementary manipulations show that a point π_0 is a Fell point if and only if there is an abelian element x such that $\pi_0(x) \neq 0$. This shows that a C*-algebra is of Type I_0 if and only if each point of its spectrum is a Fell point. Because of this it is convenient to refer to such algebras as Fell C*-algebras, and we shall use this name from now on. For any C*-algebra A, the set F of all Fell points in $\hat{A}$ is clearly open. If A is postliminal then it follows from [24; 6.2.11] that F is dense in $\hat{A}$. The corresponding essential closed ideal J of A is the largest Fell ideal of A. Since J is liminal Prim(J) is a T_1-space, from which it follows that if π is a Fell point in $\hat{A}$ then $\ker \pi$ is a minimal primitive ideal of A (see also the proof of Lemma 3.1).

Let A be a C*-algebra and let P be the set of $a \in A^+$ such that the function

$$\pi \to \operatorname{Tr} \pi(a)$$

is finite and continuous on $\hat{A}$. Then the linear span of P is a two-sided ideal of A, which is denoted $m(A)$ [12; 4.5.2], and A is said to have *continuous trace* if $m(A)$ is dense in A. A C*-algebra has continuous trace if and only if its spectrum is Hausdorff and each point of the spectrum is a Fell point [12; 4.5.3-4].

The main purpose of this section is to consider possible generalizations of the notion of continuous trace. There are three natural ways to do this, and we begin by describing these, and showing that they are all equivalent.

Following [19] we say that a net (π_α) in $\hat{A}$ is *properly convergent* if it is convergent and every cluster point of (π_α) is a limit. It follows from Lemma 1.4 that (π_α) is properly convergent in $\hat{A}$ if and only if $(\ker \pi_\alpha)$ is τ_s-convergent to a proper ideal in Id(A). (Corollary 2.7 shows that we will need to use the τ_s-topology if we want continuity of the trace functions on a dense ideal (c.f. [14], [20]. [25])).

The first possible way of generalizing continuous trace is suggested by Theorem 2.4. Define S to be the set of $a \in A^+$ such that

$$\operatorname{Tr}\pi_\alpha(a) \to \sum_{\pi\in L} \operatorname{Tr}\pi(a) < \infty$$

whenever (π_α) is a properly convergent net in $\hat{A}$ with limit set L. Note that S can also be defined as the set of $a \in A^+$ such that $f_a(P_\alpha) \to f_a(J) < \infty$ whenever (P_α) is a net in Prim (A) converging (τ_s) to a proper closed ideal J. If $\hat{A}$ is Hausdorff then clearly $S = P$.

When $\hat{A}$ is Hausdorff, Prim (A) is equal to Primal$'(A)$. This suggests a second possible way of generalizing continuous trace. Define U to be the set of $a \in A^+$ such that f_a is finite and τ_s-continuous on Primal$'(A)$.

At this stage it might seem that the demands of the second generalization are too ambitious. It is natural to view a C*-algebra as a C*-bundle over Sub(A) (see Introduction), so for the third generalization define V to be set of $a \in A^+$ such that f_a is finite and τ_s-continuous on Sub(A).

Our first aim is to show that these three generalizations of continuous trace are actually all the same. We begin with a lemma which extends [20; Theorem 3].

LEMMA 3.1. *Let A be a C*-algebra and let a be an element of A^+ such that $\operatorname{Tr}\pi(a)$ is finite and bounded for $\pi \in \hat{A}$. Suppose that $\pi \in \hat{A}$ and that $\pi(a) > 0$. Let J be a primal ideal contained in* $\ker\pi$. *Then $\{\pi\}$ is open in $(A/J)^\wedge$.*

PROOF. Note first that if $\sigma \in \hat{A}$ and $\sigma \neq \pi$ then there exists an open neighbourhood of π which does not contain σ. For, otherwise, $\pi \in \{\sigma\}^-$ and since π (regarded as an irreducible representation of $\sigma(A)$) does not annihilate the compact operator $\sigma(a)$ we have $\pi \simeq \sigma$ [8; 1.3.4].

Now by [12; 3.5.9] there exists an open neighbourhood U of π in $\hat{A}$ such that

$$\operatorname{Tr}\sigma(a) > \frac{1}{2}\operatorname{Tr}\pi(a) > 0 \quad (\sigma \in U). \tag{1}$$

Suppose that $\{\pi\}$ is not open in $(A/J)^\wedge$. By the first paragraph we can find infinitely many points in $(A/J)^\wedge \cap U$, so by (1), $f_a(J) = \infty$. But f_a is bounded on Prim(A), by assumption, and Prim(A) is dense in (Primal$(A), \tau_w$) [3; 3.1] so Theorem 2.3 implies that f_a is bounded on Primal(A). This gives a contradiction, so $\{\pi\}$ is open in $(A/J)^\wedge$.

PROPOSITION 3.2. *Let A be a C*-algebra and let P, S, U and V be the subsets of A^+ defined before Lemma* 3.1. *Then*

(i) *$P \subseteq S$, and*

(ii) *S, U, and V are equal.*

PROOF. (i) Suppose that $a \in P$ and that L is the set of limits of a (properly) convergent net (π_α) in $\hat{A}$. Let $\sigma \in L$. Using Theorem 2.4 we have

$$\sum_{\pi \in L} \operatorname{Tr} \pi(a) \leqq \liminf \operatorname{Tr} \pi_\alpha(a) = \operatorname{Tr} \sigma(a) < \infty.$$

Thus $\operatorname{Tr} \pi(a) = 0$ for all $\pi \in L \backslash \{\sigma\}$. It follows that either L is a singleton or $\operatorname{Tr} \pi(a) = 0$ for all $\pi \in L$ and that in both cases

$$\operatorname{Tr} \pi_\alpha(a) \to \sum_{\pi \in L} \operatorname{Tr} \pi(a) < \infty.$$

Hence $a \in S$.

(ii) It follows at once from the definition of U and the alternative definition of S that $U \subseteq S$.

We now show that $S \subseteq V$. Let $a \in S$. Since $\operatorname{Sub}(A) \subseteq \overline{\operatorname{Prim}(A)}^{\tau_s}$, by [3; 4.3 b)], f_a is finite on $\operatorname{Sub}(A)$. Let $J \in \operatorname{Sub}(A)$ and let $\varepsilon > 0$ be given. Since $a \in S$ there is a τ_s-neighbourhood M of J such $|f_a(P) - f_a(J)| < \varepsilon/2$ for all $P \in \operatorname{Prim}(A) \cap M$. Let $K \in \operatorname{Sub}(A) \cap M$ and let

$$N = \{I \in \operatorname{Id}(A) : f_a(I) > f_a(K) - \varepsilon/2\}.$$

N is τ_w-open, by Theorem 2.3, so $M \cap N$ is a τ_s-neighbourhood of K. Since $\operatorname{Sub}(A) \subseteq \overline{\operatorname{Prim}(A)}^{\tau_s}$, $\operatorname{Prim}(A) \cap M \cap N$ is non-empty. Let $P \in \operatorname{Prim}(A) \cap M \cap N$. Then

$$f_a(K) - \varepsilon/2 < f_a(P) < f_a(J) + \varepsilon/2.$$

Hence $f_a(K) < f_a(J) + \varepsilon$. It follows that f_a is upper semi-continuous for τ_s on $\operatorname{Sub}(A)$. Since f_a is also lower semi-continuous for τ_s (from Theorem 2.3), $a \in V$.

Finally we show that $V \subseteq U$. Let $a \in V$. Since f_a is finite on $\operatorname{MinPrimal}(A)$, f_a is finite on $\operatorname{Primal}'(A)$. Suppose that $J_\alpha \to J$ (τ_s) in $\operatorname{Primal}'(A)$. It suffices to show that some subnet of $(f_a(J_\alpha))$ is convergent to $f_a(J)$.

For each α let I_α be a minimal primal ideal contained in J_α. We may assume, by τ_s-compactness and by passing to a subnet if necessary, that (I_α) is τ_s-convergent in $\operatorname{Primal}(A)$ with limit I, say. Since $I_\alpha \subseteq J_\alpha$ for each α, we have $I \subseteq J$ and, in particular, $I \neq A$.

For each α let $F_\alpha = \{\pi \in (A/I_\alpha)^\wedge : \pi(a) \neq 0\}$. Since $f_a(I_\alpha)$ is finite, the proof of Lemma 3.1 (with J replaced by I_α) shows that $\{\pi\}$ is open in $(A/I_\alpha)^\wedge$ for each $\pi \in F_\alpha$.

Let $G_\alpha = \{\pi \in F_\alpha : \pi(J_\alpha) \neq \{0\}\} = F_\alpha \cap \hat{J}_\alpha$, and let $H_\alpha = F_\alpha \backslash G_\alpha = F_\alpha \cap (A/J_\alpha)^\wedge$. Let $K_\alpha = \cap\{\ker \pi : \pi \in G_\alpha\}$ (and if G_α is empty set $K_\alpha = A$).

Suppose that $\pi \in H_\alpha$. Since $\{\pi\}$ is open in $(A/I_\alpha)^\wedge$ and $\pi \notin G_\alpha$, we have $\pi \notin \overline{G_\alpha}$. Thus, if G_α is non-empty, $H_\alpha \cap (A/K_\alpha)^\wedge$ is empty. Hence

$$f_a(I_\alpha) = \sum_{\pi \in F_\alpha} \operatorname{Tr} \pi(a) = \sum_{\pi \in G_\alpha} \operatorname{Tr} \pi(a) + \sum_{\pi \in H_\alpha} \operatorname{Tr} \pi(a) = f_a(K_\alpha) + f_a(J_\alpha). \tag{1}$$

We may assume, by τ_s-compactness and by passing to a subnet if necessary, that (K_α) is τ_s-convergent in $\mathrm{Id}(A)$ with limit K, say. Since $I_\alpha \subseteq K_\alpha$ for all α, $I \subseteq K$.

Let $\pi \in (A/I)^\wedge$. Then there exists a subnet (I_β) of (I_α) and $\pi_\beta \in (A/I_\beta)^\wedge$ such that $\pi_\beta \to \pi$ (and hence $\ker \pi_\beta \to \ker \pi$ in $\mathrm{Prim}(A)$).

Suppose, further that $\pi(a) \neq 0$. Then eventually $\pi_\beta(a) \neq 0$, that is, $\pi_\beta \in F_\beta$. Hence there is a subnet (I_γ) of (I_β) such that either $\pi_\gamma \in G_\gamma$ for all γ or $\pi_\gamma \in H_\gamma$ for all γ. By Lemma 1.2, either $\ker \pi \supseteq K$ or $\ker \pi \supseteq J$. Hence

$$f_a(I) = \sum_{\pi \in (A/I)^\wedge} \mathrm{Tr}\, \pi(a) \leqq f_a(J) + f_a(K). \tag{2}$$

Note that since $a \in V$, $f_a(I_\alpha) \to f_a(I)$.

From (1), (2), and Theorem 2.3 we have

$$f_a(J) + f_a(K) \geqq f_a(I) = \lim f_a(I_\alpha) = \lim (f_a(K_\alpha) + f_a(J_\alpha)) = \limsup (f_a(K_\alpha) + f_a(J_\alpha))$$

$$\geqq \liminf f_a(K_\alpha) + \limsup f_a(J_\alpha) \geqq \liminf f_a(K_\alpha) + \liminf f_a(J_\alpha) \geqq f_a(K) + f_a(J).$$

Hence $\limsup f_a(J_\alpha) = \liminf f_a(J_\alpha) = f_a(J)$, as required.

From now on we prefer to work with U rather than S or V. Clearly $U + U \subseteq U$ and if $a \in A$ and $aa^* \in U$ then $a^*a \in U$. Also, if $a \in U$, $b \in A$ and $0 \leqq b \leqq a$ then by applying Theorem 2.3 to b and $a - b$ we see that $b \in U$ (c.f. [12; 4.4.2]). It follows from [12; 4.5.1] that the linear span $X(A)$ of U is a two-sided ideal of A and $X(A)^+ = U$. Our aim is to characterize those algebras for which $X(A)$ is dense in A. First we show that Fell C*-algebras belong to this class. This follows immediately from the next Proposition, see Theorem 3.6.

For $\pi, \sigma \in \hat{A}$ let $\pi \sim \sigma$ if π and σ cannot be separated by disjoint open sets in $\hat{A}$. Set $S(\pi) = \{\sigma \in \hat{A} : \sigma \sim \pi\}$.

PROPOSITION 3.3. *Let A be a C*-algebra and let $a \in A^+$ be an abelian element of A.*

(i) *Suppose that π_1 and π_2 are distinct elements of $\hat{A}$ such that $\pi_1 \sim \pi_2$. Then at least one of $\pi_1(a)$ and $\pi_2(a)$ is zero.*

(ii) $f_a(I) = \|a + I\|$ $(I \in \mathrm{Primal}(A))$.

(iii) *f_a is finite and τ_s-continuous on* $\mathrm{Primal}(A)$. *Hence $a \in U$.*

PROOF. (i) There is a net (π_α) in $\hat{A}$ such that $\pi_\alpha \to \pi_1$ and $\pi_\alpha \to \pi_2$. By passing to a subnet if necessary we may suppose that $(\ker \pi_\alpha)$ is τ_s-convergent to some ideal $J \in \mathrm{Id}(A)$. Then $J \subseteq \ker \pi_i$ $(i = 1, 2)$. There exists $\pi \in (A/J)^\wedge$ such that $\|a + J\| = \|\pi(a)\|$. Then

$$\begin{aligned}\|\pi(a)\| &= \|a + J\| = \lim \|\pi_\alpha(a)\| = \lim \operatorname{Tr} \pi_\alpha(a) \\ &\geqq \sum_{\sigma \in (A/J)^\wedge} \operatorname{Tr} \sigma(a) \quad \text{(by Theorem 2.4)} \\ &= \sum_{\sigma \in (A/J)^\wedge} \|\sigma(a)\|.\end{aligned}$$

Thus if $\sigma \in (A/J)^\wedge$ and $\sigma \neq \pi$ then $\sigma(a) = 0$.

(ii) Let $I \in \operatorname{Primal}(A)$. If $a \in I$ then

$$f_a(I) = 0 = \|a + I\|.$$

Suppose that $a \notin I$. By (i) there exists a unique $\pi \in (A/I)^\wedge$ such that $\pi(a) \neq 0$. Then

$$f_a(I) = \operatorname{Tr} \pi(a) = \|\pi(a)\| = \|a + I\|.$$

(iii) This follows from (ii).

Corollary 3.4. *Let A be a C^*-algebra and suppose that π is a Fell point of $\hat{A}$. Then π has a Hausdorff open neighbourhood in $\hat{A}$.*

Proof. There exists an abelian element $a \in A^+$ such that $\pi(a) \neq 0$. Then

$$\{\sigma \in \hat{A} : \sigma(a) \neq 0\}$$

is an open neighbourhood of π [12; 3.3.2] and is Hausdorff by 3.3 (i).

Our aim now is to show that if $X(A)$ is dense in A then A is a Fell C*-algebra.

Proposition 3.5. *Let A be a C^*-algebra and let $\pi \in \hat{A}$. Suppose that $\pi(X(A)) \neq \{0\}$. Then $\{\pi\}$ is open in $S(\pi)$.*

Proof. Suppose that $\{\pi\}$ is not open in $S(\pi)$. Then there exists a net (π_α) in $S(\pi)\backslash\{\pi\}$ such that $\pi_\alpha \to \pi$. For each α, $J_\alpha = \ker \pi \cap \ker \pi_\alpha$ is primal (because $\pi_\alpha \in S(\pi)$). By the τ_s-compactness of $\operatorname{Primal}(A)$, (J_α) has a subnet (J_β) which is τ_s-convergent to some $J \in \operatorname{Primal}(A)$, and $J \subseteq \ker \pi$, using Lemma 1.2.

By Lemma 3.1 there exists an open subset V of $\hat{A}$ such that

$$V \cap (A/J)^\wedge = \{\pi\}.$$

Let K be the closed, two-sided ideal of A such that $\hat{K} = V$. Since $\pi(K) \neq \{0\}$ and $\pi(A) \supseteq \operatorname{LC}(H_\pi)$ we have $\pi(K) \supseteq \operatorname{LC}(H_\pi)$. Let $a \in U$ such that $\pi(a) \neq 0$. Then there exists $b \in K^+$ such that $\pi(b) = \pi(a)$. Let $c = bab \in K^+ \cap X(A) = K \cap U$. Then $\pi(c) = \pi(a)^3 \neq 0$ (since $\pi(a) \geqq 0$) and so, since $c \in U$, $0 < \operatorname{Tr} \pi(c) < \infty$. We have

$$\begin{aligned}\operatorname{Tr} \pi(c) &= f_c(J) \quad (\text{since } c \in K) \\ &= \lim f_c(J_\beta) \quad (\text{since } c \in U) \\ &\geqq \limsup (\operatorname{Tr} \pi_\beta(c) + \operatorname{Tr} \pi(c)) \\ &\geqq 2 \operatorname{Tr} \pi(c) \quad \text{(by [12; 3.5.9])}.\end{aligned}$$

This contradiction shows that $\{\pi\}$ is open in $S(\pi)$.

THEOREM 3.6. *Let A be a C*-algebra and let $\pi \in \hat{A}$. Then the following conditions are equivalent:*
(i) *π is a Fell point,*
(ii) *$X(A) \nsubseteq \ker \pi$.*

PROOF. (i) $\Rightarrow$ (ii) If π is a Fell point then there exists an abelian element a such that $\pi(a) \neq 0$. Proposition 3.3 (iii) implies that $a \in X(A)$. Hence $X(A) \nsubseteq \ker \pi$.

(ii) $\Rightarrow$ (i) Suppose that (ii) holds. By [12; 4.4.2(ii)] it suffices to find $c \in A^+$ such that $\pi(c) \neq 0$ and such that the function $\sigma \to \operatorname{Tr} \sigma(c)\, (\sigma \in \hat{A})$ is finite and continuous at π.

By Proposition 3.5 there is a closed two-sided ideal K of A such that $S(\pi) \cap \hat{K} = \{\pi\}$. Choose $a \in U$ such that $\pi(a) \neq 0$, and choose $b \in K^+$ such that $\pi(a) = \pi(b)$. Then $c = bab \in K \cap U$ and $\pi(c) \neq 0$. Since $c \in U$, $\operatorname{Tr} \sigma(c) < \infty$ for all $\sigma \in \hat{A}$.

Suppose that $\pi_\alpha \to \pi$ in $\hat{A}$ and let (π_β) be any subnet of (π_α). By the τ_s-compactness of Primal(A) there exists a subnet $(\ker \pi_\gamma)$ of $(\ker \pi_\beta)$ such that $\ker \pi_\gamma \to J(\tau_s)$ for some primal ideal J of A. Lemma 1.2 shows that $J \subseteq \ker \pi$. Then

$$\begin{aligned} \lim \operatorname{Tr} \pi_\gamma(c) &= f_c(J) \quad (\text{since } c \in U) \\ &= \operatorname{Tr} \pi(c) \quad (\text{since } c \in K). \end{aligned}$$

Since the subnet (π_β) was arbitrary, $\operatorname{Tr} \pi_\alpha(c) \to \operatorname{Tr} \pi(c)$.

The next corollary follows immediately from Theorem 3.6

COROLLARY 3.7. *Let A be a C*-algebra. Then the following are equivalent:*
(i) *A is a Fell C*-algebra,*
(ii) *$X(A)$ is dense in A.*

Corollary 3.7 shows that Fell C*-algebras have a right to be thought of as C*-algebras of "generalized continuous trace'. But there is already a class of algebras claiming this name for themselves, so we now investigate the connections between these two classes.

A C*-algebra A is said to have *generalized continuous trace* (GCT or, sometimes, GTC) if the continuous trace ideal $m(A/I)$ is non-zero for every non-zero quotient A/I of A [12, 4.7.12]. It is easy to show that GCT algebras are liminal [12; 4.7.12]. If A is liminal and $\hat{A}$ is compact or if every irreducible representation of A is finite dimensional then A has GCT [13; §1], [11; Proposition 13], [1; Example 4.5]. It is easy, therefore, to give examples of GCT algebras which are not Fell algebras: for example the algebra of sequences of two-by-two complex matrices which tend to a scalar matrix at infinity.

We now give an example of a Fell C*-algebra which does not have GCT (which shows that the two classes don't have much connection). To do this we use the following characterization [12; 4.7.12]: a liminal C*-algebra A has GCT if and only if $\hat{A}$ is quasi-separated (a topological space T is *quasi-separated* if whenever F is a non-empty closed subset of T the interior of the set of separated points of F is non-empty (or, equivalently, dense)). It is, therefore. sufficient for our example to produce a Fell C*-algebra in which the set of non-separated points is dense in the spectrum.

EXAMPLE 3.8. Let H be a separable, infinite dimensional Hilbert space and let $(e_i)(1 \leqq i < \infty)$ be an orthonormal basis for H. For $i, j \in \{1, 2, \ldots\}$ let T_{ij} be the rank-one operator such that $T_{ij}\, e_i = e_j$. For each $n \in \{1, 2, \ldots\}$ let C_n be the C^*-subalgebra of $\mathrm{LC}(H)$ generated by $T_{ii}(1 \leqq i < \infty)$ and $T_{ij}(1 \leqq i, j \leqq n)$. Then $C_n \cong M_n(\mathbf{C}) \otimes c_0$. Let B be the C*-algebra of continuous functions from $[0, 1]$ into $\mathrm{LC}(H)$. If r is a rational number in $(0, 1)$ let $d(r)$ be the denominator when r is written as a fraction in its lowest terms and set $d(0) = d(1) = 1$. Let A be the C*-subalgebra of B consisting of those functions $f \in B$ such that $f(r) \in C_{d(r)}$ for each rational $r \in [0, 1]$.

If $s \in [0, 1]$ let I_s be the closed, two-sided ideal of A consisting of those $f \in A$ such that $f(s) = 0$. If s is irrational then for each $n \geqq 1$ there exists a neighbourhood of s in which each rational number r satisfies $d(r) \geqq n$. It follows that $\{f(s) : f \in A\}$ contains the norm-closure of $\bigcup_{n=1}^{\infty} C_n$ and hence is equal to $\mathrm{LC}(H)$. Thus I_s is primitive. Otherwise $A/I_s \cong C_{d(s)}$, so $\mathrm{Prim}(A/I_s)$ is a countably infinite, discrete space. If (s_n) is a sequence of irrational numbers in $(0, 1)$ converging to a rational number r then $I_{s_n} \to I_r(\tau_s)$, so I_{s_n} converges (τ_w) to each $P \in \mathrm{Prim}(A/I_r)$, by Lemma 1.1. Thus if $P \in \mathrm{Prim}(A/I_r)$ P is not a separated point of $\mathrm{Prim}(A)$. Since the ideals I_r, for r rational, have zero intersection it follows that the set of non-separated points is dense in $\mathrm{Prim}(A)$. Since A is obviously liminal, $\mathrm{Prim}(A)$ is homeomorphic to $\hat{A}$, so A does not have GCT.

For each $i \in \{1, 2, \ldots\}$ the constant function $f_i : [0, 1] \to T_{ii}$ is clearly an abelian element of A. If $\pi \in \hat{A}$ there exists i such that $\pi(f_i) \neq 0$. Hence the ideal generated by these elements is dense in A. Hence A is a Fell C*-algebra which does not have GCT.

An example has previously been given of a separable, liminal C*-algebra for which the set of non-separated points is dense in the spectrum [13], but it is more complicated to describe, and it is not easy to see whether it is a Fell C*-algebra.

We now continue to investigate separated points in the spectra of Fell C*-algebras. For $I, J \in \mathrm{Primal}'(A)$ let $I \sim J$ if I and J cannot be separated by disjoint τ_w-open sets in $\mathrm{Primal}'(A)$.

LEMMA 3.9. *Let A be a C^*-algebra and let $I, J \in \mathrm{Primal}'(A)$. Then $I \sim J$ if and only if $I \cap J$ is primal.*

PROOF. The ideal $I \cap J$ is primal if and only if there exists a net (P_α) in $\mathrm{Prim}(A)$ such that $P_\alpha \to I(\tau_w)$ and $P_\alpha \to J(\tau_w)$ [5; 3.2], [3; 3.2]. If such a net exists then clearly $I \sim J$. Conversely if $I \sim J$ then the denseness of $\mathrm{Prim}(A)$ in $(\mathrm{Primal}(A), \tau_w)$ [3; 3.1] implies that such a net exists.

The next lemma is an immediate consequence of the definition of $\sim$.

LEMMA 3.10. *Let A be a C^*-algebra Suppose that (P_α) and (Q_α) are nets in $\mathrm{Prim}(A)$ with $P_\alpha \to I\ (\tau_w)$ and $Q_\alpha \to J(\tau_w)$ for some $I, J \in \mathrm{Primal}'(A)$. If $P_\alpha \sim Q_\alpha$ for each α then $I \sim J$.*

LEMMA 3.11. *Let A be a C^*-algebra. Let (P_α) be a net in $\mathrm{Prim}(A)$ converging (τ_w) to a primal ideal I and suppose that (Q_α) is a net in $\mathrm{Prim}(A)$, with $Q_\alpha \sim P_\alpha$ for each α.*

(i) If I is minimal primal and (Q_α) converges (τ_w) to a primal ideal J then $J \supseteq I$.

(ii) If I is a closed, separated point of $\mathrm{Prim}(A)$, and $\mathrm{Prim}(A)$ is compact then $Q_\alpha \to I$.

PROOF. (i) This follows from Lemmas 3.9 and 3.10.

(ii) Let (Q_β) be any subnet of (Q_α). By the compactness of $\mathrm{Prim}(A)$, (Q_β) has a convergent subnet (Q_γ), converging to some $Q \in \mathrm{Prim}(A)$. Since I is minimal primal [3; 4.5] part (i) implies that $Q \supseteq I$. But I is a closed point of $\mathrm{Prim}(A)$, hence a maximal ideal. Hence $Q = I$, and $Q_\alpha \to I$.

THEOREM 3.12. *Let A be a C^*-algebra. Suppose that $\mathrm{Prim}(A)$ is compact and that each point of $\mathrm{Prim}(A)$ has a Hausdorff neighbourhood. Let X be the set of separated points of $\mathrm{Prim}(A)$. Then $\mathrm{Prim}(A)$ is a T_1-space and X is a dense, open subset of $\mathrm{Prim}(A)$.*

PROOF. Since each point has a Hausdorff neighbourhood, $\mathrm{Prim}(A)$ is a T_1-space. Suppose that P is a separated point in $\mathrm{Prim}(A)$. Let (P_α), (Q_α) be nets in $\mathrm{Prim}(A)$, with $P_\alpha \sim Q_\alpha$ for each α, and $P_\alpha \to P$. By Lemma 3.11 (ii) $Q_\alpha \to P$ also. But P has a Hausdorff neighbourhood, by assumption, so eventually $P_\alpha = Q_\alpha$. Hence P_α is eventually a separated point. It follows that X is open.

Let $P \in \mathrm{Prim}(A)$ and let I be a minimal primal ideal such that $I \subseteq P$. By [3; 3.1] there exists a net (P_α) in $\mathrm{Prim}(A)$ such that $P_\alpha \to I(\tau_w)$. Let (Q_α) be a net in $\mathrm{Prim}(A)$ with $Q_\alpha \sim P_\alpha$ for each α. By the τ_w-compactness of $\mathrm{Prim}(A)$ each subnet (Q_β) of (Q_α) has a subnet (Q_γ) converging (τ_w) to some Q in $\mathrm{Prim}(A)$, and $Q \supseteq I$, by Lemma 3.11 (i). Since Q has a Hausdorff neighbourhood and (P_γ) also converges (τ_w) to Q it follows that eventually $P_\gamma = Q_\gamma$. Hence P_α is eventually a separated point. Since $P_\alpha \to P$ in $\mathrm{Prim}(A)$, X is dense in $\mathrm{Prim}(A)$.

REMARKS: (i) If A is a Fell C*-algebra with compact spectrum then Corollary 3.4 shows that the theorem above applies to A. In this case, however the denseness of X in Prim(A) already follows from the fact that A is a GCT C*-algebra.

(ii) The first paragraph of the proof above actually shows that if A is a C*-algebra with Prim(A) compact and T_1 then the interior of the set of separated points of Prim(A) is precisely the set of separated points which have a Hausdorff neighbourhood.

4. Points of continuity for transition probabilities.

In this section we extend the results of [6] on continuity questions for the transition probability mapping T (see Section 1) and its restriction T_0 to $R(A)$ (the subset of $P(A) \times P(A)$ consisting of those pairs (ϕ, ψ) such that π_ϕ is equivalent to π_ψ). We equip $P(A) \times P(A)$ with the product w^*-topology. The restriction of this topology to $R(A)$ is called the product topology and is denoted τ_p. We shall also be concerned with the quotient topology τ_q on $R(A)$. This is obtained as follows (see [6]). We define $G(A)$ to be the set of extreme points of the closed unit ball of the Banach dual A^*. There is a mapping q of $(G(A), w^*)$ onto $R(A)$ given by

$$q(\phi) = (|\phi|, |\phi^*|) \quad (\phi \in G(A)).$$

The topology τ_q on $R(A)$ is defined to be the quotient topology induced by q from $(G(A), w^*)$.

In the locally convex, separated space $A^* \times A^*$ the set $P(A) \times P(A)$ is equal to $\partial_e(S(A) \times S(A))$, so it follows from [12; B14] that $P(A) \times P(A)$ is a Baire space. Since T is upper semi-continuous on $P(A) \times P(A)$, the set of points of continuity is a dense G_δ-set [12; B18]. In Theorem 4.1 we describe this set and give an alternative proof of its density. This result extends the observation of [6; p. 8, Remark 2] which was described in Section 1.

Recall that a C*-algebra is said to be *elementary* if it is isomorphic to the algebra of compact operators LC(H) on some Hilbert space H.

THEOREM 4.1. *Let A be a C*-algebra and let $T: P(A) \times P(A) \to [0,1]$ be defined by*

$$T(\phi, \psi) = \langle \phi, \psi \rangle \quad (\phi, \psi \in P(A)).$$

Let

$$E_1 = \{(\phi, \psi) \in P(A) \times P(A) : T(\phi, \psi) = 0\}$$

and let E_2 be the subset of $P(A) \times P(A)$ consisting of those pairs (ϕ, ψ) such that there exists a closed ideal J of A such that J is an elementary C-algebra and*

$\hat{J} = \{\pi_\phi\}$. *Then* $E = E_1 \cup E_2$ *is dense in* $P(A) \times P(A)$ *and is precisely the set of points at which* T *is continuous.*

PROOF. Let $(\phi, \psi) \in E$. If $(\phi, \psi) \in E_1$ then T is continuous at (ϕ, ψ), [6; p. 8]. Suppose that $(\phi, \psi) \in E_2 \setminus E_1$. Then there exists an elementary closed ideal J of A such that $\hat{J} = \{\pi_\phi\} = \{\pi_\psi\}$. Suppose that $\phi_\alpha \to \phi$ and $\psi_\alpha \to \psi$ in $P(A)$ in the w^*-topology. Then, eventually, $\phi_\alpha | J, \psi_\alpha | J \in P(J)$ and so

$$\langle \phi_\alpha, \psi_\alpha \rangle = \langle \phi_\alpha | J, \psi_\alpha | J \rangle \to \langle \phi | J, \psi | J \rangle = \langle \phi, \psi \rangle$$

by [6; p. 8, Remark 1]. Thus T is continuous at (ϕ, ψ).

Now suppose that $(\phi, \psi) \in (P(A) \times P(A)) \setminus E$. Note that $\langle \phi, \psi \rangle \neq 0$ and that ϕ and ψ are equivalent pure states. We shall complete the proof (and confirm the density of E) by showing that (ϕ, ψ) is the limit of a net in E_1. There are two cases to consider.

Case 1. Suppose that $\{\pi_\phi\}$ is not open in $\hat{A}$. Let Γ be a base of open neighbourhoods of 0 in A^*. Let $N \in \Gamma$. Then the canonical image of $(\phi + N) \cap P(A)$ in $\hat{A}$ is an open neighbourhood of π_ϕ in $\hat{A}$ and so there exists $\phi_N \in (\phi + N) \cap P(A)$ such that ϕ_N is not equivalent to ϕ or ψ and hence such that $\langle \phi_N, \psi \rangle = 0$. Directing Γ by reverse inclusion in the usual way, $(\phi_N, \psi) \to (\phi, \psi)$ in $P(A) \times P(A)$.

Case 2. Suppose that $\{\pi_\phi\}$ is open in $\hat{A}$. Then there is a closed ideal J of A such that $\{\pi_\phi\} = \hat{J}$. Since $(\phi, \psi) \notin E$ we must (in the absence of a positive resolution of Naimark's conjecture) consider the possibility that J is a simple, antiliminal, inseparable C*-algebra. There exists a unit vector $\eta \in H_\phi$ such that $\psi = \omega_\eta \circ \pi_\phi$. Since $\pi_\phi(A)$ is a prime C*-algebra and $\pi_\phi(J) \cap \mathrm{LC}(H_\phi) = \{0\}$, it follows that $\pi_\phi(A) \cap \mathrm{LC}(H_\phi) = \{0\}$. Let ρ be the unique state of $B = \pi_\phi(A) + \mathrm{LC}(H_\phi)$ which is zero on $\mathrm{LC}(H_\phi)$ and agrees with ω_η on $\pi_\phi(A)$. By Glimm's vector state theorem [15; Theorem 2], [16; Lemma 9] there is a net (η_α) of unit vectors of H_ϕ such that $\omega_{\eta_\alpha} | B \to \rho$. Then $\langle \eta_\alpha, \xi_\phi \rangle \to 0$ and so eventually $\eta_\alpha - \langle \eta_\alpha, \xi_\phi \rangle \xi_\phi$ can be normalised to a unit vector, ζ_α say. It follows that

$$\langle \phi, \omega_{\zeta_\alpha} \circ \pi_\phi \rangle = |\langle \xi_\phi, \zeta_\alpha \rangle|^2 = 0$$

and $(\phi, \omega_{\zeta_\alpha} \circ \pi_\phi) \to (\phi, \psi)$ in $P(A) \times P(A)$.

Next we give a localized version of the global continuity result for the mapping T_0 on $R(A)$ [6; Theorem 2.3]. The methods are partly similar to those in [6]. However, some extra arguments are required and we also replace the method of [6; p. 7] by using, and then developing further, a technique of Glimm [16].

THEOREM 4.2. *Let* A *be a C*-algebra and let* $T_0 : R(A) \to [0, 1]$ *be defined by*

$$T_0(\phi, \psi) = \langle \phi, \psi \rangle \quad ((\phi, \psi) \in R(A)).$$

Let $(\phi, \psi) \in R(A)$. *The following conditions are equivalent:*

(i) *T_0 is continuous at (ϕ, ψ) for the product topology on $R(A)$,*
(ii) *either $\langle \phi, \psi \rangle = 0$ or π_ϕ is a Fell point in $\hat{A}$.*

PROOF. (ii) $\Rightarrow$ (i) If π_ϕ is a Fell point in $\hat{A}$ then the argument in the proof of [6; Theorem 2.3] shows that (ϕ, ψ) is a point of continuity for T_0. If $\langle \phi, \psi \rangle = 0$ then (ϕ, ψ) is a point of continuity for the map T of Theorem 4.1, [6; p. 8]. Hence, by restriction, (ϕ, ψ) is a point of continuity for T_0.

(i) $\Rightarrow$ (ii) Suppose that (ϕ, ψ) is a point of continuity for T_0 and that $\langle \phi, \psi \rangle = \delta \neq 0$. We shall show that π_ϕ is a Fell point in $\hat{A}$.

Write $\pi = \pi_\phi$. We begin by showing that $\{\pi\}$ is open in $S(\pi)$. Suppose, on the contrary, that $\{\pi\}$ is not open in $S(\pi)$. As before let Γ be a base of open neighbourhoods of 0 in A^*. Let $N \in \Gamma$ and let $N_1 = 1/2\, N$. The canonical image of $(\phi + N_1) \cap P(A)$ in $\hat{A}$ is an open neighbourhood of π and hence contains an element σ of $S(\pi) \backslash \{\pi\}$. Thus there exists a pure state ϕ' associated with σ such that $\phi' \in \phi + N_1$. The argument in the proof of [4; Theorem 1 ((ii) $\Rightarrow$ (iii))] shows that there exists $\rho \in \hat{A}$ and unit vectors ξ and η in H_ρ such that $\phi_N = \omega_\xi \circ \rho \in \phi' + N_1$, $\psi_N = \omega_\eta \circ \rho \in \psi + N_1$ and $|\langle \xi, \eta \rangle|^2 \leqq \delta/2$. Hence $\phi_N \in \phi + N$ and $\psi_N \in \psi + N$. Letting N run through Γ we have $(\phi_N, \psi_N) \to (\phi, \psi)$ in the product topology on $R(A)$ but $\limsup \langle \phi_N, \psi_N \rangle \leqq 1/2 \langle \phi, \psi \rangle$. This contradicts the continuity of T_0 at (ϕ, ψ) and so $\{\pi\}$ is open in $S(\pi)$.

We show next that $\pi(A) \supseteq \mathrm{LC}(H_\pi)$. Suppose, on the contrary, that $\pi(A) \cap \mathrm{LC}(H_\pi) = \{0\}$. Let $B = \pi(A) + \mathrm{LC}(H_\pi)$. Let ϕ' and ψ' be the (unique) states of B which annihilate $\mathrm{LC}(H_\pi)$ and satisfy $\phi' \circ \pi = \phi$, and $\psi' \circ \pi = \psi$. Let Δ be a base of w^*-open neighbourhoods of 0 in B^* and let $M \in \Delta$. By Glimm's vector state space theorem [15], [16] there exists a unit vector ξ_M in H_π such that $\omega_{\xi_M} | B \in \phi' + M$. By the same theorem there is a net (η_α) of unit vectors in H_π such that $\omega_{\eta_\alpha} | B \to \psi'$. Let E_M be the projection onto the linear span of ξ_M. Since $\psi'(E_M) = 0$, $\langle E_M \eta_\alpha, \eta_\alpha \rangle \to 0$. Thus there exists a unit vector η_M such that $\omega_{\eta_M} | B \in \psi' + M$ and $|\langle \xi_M, \eta_M \rangle|^2 \leqq \delta/2$. As M runs through Δ, $\omega_{\xi_M} \circ \pi \to \phi$ and $\omega_{\eta_M} \circ \pi \to \psi$. Since $|\langle \xi_M, \eta_M \rangle^2 \nrightarrow \delta$, this contradicts the assumed continuity of T_0 at (ϕ, ψ). Hence $\pi(A) \supseteq \mathrm{LC}(H_\pi)$.

Since $\{\pi\}$ is open in $S(\pi)$ there exists an open neighbourhood U of π in $\hat{A}$ such that $U \cap S(\pi) = \{\pi\}$. Let J be the closed two-sided ideal of A for which $U = \hat{J}$. Since $\pi(A) \supseteq \mathrm{LC}(H_\pi)$ and $\pi(J) \neq \{0\}$, we have $\pi(J) \supseteq \mathrm{LC}(H_\pi)$. Thus there exists $a \in J^+$ such that $\pi(a)$ is a rank one projection which supports the pure state ϕ in the representation π. Since π is a separated point of $\hat{J}$, $\|\sigma(a^2 - a)\| \to 0$ as $\sigma \to \pi$ in $\hat{J}$. By a standard functional calculus argument, we may assume without loss of generality that $\sigma(a)$ is a projection for all σ in some neighbourhood V of π with $V \subseteq \hat{J}$. By shrinking V, if necessary, we may also assume that $\sigma(a) \neq 0$ for all $\sigma \in V$ [12; 3.3.2].

Suppose that π is not a Fell point. Then there exists a net (σ_α) in V such that $\sigma_\alpha \to \pi$ and, for each α, rank $(\sigma_\alpha(a)) \geqq 2$. For each α there is an orthonormal set

$\{\xi_\alpha, \zeta_\alpha\}$ in the Hilbert space for σ_α such that $\sigma_\alpha(a)\xi_\alpha = \xi_\alpha$ and $\sigma_\alpha(a)\zeta_\alpha = \zeta_\alpha$. Let $x \in A$. Then $\pi(a)\pi(x)\pi(a) = \lambda\pi(a)$ for some $\lambda \in \mathbf{C}$. Since $axa - \lambda a \in J$, $\|\sigma_\alpha(axa - \lambda a)\| \to 0$. Hence, arguing as in [16; p. 605]

$$\lim \langle \sigma_\alpha(x)\xi_\alpha, \xi_\alpha \rangle = \lim \langle \sigma_\alpha(axa)\xi_\alpha, \xi_\alpha \rangle$$
$$= \lim \langle \sigma_\alpha(\lambda a)\xi_\alpha, \xi_\alpha \rangle = \lambda = \phi(x).$$

Thus $\omega_{\xi_\alpha} \circ \sigma_\alpha \to \phi$ and similarly $\omega_{\zeta_\alpha} \circ \sigma_\alpha \to \phi$.

Recall that $\tilde{A}$ is A itself (if A is unital) or $A + \mathbf{C}1$ (if A is non-unital). There exists a unitary element $u \in \tilde{A}$ such that $\psi = \phi(u^*.u)$. For each α, let $\eta_\alpha = \tilde{\sigma}_\alpha(u)\zeta_\alpha$, where $\tilde{\sigma}_\alpha$ is the canonical extension of σ_α to $\tilde{A}$. Then $\omega_{\eta_\alpha} \circ \sigma_\alpha \to \psi$. Since T_0 is continuous at (ϕ, ψ), $|\langle \xi_\alpha, \eta_\alpha \rangle|^2 \to \langle \phi, \psi \rangle$. Thus $|\langle \xi_\alpha, \tilde{\sigma}_\alpha(u)\zeta_\alpha \rangle|^2 \to \delta \neq 0$.

On the other hand, $\pi(a)\tilde{\pi}(u)\pi(a) = \mu\pi(a)$ for some $\mu \in \mathbf{C}$. Since $aua - \mu a \in J$, $\|\sigma_\alpha(aua - \mu a)\| \to 0$. Hence

$$\lim \langle \xi_\alpha, \tilde{\sigma}_\alpha(u)\zeta_\alpha \rangle = \lim \langle \xi_\alpha, \sigma_\alpha(aua)\zeta_\alpha \rangle$$
$$= \lim \langle \xi_\alpha, \sigma_\alpha(\mu a)\zeta_\alpha \rangle = \mu \lim \langle \xi_\alpha, \zeta_\alpha \rangle = 0.$$

This contradiction shows that π is a Fell point.

PROPOSITION 4.3. *Let A be a C^*-algebra and let $u \in \tilde{A}$ be a unitary element. Define $\Phi_u : R(A) \to R(A)$ by*

$$\Phi_u(\phi, \psi) = (\phi, \psi(u\,.\,u^*)) \quad (\phi, \psi) \in R(A).$$

Then Φ_u is a bijection and
(i) *Φ_u is a homeomorphism for τ_p,*
(ii) *Φ_u is a homeomorphism for τ_q.*

PROOF. Φ_{u^*} is a two-sided inverse for Φ_u, so Φ_u is a bijection. For (i) and (ii) it suffices to prove either that Φ_u is open or that Φ_u is continuous, since this will then also hold for the inverse map Φ_{u^*}.

(i) Suppose that $(\phi_\alpha, \psi_\alpha) \to (\phi, \psi)(\tau_p)$ in $R(A)$. Then $\phi_\alpha \to \phi$, $\psi_\alpha \to \psi$, $\psi_\alpha(u\,.\,u^*) \to \psi(u\,.\,u^*)$, all in the w^*-topology, and so $(\phi_\alpha, \psi_\alpha(u\,.\,u^*)) \to (\phi, \psi(u\,.\,u^*))$. This shows that Φ_u is τ_p-continuous.

(ii) Consider the map $q: (G(A), w^*) \to (R(A), \tau_q)$ given by $q(\phi) = (|\phi|, |\phi^*|)$ and recall that if $\phi = \langle \pi(.)\xi, \eta \rangle$ (where π is an irreducible representation and ξ and η are unit vectors in H_π) then $|\phi| = \omega_\xi \circ \pi$ and $|\phi^*| = \omega_\eta \circ \pi$.

Let U be a non-empty open subset of $(R(A), \tau_q)$. Then $q^{-1}(U)$ is an open, q-saturated subset of $G(A)$. Let $V = \{\psi \in G(A) : \psi = \phi(u.) \text{ for some } \phi \in q^{-1}(U)\}$. Then V is an open, q-saturated subset of $G(A)$ (note that if $\phi, \psi \in G(A)$ then $q(\phi) = q(\psi)$ if and only if $\phi = \lambda\,\psi$ for some $\lambda \in \mathbf{C}$ with $|\lambda| = 1$). Using the vector functional representation of the previous paragraph, one may check that $q(V) = \Phi_u(U)$. This shows that Φ_u is τ_q-open.

Another way of viewing the proof of (ii) is to observe that Φ_u lifts to a w^*-homeomorphism of $G(A)$ given by $\phi \to \phi(u.)(\phi \in G(A))$.

COROLLARY 4.4. *Let A be a C^*-algebra and let*

$$i:(R(A), \tau_p) \to (R(A), \tau_q)$$

be the identity map. Let $(\phi, \psi) \in R(A)$. Then i is continuous at (ϕ, ψ) if and only if π_ϕ is a Fell point in $\hat{A}$.

PROOF. Suppose that i is continuous at (ϕ, ψ). There exists a unitary $u \in \tilde{A}$ such that $\psi = \phi(u^* . u)$. Since Φ_{u*} is τ_p-continuous on $R(A)$ and Φ_u is τ_q-continuous on $R(A)$, Proposition 4.3, it follows that i is continuous at (ϕ, ϕ). Since the transition probability map is τ_q-continuous on $R(A)$ [6; Proposition 3.2], (ϕ, ϕ) is a point of τ_p-continuity for T_0. Hence π_ϕ is a Fell point, by Theorem 4.2.

Conversely, suppose that π_ϕ is a Fell point. Then, by Theorem 4.2, (ϕ, ϕ) is a point of τ_p-continuity for T_0. By [6; 3.3], (ϕ, ϕ) is a point of continuity for i. Since Φ_u is τ_p-continuous on $R(A)$ and Φ_{u*} is τ_q-continous on $R(A)$, i is continuous at (ϕ, ψ).

5. Fell points in C*-algebras of separated topological equivalence relations.

The work in this section was done jointly with Mark Priest, and we would like to thank him for allowing it to appear here.

The main result of this section is that the Fell points in the spectrum of the reduced C*-algebra of a separated topological equivalence relation can be characterized by the points of continuity of a certain map (Theorem 5.7), exactly as in Corollary 4.4.

Let R be an equivalence relation on a set X, with diagonal R^0. Let r, s be the projection maps from R to R^0 defined by $r((x, y)) = (x, y)$ and $s((x, y)) = (y, y)((x, y) \in R)$. Then R is said to be a *separated topological equivalence relation* if there is a second countable locally compact, Hausdorff topology τ_0 on R such that r (or, equivalently, s) is a local homeomorphism. The separated topological equivalence relations are precisely the second countable r-discrete principal groupoids which admit Haar systems [27; I.2.7, I.2.8]. For $(x, y) \in R$ an R^0-*strip neighbourhood* of (x, y) is an open neighbourhood of (x, y) on which both the projection maps, r and s, are homeomorphisms onto open subsets of R^0. When R is a separated topological equivalence relation the R^0-strip neighbourhoods form a base for the topology. For $x \in X$ the equivalence class of x in X will be denoted $[x]$. The set $\{(y, y): y \in [x]\}$ is called the *orbit* of (x, x) in R^0. It will be convenient to refer to the set $\{(y, z): y, z \in [x]\}$ as an *equivalence class in R*. A subset of R^0 is said to be *invariant* if it contains the orbit of each of its points.

Now let R be a separated topological equivalence relation and let $C_c(R)$ be the

$*$-algebra of continuous complex functions on R of compact support, with involution given by

$$f^*(u,v) = \overline{f(v,u)},$$

with pointwise addition, and with multiplication given by

$$f * g(u,v) = \sum_{w \sim u} f(u,w)g(w,v) \quad f,g \in C_c(R).$$

The topology ensures that the sum has only finitely many non-zero terms. Let $\|.\|_{\text{red}}$ be the C*-norm on $C_c(R)$ in [27; II.2.8], [21; 6.3]. The completion of $C_c(R)$ with respect to this norm is denoted $C^*_{\text{red}}(R)$. We shall give an alternative description of $\|.\|_{\text{red}}$ after Proposition 5.2. The norm $\|.\|_{\text{red}}$ dominates the uniform norm on R so we can regard the elements of $C^*_{\text{red}}(R)$ as continuous functions on R vanishing at infinity [27; II.4.2].

When R is a separated topological equivalence relation R^0 is a clopen subset of R [27; I.2.8], so the C^*-algebra $C_0(R^0)$ is an abelian subalgebra of $A = C^*_{\text{red}}(R)$. In fact it is a *Cartan* (or *diagonal*) subalgebra of A [28], [18], [21]. (The definition of "Cartan subalgebra" given in [27; II.4.13] is slightly different, and is no longer used.) For each $x \in X$, (x,x) can be viewed as a pure state of $C_0(R^0)$ (given by evaluation at (x,x)), and (x,x) has a unique extension to a pure state ϕ_x on A (which is also given by evaluation at (x,x)).

Recall that if ϕ is a state on a C*-algebra A then the *left kernel* of ϕ is the closed left ideal

$$L_\phi = \{a \in A : \phi(a^*a) = 0\}.$$

For $a \in A$ let ξ_a denote the image of a in the quotient space A/L_ϕ. Then $\langle \xi_a, \xi_b \rangle = \phi(b^* a)$ is a sesquilinear form on A/L_ϕ defining a pre-Hilbert space structure. Let H_ϕ denote the completed Hilbert space. For $a,b \in A$ define $\pi_\phi(a)\xi_b = \xi_{ab}$. Then (π_ϕ, H_ϕ) is the GNS representation corresponding to ϕ.

LEMMA 5.1. *Let R be a separated topological equivalence on a set X. Let $x \in X$ and for each point $y \in [x]$ let f_y be a function in $C_c(R)$ whose support is contained in an R^0-strip neighbourhood of (y,x) and such that $f_y((y,x)) = 1$. Then the set*

$$B = \{\xi_{f_y} : y \in [x]\}$$

is an orthonormal basis for H_{ϕ_x}.

PROOF. Let $f_y \in B$. Then

$$\langle \xi_{f_y}, \xi_{f_y} \rangle = \phi_x((f_y)^* f_y) = ((f_y)^* f_y)(x,x)$$

$$= \sum_{w \sim x} |f_y(w,x)|^2 = |f_y(y,x)|^2 = 1.$$

If f_z is a different element of B then

$$\begin{aligned}\langle \xi_{f_y}, \xi_{f_x} \rangle &= \phi_x((f_z)^* f_y)\\ &= ((f_z)^* f_y)(x, x)\\ &= \sum_{w \sim x} \overline{f_z(w, x)} f_y(w, x)\\ &= 0\end{aligned}$$

since $\operatorname{supp} f_y \cap \operatorname{supp} f_z \cap s^{-1}((x, x)) = \emptyset$. Hence the set B is orthonormal.

To show that the linear span of B is dense in H_{ϕ_x} suppose that a is a function in $C_c(R)$. Then $a \in L_{\phi_x}$ if and only if $a|_{s^{-1}((x,x))} = 0$. Indeed

$$a^* a(x, x) = \sum_{w \sim x} |a(w, x)^2$$

and so $a^* a(x, x) = 0$ if and only if $a|_{s^{-1}((x,x))} = 0$. The support of a is compact so there exist only finitely many distinct elements $y(1), \ldots, y(n)$ in X for which $a(y(i), x) \neq 0$ $(1 \leqq i \leqq n)$. Hence

$$\left(a - \sum_{i=1}^{n} a(y(i), x) f_{y(i)}\right)\Big|_{s^{-1}((x,x))} = 0$$

and we have that

$$\xi_a = \sum_{i=1}^{n} a(y(i), x) \xi_{f_{y(i)}},$$

by the preceding observation. Hence B spans $C_c(R)/L_{\phi_x}$ which is dense in H_{ϕ_x}.

For each $x \in X$ let $H_{[x]}$ be a Hilbert space of dimension equal to the cardinality of $[x]$. Let $\{e_y\}$ $(y \in [x])$ be an orthonormal basis for $H_{[x]}$. Then Lemma 5.1 shows that the map

$$e_y \to \xi_{f_y} \quad (y \in [x])$$

extends to an unitary operator U from $H_{[x]}$ to H_{ϕ_x}. Let $\pi_{[x]}$ be the irreducible representation of A on $H_{[x]}$ defined by

$$\pi_{[x]}(.) = U^* \pi_{\phi_x}(.) U.$$

PROPOSITION 5.2. *Let R be a separated topological equivalence relation on a set X and let $A = C^*_{\mathrm{red}}(R)$. Let $x \in X$ and let $\pi_{[x]}$ be the irreducible representation of A defined above. Then, for all $a \in A$, $\langle \pi_{[x]}(a) e_y, e_z \rangle = a(z, y)$ $(y, z \in [x])$.*

PROOF. Since $\|.\|_{\mathrm{red}}$ dominates the uniform norm on R [27; II.4.2] it is enough to prove the result for $a \in C_c(R)$. So let $a \in C_c(R)$, $y, z \in [x]$. Then

$$\langle \pi_{[x]}(a)e_y, e_z\rangle = \langle \pi_{\phi_x}(a)\xi_{f_y}, \xi_{f_z}\rangle = \langle \xi_{af_y}, \xi_{f_z}\rangle$$
$$= \phi((f_z)^* af_y) = ((f_z)^* af_y)(x, x)$$
$$= \sum_{w_1, w_2 \sim x} (f_z)^*(x, w_1)a(w_1, w_2)f_y(w_2, x)$$
$$= \sum_{w_1, w_2 \sim x} \overline{(f_z)(x, w_1)}a(w_1, w_2)f_y(w_2, x) = a(z, y).$$

REMARK. It follows at once from Lemma 5.2 that $\cap_{x\in X} \ker \pi_{[x]} = \{0\}$. Hence for all $a \in C^*_{\text{red}}(R)$

$$\|a\|_{\text{red}} = \sup\{\|\pi_{[x]}(a)\| : x \in X\}.$$

The next lemma follows immediately from the fact that $\{e_y\}$ ($y \in [x]$) is an orthonormal basis for $H_{[x]}$.

LEMMA 5.3. *Let R be a separated topological equivalence relation on a set X and let $A = C^*_{\text{red}}(R)$. If $x \in X$ and $a \in A^+$ then*

$$\operatorname{Tr} \pi_{[x]}(a) = \sum_{y \sim x} a(y, y).$$

LEMMA 5.4. *Let R be a separated topological equivalence relation on a set X and let $A = C^*_{\text{red}}(R)$.*

(i) *If $x \in X$ then $\pi_{[x]}(A) \supseteq \operatorname{LC}(H_{[x]})$ if and only if the orbit of (x, x) in R^0 is discrete.*

(ii) *If $\pi \in \hat{A}$ and if $\pi(A) \supseteq \operatorname{LC}(H_\pi)$ then there exists $x \in X$ such that π is equivalent to $\pi_{[x]}$.*

PROOF. (i) The local homeomorphism property of the projection maps r and s implies that either every point of the orbit of (x, x) is isolated, or the orbit of (x, x) has no isolated points. In the first case there exist elements a of A^+ such that $0 < \operatorname{Tr} \pi_{[x]}(a) < \infty$; in the second case there do not. Hence in the first case $\pi_{[x]}(A) \supseteq \operatorname{LC}(H_{[x]})$, in the second case $\pi_{[x]}(A) \not\supseteq \operatorname{LC}(H_{[x]})$.

(ii) Let S be the closed invariant subset of R^0 such that $(\ker \pi)^+ = \{a \in A^+ : a(x, x) = 0 \text{ for all } (x, x) \in S\}$ [27; II.4.6]. Let $a \in A^+$ such that $\pi(a)$ is a non-zero compact operator. Since $a \notin (\ker \pi)^+$ there exists $(x, x) \in S$ such that $a(x, x) \neq 0$. Hence $\pi_{[x]}(a)$ is non-zero. Since $\ker \pi \subseteq \ker \pi_{[x]}$ it follows from [8; 1.3.4] that π is equivalent to $\pi_{[x]}$.

REMARK. It follows from 5.4 (ii) that if $A = C^*_{\text{red}}(R)$ is a postliminal C*-algebra then $\hat{A} = \{\pi_{[x]} : x \in X\}$. If A is a UHF algebra, on the other hand, then A is isomorphic to $C^*_{\text{red}}(R)$ for some separated topological equivalence relation R, but it is known that, for each $x \in X$, ϕ_x is a product state. Since A has pure states which are not unitarily equivalent to product states it follows that the set $\{\pi_{[x]} : x \in X\}$ is not the whole of $\hat{A}$. Maybe $\{\pi_{[x]} : x \in X\}$ is equal to $\hat{A}$ if and only if A is postliminal.

We now describe a second topology on a separated topological equivalence relation R. Under the original topology, τ_o, the diagonal R^0 is a topological space. We transfer the topology to X, using the natural bijection from R^0 to X, and then give R a new topology, τ_p, by restricting the product topology on $X \times X$. It is straightforward to check, using the base of R^0-strip neighbourhoods for τ_o, that τ_o is finer than τ_p.

If $A = C^*_{\text{red}}(R)$ then the mapping $(x, y) \to (\phi_x, \phi_y)$ embeds R as a subset of $R(A)$ in such a way that the τ_q-topology on $R(A)$ restricts to the τ_o-topology on R, and the τ_p-topology on $R(A)$ restricts to the τ_p-topology on R [28]. In the light of Section 4 it is natural to investigate the points of continuity for the inclusion map from $(R. \tau_p)$ to (R, τ_o).

LEMMA 5.5. *Let R be a separated topological equivalence relation and let $j : (R, \tau_p) \to (R, \tau_o)$ be the inclusion map. Let $(x, y) \in R$. The following are equivalent:*

(i) *(x, y) is a point of continuity for j,*

(ii) *(x, y) has a τ_o-open neighbourhood which contains at most one point from each equivalence class in R.*

PROOF. (i) $\Rightarrow$ (ii) Suppose that each neighbourhood of (x, y) contains two points from the same equivalence class in R. Let U be any fixed R^0-strip neighbourhood in τ_0 of (x, y). By hypothesis there are nets (a_α, b_α) and (c_α, d_α) both converging (τ_o) to (x, y) with (a_α, b_α) and (c_α, d_α) in the same equivalence class in R but not equal, for each α. Since each net is eventually inside U it follows that eventually $a_\alpha \neq c_\alpha$ for each α. The continuity of the projection maps r and s implies that

$$(a_\alpha, a_\alpha) \to (x, x) \quad (\tau_o)$$

and

$$(d_\alpha, d_\alpha) \to (y, y) \quad (\tau_o).$$

Therefore, by definition

$$(a_\alpha, d_\alpha) \to (x, y) \quad (\tau_p).$$

But s is injective on U and (c_α, d_α) is eventually in U, with $a_\alpha \neq c_\alpha$), so eventually (a_α, d_α) is not in U. Hence $(a_\alpha, d_\alpha) \not\to (x, y)$ (τ_o), as required.

(ii) $\Rightarrow$ (i) Conversely, suppose that (x, y) has a τ_o-neighbourhood U containing at most one point from each equivalence class in R. Let (x_α, y_α) be a net converging (τ_p) to (x, y). This means, by definition, that $(x_\alpha, x_\alpha) \to (x, x)$ and

$(y_\alpha, y_\alpha) \to (y, y)$ (τ_o). Hence eventually (x_α, x_α) is in $r(U)$, so there is a point $(x_\alpha, z_\alpha) \in U$ for all α sufficiently large. Similarly there is a point $(w_\alpha, y_\alpha) \in U$ for all α sufficiently large. But U contains at most one point from each equivalence class, which implies that $x_\alpha = w_\alpha$ and $y_\alpha = z_\alpha$, that is that (x_α, y_α) is eventually in U. The same argument shows that (x_α, y_α) is eventually inside any τ_o-neighbourhood of (x, y) contained in U, and hence that $(x_\alpha, y_\alpha) \to (x, y)$ (τ_o).

REMARK. The local homeomorphism property of r and s implies that either every point in an equivalence class in R satisfies (ii) above, or no point does. Hence either every point in an equivalence class is a point of continuity, or no point is (c.f. Corollary 4.4).

LEMMA 5.6. *Let A be a C^*-algebra and let $b \in A^+$. Then the function from $\hat{A}$ to the extended interval $[0, \infty]$ defined by $\pi \to \operatorname{rank} \pi(b)$ is lower semi-continuous on $\hat{A}$.*

PROOF. Let $B = (bAb)^-$ be the hereditary subalgebra of A generated by b. By [24; 4.1.9] the map $(\pi, H) \to (\pi|B, \pi(B)H)$ induces a homeomorphism from $\hat{A}\backslash \operatorname{hull}(B)$ onto $\hat{B}$. If $\pi \in \hat{A}$ then $\dim \pi|B = \operatorname{rank} \pi(b)$ so for each $n \in \mathbb{N}$ the set $\{\pi \in \hat{A} : \operatorname{rank} \pi(b) > n\}$ is equal to the set $\{\pi \in \hat{A} : \dim \pi|B > n\}$, which is open by [12; 3.6.3].

THEOREM 5.7. *Let R be a separated topological equivalence relation and let $A = C^*_{\text{red}}(R)$. Let j be the map defined in Lemma 5.5.*

(i) Let $(x, y) \in R$. If (x, y) is a point of continuity for j then $\pi_{[x]}$ is a Fell point in $\hat{A}$.

(ii) If π is a Fell point in $\hat{A}$ then there exists $x \in X$ such that $\pi \simeq \pi_{[x]}$, and (y, z) is a point of continuity for j for all $y, z \in [x]$.

PROOF. (i) Suppose that (x, y) is a point of continuity for j. By Lemma 5.5 (x, y) has a τ_o-open neighbourhood U which contains at most one point from each equivalence class of R. Let $f \in C_c(R)$ be supported in U with $f(x, y) \neq 0$. Then for each $z \in X$, $\pi_{[z]}(f^*f)$ is an operator of rank less than or equal to one. Since the set $\{\pi_{[z]} : z \in X\}$ is dense in $\hat{A}$ it follows from Lemma 5.6 that f^*f is an abelian element. Since $\pi_{[x]}(f^*f) \neq 0$, $\pi_{[x]}$ is a Fell point in $\hat{A}$.

(ii) Let π be a Fell point in $\hat{A}$. By Lemma 5.4 (ii) there exists $x \in X$ such that π is equivalent to $\pi_{[x]}$. Let a be an abelian element in A^+ such that $\|a\| = \|\pi(a)\| = 1$. Let e be the unit vector in $H_{[x]}$ which spans the range of $\pi(a)$. By Kadison's Transitivity Theorem [24; 3.13.2] there exists $b \in A$ with $\|b\| = 1$ such that $\pi(b)e_x = e$. Set $c = b^*ab$. Then c is an abelian element in A^+ and $\|c\| = \|\pi(c)\| = 1$. In fact $\pi(c)$ is the orthogonal projection onto the span of e_x. Hence $c(x, x) = 1$ and c vanishes on all other points in the equivalence class of (x, x) in R. Let N be the τ_o-open neighbourhood of (x, x) in R defined by

$$N = \{(y, y) \in R^0 : c(y, y) > 1/2\}.$$

The N contains at most one point from each equivalences class in R, for if (y, y), $(z, z) \in N$ are in the same orbit, but not equal, then

$$\operatorname{Tr} \pi_{[y]}(c) \geqq c(y, y) + c(z, z) > 1,$$

contradicting the fact that c is abelian with $\|c\| = 1$. It follows from Lemma 5.5 that (x, x) is a point of continuity for j. Hence (y, z) is a point of continuity for j for all $y, z \in [x]$.

An alternative proof of Theorem 5.7 (ii) can be given, based on Corollary 4.4 and the remarks before Lemma 5.5.

LEMMA 5.8. *Let R be a separated topological relation and let $A = C^*_{\mathrm{red}}(R)$. If $(u_\lambda)_{\lambda \in \Lambda}$ is an approximate identity for $C_0(R^0)$ then it is an approximate identity for A.*

PROOF. Let $a \in A$ with $\|a\| = 1$, and let $\varepsilon > 0$ be given. Let $b \in C_c(R)$ with $\|a - b\| < \varepsilon/3$ and $\|b\| \leqq 1$. Let S denote the support of b and set $T = r(S)$. Then T is a compact subset of R^0. Let $f \in C_0(R^0)$ with $f(t) = 1$ for all $t \in T$. Then $fb = b$. Choose $\lambda \in \Lambda$ such that $\|u_\lambda f - f| < \varepsilon/3$. Then

$$\begin{aligned} \|u_\lambda a - a\| &= \|u_\lambda a - u_\lambda b + u_\lambda b - fb + fb - a\| \\ &= \|u_\lambda a - u_\lambda b + u_\lambda fb - fb + b - a\| \\ &\leqq \|u_\lambda\| \, \|a - b\| + \|u_\lambda f - f\| \, \|b\| + \|b - a\| < \varepsilon. \end{aligned}$$

COROLLARY 5.9. *Let R be a separated topological equivalence relation and let $A = C^*_{\mathrm{red}}(R)$. Then A is a Fell C*-algebra if and only if the topologies τ_p and τ_o coincide on R.*

PROOF. If A is a Fell C*-algebra then for every point (x, y) of R, $\pi_{[x]}$ is a Fell point of $\hat{A}$, so (x, y) is a point of continuity for j, by Theorem 5.7. Hence the topologies τ_p and τ_o coincide on R.

Conversely, suppose that τ_p and τ_o coincide on R. Let x and y be distinct points of X. By Lemma 5.5 there is a τ_o-open neighbourhood U of (x, x) containing at most one point from each equivalence class in R. We may assume that $U \subseteq R^0$ and that $(y, y) \notin U$. Let $f \in C_c(R^0)^+$ such that $f(x, x) > 0$ and such that the support of f is contained in U. Then the argument in the proof of Theorem 5.7 (i) shows that f is an abelian element of A.

By the Stone-Weierstrass theorem the set of elements in $C_0(R^0)$ which are abelian in A generates $C_0(R^0)$ as a C*-algebra. Since $C_0(R^0)$ contains an approximate identity for A, Lemma 5.8, it follows that the smallest closed ideal of A containing the abelian elements is A itself. This implies, by [24; 6.1.7], that A is a Fell C*-algebra.

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