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MULTIVALUED HARMONIC MORPHISMS

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Abstract.

We define a notion of *multivalued harmonic morphism* Φ from a Riemannian manifold (M^m, g_M) to a surface N^2. We show how Φ defines a manifold $\tilde{M}^m$, a map $\pi: \tilde{M}^m \to M^m$, which is a local diffeomorphism except on a closed subset $\tilde{E}$ of $\tilde{M}^m$, and a single valued harmonic morphism $\psi: \tilde{M}^m \to N^2$ covering all values of Φ. In the case of a space form M^3 we give some examples, a classification theorem and discuss the behaviour of π on $\tilde{E}$. Higher dimensional examples are mentioned in the last section.

1. Introduction.

In complex analysis one considers equations of the form

$$F(z, x) = w, \tag{1.1}$$

where $F: N^2 \times M^2 \to P^2$ is a non-constant holomorphic map from the product of two Riemann surfaces M^2 and N^2 to a Riemann surface P^2, and w a fixed point of P^2. Away from points where $\partial F/\partial z = 0$, (1.1) has local smooth solutions $z = \phi(x)$. Any such solution is holomorphic but, in general, not unique. Instead we think of equation (1.1) as defining a "multivalued holomorphic function" $z(x)$ from M^2 to N^2. Assuming that $dF \neq 0$ at points satisfying (1.1), the latter equation defines a smooth surface

$$\tilde{M}^2 := \{(z, x) \in N^2 \times M^2 \mid F(z, x) = w\}.$$

This is the Riemann surface of $z(x)$ in the sense that if $\pi: \tilde{M}^2 \to M^2$ denotes the restriction of the natural projection $(z, x) \mapsto x$ to $\tilde{M}^2$, then there is a holomorphic map $\psi: \tilde{M}^2 \to N^2$, such that any local solution $z = \phi(x)$ of (1.1) on an open subset U of M^2, satisfies $\psi = \phi \circ \pi$ on $\pi^{-1}(U)$. In fact ψ is simply the restriction of the

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other natural projection $(z, x) \mapsto z$ to $\tilde{M}^2$. The map π is locally diffeomorphic except at the isolated points where $\partial F/\partial z = 0$. There it is locally of the form $z \mapsto z^k$, for some $k \in \mathbf{N}$. No local solution exists at the images of such points and analytic continuation of a germ of a solution $z = \phi(x)$ around such a point gives a different germ. Examples are: $M^2 = N^2 = P^2 = \mathbf{C}$ (or $\mathbf{C} \cup \{\infty\}$) with $F(z, x) = x - z^2 = 0$, and $M^2 = \mathbf{C} \backslash \{0\}$, $N^2 = P^2 = \mathbf{C}$ with $F(z, x) = x - e^z = 0$, defining the multivalued holomorphic functions $z = \sqrt{x}$, $z = \log x$ and their Riemann surfaces, respectively.

In this paper we generalize the above situation to higher dimensions by using the idea of a harmonic morphism. *Harmonic morphisms* $\pi : (M^m, g) \to (N^n, h)$ between Riemannian manifolds are maps which pull back germs of real-valued harmonic functions on N to germs of real-valued harmonic functions on M. They can be characterized as those harmonic maps which are horizontally conformal, see Lemma 2.3.

Harmonic morphisms $\pi : M^m \to N^2$ to *surfaces* have many nice properties. For example: (i) The condition of π being a harmonic morphism depends only on the conformal structure of N^2, so if N^2 is oriented we can take it to be a Riemann surface. (ii) Every regular fibre of such a map is a minimal submanifold of M^m, see [Bai-Eel]. (iii) Holomorphic maps from Kähler manifolds to Riemann surfaces are harmonic morphisms ([Fug] §11). (iv) When M^2 and N^2 are Riemann surfaces, the harmonic morphisms $\pi : M^2 \to N^2$ are precisely the $\pm$holomorphic (i.e. holomorphic or antiholomorphic) maps. Harmonic morphisms can therefore be seen as a higher dimensional generalization of $\pm$holomorphic maps between Riemann surfaces.

Our first observation is the following:

THEOREM 1.1. *Let (M^m, g_M), (N^2, g_N) and (P^2, g_P) be Riemannian manifolds with* $\dim N^2 = \dim P^2 = 2$, *and let $G : N^2 \times M^m \to P^2$ be a harmonic morphism in each variable separately. If w is a fixed point on P^2 and $dG \neq 0$ on $G^{-1}(w)$, then any smooth local solution $\phi : U \to N^2$, $z = \phi(x)$ to the equation*

$$G(z, x) = w, \tag{1.2}$$

defined on an open subset U of M^m, is a harmonic morphism.

In general there is not a unique solution to equation (1.2); we need the following concept:

DEFINITION 1.2. Let $\mathscr{C}(N^n)$ be the set of all closed subsets of N^n. By a *multivalued harmonic morphism Φ from* M^m to N^n we shall mean a mapping $\Phi : M^m \to \mathscr{C}(N^n)$, such that any smooth map $\phi : U \to N^n$ defined on an open subset U of M^m satisfying $\phi(x) \in \Phi(x)$ for all $x \in U$ is a harmonic morphism. Such a map ϕ is called a *branch* of Φ.

Theorem 1.1 says that the set-valued mapping $\Phi: M^m \to \mathscr{C}(N^2)$ with $\Phi(x) = \{z \in N^2 \mid G(z,x) = w\}$ is a multivalued harmonic morphism.

DEFINITION 1.3. We call $\Phi: M^m \to \mathscr{C}(N^2)$ given by $\Phi(x) := \{z \in N^2 \mid G(z,x) = w\}$ the *multivalued harmonic morphism* defined by equation (1.2).

In the case when M^m is an open subset of R^m and $N^2 = \mathsf{C}$, a harmonic morphism $\phi: M^m \to \mathsf{C}$ is simply a map satisfying:

$$\sum_{i=1}^{m} \frac{\partial^2 \phi}{\partial x_i^2} = 0, \tag{1.3}$$

and

$$\sum_{i=1}^{m} \left(\frac{\partial \phi}{\partial x_i}\right)^2 = 0. \tag{1.4}$$

For this case with $m = 3$, Theorem 1.1 was essentially observed by Jacobi, see [Jac]. Note that for $m = 2$, (1.4) is equivalent to the $\pm$Cauchy-Riemann equations and implies (1.3), confirming that the harmonic morphisms $\phi: M^2 \subset \mathsf{R}^2 = \mathsf{C} \to \mathsf{C}$ are simply the $\pm$holomorphic maps.

We next construct the "Riemannian covering manifold" of our multivalued harmonic morphism.

THEOREM 1.4. *Let (M^m, g_M), (N^2, g_N) and (P^2, g_P) be Riemannian manifolds, and equip the product manifold $N^2 \times M^m$ with the product metric $g_N \times g_M$. Further, let $G: N^2 \times M^m \to P^2$ be a harmonic morphism in each variable separately. Suppose that for some fixed $w \in P^2$, $dG \neq 0$ along $\tilde{M}^m := G^{-1}(w) = \{(z,x) \in N^2 \times M^m \mid G(z,x) = w\}$. Then $\tilde{M}^m$ is a smooth submanifold of $N^2 \times M^m$. Further, there exists a smooth map $\pi: \tilde{M}^m \to M^m$ and a harmonic morphism $\psi: \tilde{M}^m \to N^2$ such that any smooth local solution $\phi: U \subset M^m \to N^2$ to $G(z,x) = w$ (necessarily a harmonic morphism by Theorem* 1.1) *satisfies $\psi = \phi \circ \pi$ on $\pi^{-1}(U)$.*

The maps π and ψ are the restrictions of the natural projections of $N^2 \times M^m$ to $\tilde{M}^m$. π is a local diffeomorphism except on the branching set or envelope $\tilde{E} := \{(z,x) \in \tilde{M}^m \mid dG_x = 0\}$, a closed subset of $\tilde{M}^m$ (see Definition 2.2 for notation). If $(z_0, x_0) \in \tilde{E}$ then there are no local solutions $z = \phi(x)$ to (1.2) with $z_0 = \phi(x_0)$. Analytic continuation of a local solution around $E = \pi(\tilde{E})$ in M^m can give rise to different solutions.

REMARK 1.5. Let $\Phi: M^m \to \mathscr{C}(N^2)$ be the multivalued harmonic morphism defined by equation (1.2). Then our construction provides a Riemannian manifold $\tilde{M}^m$, a map $\pi: \tilde{M}^m \to M^m$ and a single-valued harmonic morphism $\psi: \tilde{M}^m \to N^2$ which covers all branches $\phi: U \to N^2$ of Φ in the sense that $\psi = \phi \circ \pi$ on $\pi^{-1}(U)$. In fact $\tilde{M}^m$ is the *graph* of Φ, i.e. $\tilde{M}^m = \{(z,x) \in M^m \mid z \in \Phi(x)\}$.

In §3 we discuss a more convenient but essentially equivalent approach in the case that M^3 is a simply connected space form $\mathbb{R}^3, S^3$ or H^3. In this case, harmonic morphisms to Riemann surfaces N^2 are given locally by non-constant holomorphic maps $\eta: N^2 \to \mathscr{G}_{M^3}$, where $\mathscr{G}_{M^3}$ is the space of oriented geodesics in M^3. Conversely we shall think of such a map η as defining a multivalued harmonic morphism from M^3 to N^2 and will similarly construct a covering Riemannian manifold $\tilde{M}^3$ and a single-valued harmonic morphism $\psi: \tilde{M}^3 \to N^2$.

Note that our restriction to space forms is not artificial: It was proved in Corollary 4.6 of [Bai-Woo-3], that for other Riemannian 3-manifolds there can (even locally) exist (up to post-composition with a weakly conformal map) at most two harmonic morphisms to surfaces. We therefore do not expect a very rich theory of multivalued harmonic morphisms in this case!

In §§4 and 5 we give some examples illustrating various sorts of behaviour for the projection maps $\pi: \tilde{M}^3 \to M^3$ for $M^3 = \mathbb{R}^3$ and S^3. On the way, we characterize an interesting class of harmonic morphisms, namely radial projections and members of the "outer disk family", see Theorem 4.6. Then in §6 we describe all the possible behaviour of π on its branching set $\tilde{E}$.

Finally in §7 we discuss briefly some higher dimensional constructions, especially those coming from the second author's description of all harmonic morphisms from open subsets of $\mathbb{R}^4$ to a surface in terms of holomorphic functions, see [Woo].

The idea that a multivalued harmonic morphism should be covered by a single-valued one defined on a Riemannian manifold covering its domain, is contained in P. Baird's articles [Bai-1] and [Bai-2]. The present paper can be seen as a realization of that idea. We are grateful to P. Baird and J. Eells for comments on this work, to G. Dethloff and J. Birman for useful conversations and to V. Parmar for help with Superpaint. The second author thanks J.-P. Bourguignon, M. Berger and J.-M. Coron for making his year-long stay in France possible. Both authors thank the staff of the Institut des Hautes Etudes Scientifiques where most of this work was done.

2. The covering construction.

In this section we show explicitly how a multivalued harmonic morphism from M^m to N^2 gives to a "covering Riemannian manifold" $\tilde{M}^m$, a map $\pi: \tilde{M}^m \to M^m$ and a single-valued harmonic morphism $\psi: \tilde{M}^m \to N^2$.

Throughout this paper we assume that all our objects such as manifolds, metrics and maps are smooth, that, in the C^∞-category. Let $M = (M^m, g_M)$, $N = (N^n, g_N)$ and $P = (P^p, g_P)$ be Riemannian manifolds of dimensions m, n, p respectively. Further let the product manifold $N^n \times M^m$ be equipped with the product metric $g_N \times g_M$.

DEFINITION 2.1. For a smooth map $\phi: M^m \to N^n$ and $x \in M^m$ let $V_x^\phi := \operatorname{Ker} d\phi_x \subset T_x M^m$ and $H_x^\phi = (V_x^\phi)^\perp \subset T_x M^m$. Further let $C_\phi := \{x \in M^m \mid d\phi_x = 0\}$ and $M^* := M^m \setminus C_\phi$. Then C_ϕ is called the *critical set* of ϕ. The map ϕ is said to be *horizontally weakly conformal* on M^m if there exists a function $\lambda: M^* \to \mathrm{R}^+$ such that $\lambda^2 g_M(X, Y) = g_N(d\phi(X), d\phi(Y))$ for all $X, Y \in H_x^\phi$, and $x \in M^*$. If ϕ is horizontally conformal, then $V^\phi := \{V_x^\phi \mid x \in M^*\}$ and $H^\phi := \{H_x^\phi \mid x \in M^*\}$ are distributions on M^*, called the *vertical* and *horizontal distributions* of ϕ. Setting $\lambda \equiv 0$ on C_ϕ determines a continuous function $\lambda: M^m \to \mathrm{R}$, called the *dilation* of ϕ, with $\lambda^2 = |d\phi|^2/n$ smooth.

We shall frequently use the following fact proved independently in [Fug] and [Ish]:

LEMMA 2.2. *A smooth map $\phi: M^m \to N^n$ is a harmonic morphism if and only if*
(1) *ϕ is a harmonic map, and*
(2) *ϕ is horizontally weakly conformal.*

DEFINITION 2.3. Let $G: N^n \times M^m \to P^p$ be a map. For $z \in N^n$ and $x \in M^m$ we denote by $G_z: M^m \to P^p$ and $G_x: N^n \to P^p$ the maps given by $G_z: y \mapsto G(z, y)$ and $G_x: w \mapsto G(w, x)$, respectively. G is said to be a *harmonic morphism in each variable separately* if $G_z: M^m \to P^p$ and $G_x: N^n \to P^p$ are harmonic morphisms for every $z \in N^n$ and $x \in M^m$.

REMARK 2.4. If $f: U \subset P^p \to \mathrm{R}$ is a function and $G: N^n \times M^m \to P^p$ a smooth map, then it is easily verified that

$$\Delta_{N \times M}(f \circ G)(z, x) = \Delta_N(f \circ G_x)(z) + \Delta_M(f \circ G_z)(x)$$

for all $(z, x) \in N^n \times M^m$. Thus if G is a harmonic morphism in each variable separately, then it is a harmonic morphism as a map from the product manifold.

Note that if $n = p = 2$, then the hypothesis that $G_x: N^2 \to P^2$ be a harmonic morphism is equivalent to G_x being weakly conformal.

We now establish a result which is actually more general than that of Theorem 1.1.

PROPOSITION 2.5. *Let (M^m, g_M), (N^2, g_N) and (P^2, g_P) be Riemannian manifolds and $G: N^2 \times M^m \to P^2$ a harmonic morphism in each variable separately. Let $\phi: U \subset P^2 \times M^m \to N^2$, be a local solution to the equation*

$$G(z, x) = w, \tag{2.1}$$

that is,

$$G(\phi(w, x), x) = w \tag{2.2}$$

for all (w, x) on some open subset U of $P^2 \times M^m$. Then ϕ is a harmonic morphism in each variable separately.

PROOF. Let $z = \phi(w, x)$ be a local solution to (2.1) through (w_0, x_0); write $z_0 = \phi(w_0, x_0)$. If we differentiate equation (2.2) with respect to w we obtain $dG_x \circ d\phi_x = \mathrm{Id}_{TP}$. Hence $dG_x \neq 0$ for all (z, x) in some neighbourhood of (z_0, x_0). Further since G_x is a harmonic morphism, dG_x is conformal, so $d\phi_x$ is conformal. Thus, since $\dim N^2 = \dim P^2 = 2$, ϕ_x is a harmonic morphism.

To prove that $\phi_w : M^m \to N^2$ is a harmonic morphism, we show that it is horizontally conformal and harmonic. Firstly, since G_x is conformal and non-constant in a neighbourhood of (z_0, x_0) and our problem is local, we can without loss of generality assume that N^2 and P^2 are oriented and that G_x is holomorphic. Then we may choose local complex coordinates z and w on N^2 and P^2 in neighbourhoods of z_0 and w_0 respectively, and normal coordinates $x = (x_1, \ldots, x_n)$ centred at the point $x_0 \in M^m$. In a neighbourhood of (x_0, w_0) a local solution $\phi : (x, w) \mapsto z$ satisfies

$$G(z(x, w), x) = w. \tag{2.3}$$

Differentiating with respect to x_i gives:

$$\frac{\partial G}{\partial z}\frac{\partial z}{\partial x_i} + \frac{\partial G}{\partial x_i} = 0. \tag{2.4}$$

Now as noted above $dG_x = \partial G/\partial z \neq 0$, so

$$\frac{\partial z}{\partial x_i} = -\left(\frac{\partial G}{\partial z}\right)^{-1}\frac{\partial G}{\partial x_i}.$$

Since G_z is a harmonic morphism it is horizontally conformal, that is $\sum_{i=1}^m (\partial G/\partial x_i)^2 = 0$ at x_0, thus at that point,

$$\sum_{i=1}^m \left(\frac{\partial z}{\partial x_i}\right)^2 = 0. \tag{2.5}$$

Differentiating (2.4) with respect to x_i gives:

$$\frac{\partial^2 G}{\partial z^2}\left(\frac{\partial z}{\partial x_i}\right)^2 + \frac{\partial G}{\partial z}\frac{\partial^2 z}{\partial x_i^2} + \frac{\partial^2 G}{\partial x_i^2} = 0.$$

Summing using (2.5), we have at x_0,

$$\frac{\partial G}{\partial z}\sum_{i=1}^m \frac{\partial^2 z}{\partial x_i^2} + \sum_{i=1}^m \frac{\partial^2 G}{\partial x_i^2} = 0.$$

Since $\partial G/\partial z \neq 0$ and the last term vanishes we conclude that

$$\sum_{i=1}^m \frac{\partial^2 z}{\partial x_i^2} = 0. \tag{2.6}$$

At the point x_0, (2.5) and (2.6) are the conditions for horizontal conformality and harmonicity respectively. Thus by Lemma 2.2, ϕ_w is a harmonic morphism.

The proof of Theorem 1.1 is similar, except that now $dG \neq 0$ is required to ensure that $\partial G/\partial z \neq 0$, for if $\partial G/\partial z = 0$ then by (2.4) $\partial G/\partial x_i = 0$ for all i, so that $dG = 0$, contradicting the hypothesis.

For the map $G: N^2 \times M^m \to P^2$ and a point $w \in P^2$, define

$$\tilde{M}^m = \tilde{M}^m_w := \{(z,x) \in N^2 \times M^m \mid G(z,x) = w\},$$

and let $\psi = \pi_1|_{\tilde{M}^m}: \tilde{M}^m \to N^2$ and $\pi = \pi_2|_{\tilde{M}^m}: \tilde{M}^m \to M^m$ be the restrictions of the natural projections $\pi_1: N^2 \times M^m \to N^2$ and $\pi_2: N^2 \times M^m \to M^m$ to $\tilde{M}^m$. We call the closed subset $\tilde{E} := \{(z,x) \in \tilde{M}^m \mid dG_x = 0\}$ of $\tilde{M}^m$ the *envelope* (or *branching set upstairs*) of G and its image $E = \pi(\tilde{E})$ in M^m, the *geometric envelope* (or *branching set downstairs*). Further let $\tilde{F} := \{(z,x) \in \tilde{M}^m \mid dG_z = 0\}$.

The next result is an expanded version of Theorem 1.4, and explains how $\tilde{M}^m$ is the "covering Riemannian manifold" of the multivalued harmonic morphism defined by equation (1.2).

PROPOSITION 2.6. *Let $G: N^2 \times M^m \to P^2$ be a harmonic morphism in each variable separately. Suppose that for some $w \in P^2$, $dG \neq 0$ along $\tilde{M}^m := G^{-1}(w)$. Then, with notations as above,*

(1) *$\tilde{M}^m$ is an m-dimensional minimal submanifold of $N^2 \times M^m$,*

(2) *$\pi: \tilde{M}^m \to M^m$ is a local diffeomorphism except on $\tilde{E}$,*

(3) *$\psi: \tilde{M}^m \to N^2$ is a harmonic morphism with critical set $\tilde{F}$,*

(4) *any local solution $\phi: U \subset M^m \to N^2$ of equation* (1.2), *necessarily a harmonic morphism by Theorem* 1.1, *satisfies $\psi = \phi \circ \pi$ on $\pi^{-1}(U)$.*

PROOF. First note that the tangent space of $\tilde{M}$ at (z,x) is given by

$$\begin{aligned} T_{(z,x)}\tilde{M} &= \{(Z,X) \in T_zN^2 \times T_xM^m \mid dG(Z,X) = 0\} \\ &= \{(Z,X) \in T_zN^2 \times T_xM^m \mid dG_z(X) + dG_x(Z) = 0\}. \end{aligned}$$

(1) Since G is a harmonic morphism (see Remark 2.4), it is horizontally weakly conformal, so the fact that $dG \neq 0$ on $\tilde{M}^m$ implies that dG is surjective. It follows from the implicit function theorem that $\tilde{M}^m$ is an m-dimensional submanifold of $N^2 \times M^m$. That $\tilde{M}^m$ is minimal is a consequence of Theorem 5.2 of [Bai-Eel] as mentioned in the introduction.

(2) Since M^m and $\tilde{M}^m$ have the same dimension, we only have to show that $d\pi$ is surjective outside $\tilde{E}$. Let $(z,x) \in \tilde{M}^m \setminus \tilde{E}$ and let $X \in T_xM^m$ be non-zero. Since G_x is a harmonic morphism, $dG_x \neq 0$ means that dG_x is non-singular. Let $Z := -(dG_x)^{-1} \circ dG_z(X) \in T_zN^2$, then $dG_x(Z) + dG_z(X) = 0$, so $(Z,X) \in T_{(z,x)}\tilde{M}^m$ and $d\pi(Z,X) = X$. Thus $d\pi$ is surjective.

(3) We look first at points $(z,x)\in\tilde{F}$. There the tangent space of $\tilde{M}^m$ is $\{(0,X)\in T_zN^2\times T_xM^m\}$, so clearly $d\psi=0$ on $\tilde{F}$.

On the other hand let $(z,x)\in\tilde{M}^m\setminus\tilde{F}$, then we have $\psi^{-1}(z)=\{(z,x)\in\tilde{M}^m \mid G(z,x)=w\}$, so that the vertical space $V^{\psi}_{(z,x)}$ at $(z,x)\in\tilde{M}^m$ with respect to ψ is given by $V^{\psi}_{(z,x)}=\{(0,X)\in T_{(z,x)}\tilde{M}^m \mid X\in V^{G_z}_x\}$. Hence $H^{\psi}_{(z,x)}=\{(Z,X)\in T_{(z,x)}\tilde{M}^m \mid X\in H^{G_z}_x\}$. Given $Z\in T_zN^2$, there exists exactly one $X\in H^{G_z}_x$ such that $dG_z(X)+dG_x(Z)=0$, namely $X=-(dG_z|_{H^{G_z}_x})^{-1}\circ dG_x(Z)$. From this it is clear that $d\psi|_{H^{\psi}_{(z,x)}}: H^{\psi}_{(z,x)}\to T_zN^2$ is given by

$$(Z,X)=(Z,-(dG_z|_{H^{G_z}_x})^{-1}\circ dG_x(Z))\mapsto Z.$$

Since $dG_z|_{H^{G_z}_x}$ and dG_x are conformal, this shows that ψ is horizontally conformal.

As regards the harmonicity of ψ, note that ψ is the composition of the inclusion i and the projection π_2,

$$\psi:\tilde{M}^m \xrightarrow{i} M^m\times N^2 \xrightarrow{\pi_2} N^2.$$

Now the composition law for the tension field (see [Eel-Sam]) is:

$$\tau(\psi)=\operatorname{trace}\nabla d\pi_2(di,di)+d\pi_2(\tau(i)).$$

The first term is zero since π_2 is totally geodesic; also $\tilde{M}^m$ is minimal in $N^2\times M^m$ so that $\tau(i)=0$. Hence ψ is harmonic, and since horizontally weakly conformal, a harmonic morphism.

(4) To say that ϕ is a local solution on $U\subset M^m$, means that $G(\phi(x),x)=w$ for all $x\in U$. Thus $(\phi(x),x)\in\tilde{M}^m$ and $\psi(\phi(x),x)=\phi(x)$, i.e. $\psi=\phi\circ\pi$ on $\pi^{-1}(U)$.

REMARKS 2.7. (i) For an interpretation of the last result in terms of the multivalued harmonic morphism $\Phi: M^m\to\mathscr{C}(N^n)$ defined by equation (2.1) see Remarks 1.5. Following on from that, we shall call $\tilde{M}^m$ the *covering manifold*, π the *projection* and ψ the *covering harmonic morphism* of the multivalued harmonic morphism defined by equation (2.1).

(ii) If (M^m,g_M) is real analytic, then $\tilde{E}$ is a real-analytic subset of $\tilde{M}^m$. Further, analytic continuation of a branch ϕ of Φ (i.e. local solution of (2.1)) clearly gives another one. We might therefore hope to construct $\tilde{M}^m$ by analytic continuation of branches of Φ in the manner of §16.A. of [Ahl-Sar]. However this procedure cannot give the envelopes $\tilde{E}$ and E since analytic continuation in M^m to a point on the envelope E is not, in general, possible. Our procedure, gives the whole of the covering manifold $\tilde{M}^m$.

(iii) For a geometrical interpretation of the envelope $\tilde{E}$ see Remarks 3.3 (ii) and 6.3.

Regarding the various dilations and their behaviour near $\tilde{E}$ and $\tilde{F}$, let $(z, x) \in \tilde{M}^m$ lie outside the two disjoint subsets $\tilde{E}$ and $\tilde{F}$ of $\tilde{M}^m$ and let $(Z, X) \in H^{\psi}_{(z,x)}$. Then there exists a local solution $\phi : U \subset M^m \to N^2$, such that $x \in U$ and $\psi = \phi \circ \pi$ on an appropriate open neighbourhood V of (z, x). For the tangent maps we have the following commutative diagram:

$$\begin{array}{ccc} & (Z, X) \in H^{\psi}_{(z,x)} & \\ d\pi|_{H^{\psi}_{(z,x)}} \swarrow & & \searrow d\psi \\ X \in H^{G_z}_x = H^{\phi}_x & \xrightarrow{d\phi} & Z \in T_z N \end{array}$$

For the dilation $\lambda_\phi(x) \in \mathbb{R}^+$ we have $\lambda_\phi \cdot |X| = |Z|$. From the above diagram it is easy to see that the dilation λ_ψ of ψ and the conformality factor λ_{π_H} of $d\pi|_{H^{\psi}_{(z,x)}}$ satisfy:

$$(2.7) \quad \lambda_{\pi_H}(z, x) = \frac{1}{\sqrt{1 + \lambda_\phi^2(x)}} \in (0. 1) \text{ and } \lambda_\psi(z, x) = \frac{\lambda_\phi(x)}{\sqrt{1 + \lambda_\phi^2(x)}} \in (0, 1).$$

From these last equations, we can see how the functions λ_{π_H} and λ_ϕ behave near the envelope $\tilde{E}$ and near $\tilde{F}$. As $(z, x) \to \tilde{E}$, then $\lambda_{\pi_H}(z, x) \to 0$ confirming that points of $\tilde{E}$ are critical points of π, the dilation of any local solution must tend to infinity but the dilation λ_ψ stays finite and is, in fact, 1 on $\tilde{E}$. On the other hand at a point (z, x) of $\tilde{F}$, the dilation of ψ and of any local solution ϕ is zero, these having branch points, but π is a local diffeomorphism with $\lambda_{\pi_H}(z, x) = 1$.

REMARK 2.8. For later use, note that π maps the fibre $\psi^{-1}(z)$ isometrically on to the fibre $\phi^{-1}(z)$, and at a point $(z, x) \in \tilde{E}$, $(\operatorname{Ker} d\pi)_{(z,x)} = H^{\psi}_{(z,x)}$, so that π has rank $m - 2$ at (z, x).

3. Multivalued harmonic morphisms from 3-dimensional space forms.

In [Bai-Woo-3] it is noted that, if M^3 is a space form, then there exist locally many harmonic morphisms from M^3 to a surface. On the other hand, it was shown in [Bai-Woo-3] that if M^3 does not have constant curvature, there are at most two. Here we develop a theory of multivalued harmonic morphisms from such a space form in a related, but slightly different, fashion to that of §2. It will be convenient to have both desciptions when presenting examples later on.

Let M^3 be a simply-connected space form, that is, $\mathbb{R}^3$, S^3 or H^3. Then $\mathcal{G}_{M^3}$, the space of oriented geodesics of M^3, is a complex manifold of dimension 2. For example the space $\mathcal{G}_{\mathbb{R}^3}$ can be identified with the tangent bundle of the 2-sphere TS^2. As in [Hit], a point $(\gamma, c) \in TS^2$ corresponds to the oriented line with direction $\gamma \in S^2$ and "origin" $c \in T_\gamma S^2$, this giving the point on the line nearest to the origin $0 \in \mathbb{R}^3$. Similarly, since every oriented geodesic of S^3 corresponds to an oriented 2-plane of R^4 through the origin, $\mathscr{G}_{S^3}$ can be identified with the Grassmannian $\tilde{G}_2(\mathsf{R}^4)$ of such 2-planes. The space $\tilde{G}_2(\mathsf{R}^4)$ can then be identified with $S^2 \times S^2$, see [Hof-Oss]. Finally $\mathscr{G}_{H^3}$ can be identified with $S^2 \times S^2 \setminus \Delta$, where Δ is the diagonal in $S^2 \times S^2$, for this see [Bai-Woo-2].

Now let $\phi : U \subset M^3 \to N^2$ be a non-constant harmonic morphism from a convex open subset U of M^3 to a Riemann surface N^2. We can assume without loss of generality that ϕ is surjective and has connected fibres; indeed, an arbitrary non-constant harmonic morphism can be factorized into the composition of such a map and a weakly conformal map between surfaces – for this see Theorem 2.20 of [Bai-Woo-2]. Then we can define a map $\eta : N^2 \to \mathscr{G}_{M^3}$ by setting $\eta(z)$ to be the oriented geodesic containing the geodesic segment $\phi^{-1}(z)$ endowed with the orientation which, together with the lift of the orientation of N^2 to the horizontal spaces, gives the standard one on M^3. Because of the conformality of the foliation whose leaves are the fibres of ϕ, η is holomorphic, see Proposition 2.4 and Lemma 2.7 of [Bai-Woo-2].

LEMMA 3.1. *Every non-constant harmonic morphism up to post-composition with weakly conformal maps $N^2 \to N^2$, arises in this way for a suitable $\eta : \mathrm{N}^2 \to \mathscr{G}_{M^3}$.*

Conversely, let $\eta : N^2 \to \mathscr{G}_{M^3}$ be a non-constant holomorphic map. Then every smooth map $\phi : U \to N^2$, from an open subset U of M^3, such that $\phi^{-1}(z) = \eta(z) \cap U$ (or equivalently, writing $z = \phi(x)$, such that

$$x \in \eta(z) \tag{3.1}$$

where $x \in U$ and $z \in N^2$) is a *submersive* harmonic morphism. We call such a map ϕ a *local solution* to η (or to (3.1)). Thus a local solution has fibres given by η. However, since any $x \in U$ may lie on several geodesies $\eta(z)$, local solutions are not, in general, unique. We therefore make a similar construction to that in Proposition 2.4:

THEOREM 3.2. *Let $\eta : N^2 \to \mathscr{G}_{M^3}$ be a non-constant holomorphic map. Set $\tilde{M}^3 := \{(z, x) \in N^2 \times M^3 \mid x \in \eta(z)\}$ and let $\pi : \tilde{M}^3 \to M^3$ and $\psi : \tilde{M}^3 \to N^2$ be the natural projections $(z, x) \mapsto z$ respectively. Then*

(1) *$\tilde{M}^3$ is a 3-dimensional submanifold of $N^2 \times M^3$,*

(2) *π is a local diffeomorphism except on a real-analytic subset $\tilde{E}$ of $\tilde{M}^3$, called the envelope,*

(3) *ψ is a submersive harmonic morphism,*

(4) *any local solution $\phi : U \subset M^3 \to N^2$ to η satisfies $\psi = \phi \circ \pi$ on $\pi^{-1}(U)$.*

PROOF. This can be given an invariant proof, but it is useful to demonstrate the construction in the case of $M^3 = \mathsf{R}^3$; the S^3 and H^3 cases are similar.

Given a non-constant holomorphic map $\eta: N^2 \to \mathcal{G}_{\mathbb{R}^3} = TS^2$, we can write η as $\eta(z) = (\gamma(z), c(z))$, where $\gamma: N^2 \to S^2$ is a holomorphic map and c is a holomorphic vector field along γ. If $\sigma: S^2 \to \mathbb{C} \cup \{\infty\}$ is the stereographic projection from the south pole $(0,0,-1) \in S^2$, setting $g = \sigma \circ \gamma$ and $h = d\sigma_\gamma \circ c$ gives meromorphic functions $g, h: N^2 \to \mathbb{C} \cup \{\infty\}$ such that

(3.2) $$h \text{ is finite if } g \text{ is, and } \lim_{x \to z_0} h(z)/g^2(z) \text{ is finite if } g(z_0) = \infty.$$

Then by [Bai-Woo-1], condition (3.1) is equivalent to:

(3.3) $$G(z,x) := (1 - g^2(z))x_1 + i(1 + g^2(z))x_2 - 2g(z)x_3 - 2h(z) = 0.$$

To make sense of this equation at a pole $z_0 \in N^2$ of g we must divide through by $g^2(z)$ and treat it as a limit. Away from the poles of g, G defines a map $S^2 \times \mathbb{R}^3 \to \mathbb{C}$ which is a harmonic morphism in each variable separately. In a neighbourhood of a pole of g we replace G by G/g^2 and the same is true. Then our Theorem follows by applying Proposition 2.6 (with $w = 0$) "locally".

REMARKS 3.3. (i) By *the multivalued harmonic morphism defined by* η we mean the set valued map $\Phi: M^3 \to \mathcal{C}(N^2)$ with $\Phi(x) := \{z \in N^2 \mid x \in \eta(z)\}$. Then we have constructed a Riemannian manifold $\tilde{M}^3$, a map $\pi: \tilde{M}^3 \to M^3$ and a harmonic morphism $\psi: \tilde{M}^3 \to N^2$ which covers every value of Φ. As before we call $\tilde{M}^3$ the covering manifold, π the projection and ψ the covering harmonic morphism of the multivalued harmonic morphism Φ.
(ii) The geometric envelope E can be interpreted as the envelope of the parametrized family of geodesies $\{\eta(z) \mid z \in N^2\}$ in a classical sense, namely, the points where geodesies "get infinitesimally close for adjacent values of the parameter" (cf. [Bru-Gib]). In particular, at such points, the family of geodesies does not, in general, form a smooth foliation (see §6 especially Remark 6.3). In contrast, if $(z_0, x_0) \notin \tilde{E}$, then the geodesies $\eta(z)$ for z close to z_0 *do* form a smooth foliation near x_0 and a local solution ϕ to η is given by the natural projection of this foliation (cf. [Woo] Proposition 2.1).

It is useful to note that, under conditions (i) and (ii) below, a multivalued harmonic morphism defined by η and a multivalued harmonic morphism defined by a harmonic morphism G in each variable separately, are locally the same thing. For let $G: N^2 \times M^3 \to P^2$ be a harmonic morphism in each variable separately. Let $w \in P$ and suppose that (i) $dG_z \neq 0$ for all $(z,x) \in \tilde{M}^3 = G^{-1}(w)$. Let $(z_0, x_0) \in M^3$. Then there is a neighbourhood U of x_0 and a holomorphic map $\eta: V \subset N^2 \to \mathcal{G}_{M^3}$ defined on a neighbourhood V of z_0, such that $\eta(z)$ is the goedesic containing $\{x \in U \mid G(z,x) = w\}$. Assuming that (ii) η is non-constant, the constructions of Theorem 3.1 and Proposition 2.4 then coincide. We shall find both approaches useful in the sequel.

4. Multivalued harmonic morphisms from $\mathbb{R}^3$.

As explained in §3, given a surjective submersive harmonic morphism $\phi: U \subset \mathbb{R}^3 \to N^2$ with connected fibres from a convex open subset of $\mathbb{R}^3$ to a Riemann surface N^2, we obtain a non-constant holomorphic map $\eta: N^2 \to \mathcal{G}_{\mathbb{R}^3}$. The map η can be represented by two meromorphic functions g, h on N^2 satisfying conditions (3.2). Conversely, such a pair (g, h) determines η and thus a multivalued harmonic morphism. We can therefore carry out the construction of Theorem 3.1 to obtain a 3-dimensional Riemannian manifold $\tilde{\mathbb{R}}^3$ covering $\mathbb{R}^3$ (or a subset thereof) and a harmonic morphism $\psi: \tilde{\mathbb{R}}^3 \to N^2$, such that every local solution to (3.1) with $x \in \mathbb{R}^3$ and $z \in N^2$, is covered by ψ. Indeed $\tilde{\mathbb{R}}^3$ is the submanifold of $N^2 \times \mathbb{R}^3$ given by equation (3.3) away from poles of g and, as usual, the covering map $\pi: \tilde{\mathbb{R}}^3 \to \mathbb{R}^3$ and the harmonic morphism $\psi: \tilde{\mathbb{R}}^3 \to N^2$ are the restrictions of the canonical projections. We note the following:

PROPOSITION 4.1. *The harmonic morphism $\psi: \tilde{\mathbb{R}}^3 \to N^2$ is a trivial principal $\mathbb{R}$-bundle over N^2.*

PROOF. First note that the fibre of ψ at $z \in N^2$ is the line $\{z\} \times L_z \subset N^2 \times \mathbb{R}^3$, where L_z is the line in $\mathbb{R}^3$ defined by $\eta(z)$. As in §3, for each $z \in N^2$, let $\gamma(z) = \sigma^{-1} \circ g(z)$ and $c(z) = d\sigma_g^{-1} \circ h(z)$. Then L_z is the line in $\mathbb{R}^3$ with direction $\gamma(z)$ and "origin" $c(z) \in T_{\gamma(z)}S^2$. Now we can define the action of $t \in \mathbb{R}$ on $\tilde{\mathbb{R}}^3$ to be the translation through $t\gamma(z)$, so $\psi: \tilde{\mathbb{R}}^3 \to N^2$ is a principal $\mathbb{R}$-bundle. Further $(z, t) \mapsto c(z) + t\gamma(z)$ provides a global trivialization of it.

For the sake of examples, note that we have the following explicit formulae:

$$\gamma = \sigma^{-1}(g) = \left(\frac{2g}{1 + |g|^2}, \frac{1 - |g|^2}{1 + |g|^2}\right) \in \mathbb{C} \times \mathbb{R} = \mathbb{R}^3,$$

$$c = 2\left(\frac{h - \bar{h}g^2}{(1 + |g|^2)^2}, \frac{\bar{h}g + h\bar{g}}{-(1 + |g|^2)^2}\right) \in \mathbb{C} \times \mathbb{R} = \mathbb{R}^3.$$

Further, the envelope $\tilde{E}$, and so $E = d\pi(\tilde{E})$, is determined by solving the equations

$$G(z, x) = 0, \tag{4.1}$$

$$\frac{\partial G}{\partial z}(z, x) = 0 \tag{4.2}$$

simultaneously with the usual replacement of G by G/g^2 near the poles of g. We can now give some examples. For some of these examples the projection π is a branched covering in the following sense (note that most authors require further conditions on a branched covering, cf. [Rol]):

DEFINITION 4.2. We call a smooth map $\pi:\tilde{M}^3 \to M^3$ a *branched covering* if it is a local diffeomorphism except on a submanifold $\tilde{E}$ of $\tilde{M}^3$ of codimension 2. Furthermore, we say that π is *branched at* $p \in \tilde{E}$ *with branching order* k if there exist local coordinates $(z,t) \in \mathbb{C} \times \mathbb{R}$ centred on p and $(w,t) \in \mathbb{C} \times \mathbb{R}$ centred on $\pi(p)$ such that π has the form

$$(z,t) \mapsto (z^k, t). \tag{4.3}$$

EXAMPLE 4.3. (Orthogonal Projection). Let $M^3 = \mathbb{R}^3$, $N^2 = \mathbb{C}$ and $g,h:\mathbb{C} \to \mathbb{C}$ be given by $g:z \mapsto 0$ and $h:z \mapsto z/2$. Then equation (3.3) becomes $x_1 + ix_2 - z = 0$. This has a global solution the single -valued harmonic morphism $z = x_1 + ix_2$ which is an orthogonal projection. This means that the covering manifold $\tilde{\mathbb{R}}^3 = \{(z,(x_1,x_2,x_3)) \in \mathbb{C} \times \mathbb{R}^3 \mid x_1 + ix_2 = z\}$ is a 3-dimensional subspace of $\mathbb{R}^5$. The envelope $\tilde{E} \subset \tilde{\mathbb{R}}^3$ is the empty set. The covering harmonic morphism $\psi:\tilde{\mathbb{R}}^3 \to \mathbb{C}$ can be thought of as an orthogonal projection $\mathbb{R}^3 \to \mathbb{R}^2$, where the metric on the horizontal spaces has been multiplied by the constant factor 2.

EXAMPLE 4.4. (Radial Projection). Let $M^3 = \mathbb{R}^3$, $N^2 = \mathbb{C} \cup \{\infty\} = S^2$ and $g,h:\mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ be given by $g:z \mapsto z$ and $h:z \mapsto 0$. Equation (4.1) becomes

$$(x_1 - ix_2)z^2 + 2x_3 z - (x_1 + ix_2) = 0. \tag{4.4}$$

Solving this equation gives two local solutions $\sigma^{-1} \circ z^{\pm}: \mathbb{R}^3 \backslash \{0\} \to S^2$ given by $x \mapsto \pm x/|x|$ which are radial projection and its negative, well-known harmonic morphisms, see [Fug]. We can think of (4.4) as defining a multivalued harmonic morphism $z(x)$, 2-valued away from 0, with these local solutions as branches.

It is easily seen that the covering manifold is

$$\tilde{\mathbb{R}}^3 = \{(v,x) \in S^2 \times \mathbb{R}^3 \mid x = tv \quad \text{for some} \quad t \in \mathbb{R}\},$$

that is, the tautological bundle over S^2; this has an explicit trivialization $S^2 \times \mathbb{R} \to \tilde{\mathbb{R}}^3$ given by $(v,t) \mapsto (v,tv)$. The covering harmonic morphism ψ is the projection map of this bundle. The projection $\pi:\tilde{\mathbb{R}}^3 \to \mathbb{R}^3$ is a double cover with $\pi(v,x) = \pi(-v,x)$ except on the envelope $\tilde{E} = S^2 \times \{0\}$. This means that the geometric envelope E is the single point $\{0\}$. Thus on passing from $\mathbb{R}^3$ to $\tilde{\mathbb{R}}^3$ the origin $0 \in \mathbb{R}^3$ is "blown up" to an S^2.

EXAMPLE 4.5. (The Outer Disk Family). Let $M^3 = \mathbb{R}^3$, $N^2 = \mathbb{C} \cup \{\infty\} = S^2$ and $g,h:\mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ be given by $g:z \mapsto z$ and $h:z \mapsto irz$ for some $r \in \mathbb{R}^+$. Equation (3.3) is now:

$$(1 - z^2)x_1 + i(1 + z^2)x_2 - 2zx_3 - 2irz = 0. \tag{4.5}$$

We can think of this as defining a multivalued harmonic morphism $z(x)$ which is

2-valued except on the zero set of the discriminant of this quadratic equation in z where the two values coincide. This set gives the geometric envelope $E = \{x \in \mathbb{R}^3 \mid x_1^2 + x_2^2 = r^2 \text{ and } x_3 = 0\}$. It is the circle of radius r in the (x_1, x_2)-plane, whose centre is the origin $0 \in \mathbb{R}^3$. In this case $\tilde{E} = \pi^{-1}(E)$, and $\tilde{E}$ is also a circle. Outside E equation (3.3) has the two solutions:

$$z_r^{\pm} = \frac{-(x_3 + ir) \pm \sqrt{x_1^2 + x_2^2 + x_3^2 - r^2 + 2irx_3}}{x_1 - ix_2}. \tag{4.6}$$

If we choose $\sqrt{\ }$ to be the principal square root ($\sqrt{\rho e^{i\theta}} := \sqrt{\rho} e^{i\theta/2}$, where $\theta \in (-\pi, \pi)$) on $\mathbb{C} \backslash \mathbb{R}_0^-$, then we obtain two harmonic morphisms z_r^+ and z_r^- defined on $\mathbb{R}^3 \backslash D_r$, where $D_r := \{x \in \mathbb{R}^3 \mid x_1^2 + x_2^2 \leq r^2 \text{ and } x_3 = 0\}$ is the disk in the (x_1, x_2)-plane with E as its boundary. For every $(x_1, x_2, x_3) \in \mathbb{R}^3 \backslash D_r$ we have

$$z_r^+(x_1, x_2, x_3) = \overline{z_r^-(-x_1, x_2, -x_3)}$$

so z_r^+ and z_r^- are, up to isometries of $\mathbb{R}^3 \backslash D_r$ and S^2, the same map. We call $\{z_r^+ : \mathbb{R}^3 \backslash D_r \to S^2 \mid r \in \mathbb{R}^+\}$ the *outer disk family*.

The map $(\gamma, c) : S^2 \to TS^2 \subset (\mathbb{C} \times \mathbb{R})^2$ is given by

$$(\gamma, c) : \sigma^{-1}(z) \mapsto \left(\left(\frac{2z}{1 + |z|^2}, \frac{1 - |z|^2}{1 + |z|^2} \right), \left(\frac{2irz}{1 + |z|^2}, 0 \right) \right).$$

In spherical polar coordinates $(\theta, t) \in [0, 2\pi] \times [-\pi/2, \pi/2] \mapsto (e^{i\theta} \cos t, \sin t) \in S^2$, this reads

$$(\gamma, c) : (\theta, t) \mapsto ((e^{i\theta} \cos t, \sin t), (ire^{i\theta} \cos t, 0)).$$

The covering manifold $\tilde{\mathbb{R}}^3$ is therefore parametrized by

$$(\theta, t, s) \mapsto ((e^{i\theta} \cos t, \sin t), (ire^{i\theta} \cos t, 0) + s(e^{i\theta} \cos t, \sin t))$$

where $\theta \in [0, 2\pi]$, $t \in [-\pi/2, \pi/2]$ and $s \in \mathbb{R}$. The fibre of z_r^+ at $(e^{i\theta} \cos t, \sin t) \in S^2$ is

$$s \mapsto L(\theta, t, s) := (ire^{i\theta} \cos t, \sin t), \quad s \in \mathbb{R}^+, \tag{4.7}$$

and the fibres of z_r^- are given by the same formula with $s \in \mathbb{R}^-$. Note that z_r^+ and z_r^- map a point $(ire^{i\theta} \cos t, 0) + s(e^{i\theta} \cos t, \sin t) \in \mathbb{R}^3 \backslash D_r$, to $(e^{i\theta} \cos t, \sin t) \in S^2$ so they are both surjective.

Note that as $r \to 0$, $L(\theta, t, s) \to s(e^{i\theta} \cos t, \sin t)$, which shows that the foliation of half-lines given by the fibres of z_r^+ approaches the corresponding foliation for the radial projection.

Let $(x, 0)$ be an arbitrary point in $\mathbb{C} \times \mathbb{R}$. Then for $|x| \leq r$ setting $t = \cos^{-1}(|x|/r)$, we see that the fibres f^u and f^l of z_r^+ in the upper and lower half spaces, with boundaries $\partial f^u = \partial f^l = (x, 0)$ are orthogonal to the radius from $(0, 0)$

to $(x, 0)$ and make an angle t with the (x_1, x_2)-plane. As $|x|$ increases from 0 to r, t decreases from $\pi/2$ to 0. The fibres through a point $(x, 0)$ with $|x| > r$ lie in the (x_1, x_2)-plane and are tangent to the envelope E. Note that the direction of the fibres changes discontinuously as we cross the disk D_r.

Keeping x fixed and letting $r \to \infty$, we have that $t \to \pi/2$. In this sense the foliation of z_r^+ approaches the corresponding one for the orthogonal projection. Thus the outer disk family "interpolates" between the radial projection and orthogonal projection.

It is easily seen that π is a branched covering with branching order 2 on the circle $\tilde{E}$.

It follows from Proposition 4.1 that the covering manifold $\tilde{\mathbb{R}}^3$ is homeomorphic to $S^2 \times \mathbb{R}$. This can also be seen directly as follows: The map $L^+ : S^2 \times \mathbb{R}^+ \to \mathbb{R}^3 \backslash D_r$ given by (4.6) is a homeomorphism, and so is $L^- : S^2 \times \mathbb{R}^- \to \mathbb{R}^3 \backslash D_r$ given by the same formula. Extending these maps continuously to $S^2 \times \{0\}$ and glueing them together along this manifold gives a homeomorphism $L : S^2 \times \mathbb{R} \to \tilde{\mathbb{R}}^3$. We can thus think of $\tilde{\mathbb{R}}^3$ as being obtained by glueing the two copies of $\mathbb{R}^3 \backslash D_r$ across D_r in an analogous way to that used in Riemann surface theory, the disk D_r playing the role of a cut joining two branch points. Indeed $\partial D_r = E$ and D_r is a *Seifert surface* in the sence of [Rol] where this procedure of glueing across a Seifert surface to obtain a branched covering is discussed. Note finally that the covering harmonic morphism $\psi : \tilde{\mathbb{R}}^3 \to S^2$ is, via the homeomorphism L, just natural projection $S^2 \times \mathbb{R} \to S^2$, showing how the two branches $z_r^\pm$ can be glued together to form ψ.

Note that the solution $z_r^+ : \mathbb{R}^3 \backslash D_r \to S^2$ of (4.5) satisfies the following properties:

(1) it is surjective,
(2) it has connected fibres, and
(3) no two fibres are parallel as oriented line segments.

This is also true for the radial projections of Example 4.4. The following result shows that, up to equivalence, any harmonic morphism satisfying these conditions is one of these two examples.

THEOREM 4.6. *Let $\phi : U \to N^2$ be a harmonic morphism from an open subset of $\mathbb{R}^3$ to a closed Riemann surface, satisfying conditions* (4.8). *Then N^2 is conformally equivalent to S^2 and, up to isometries of $\mathbb{R}^3$ and conformal transformations of S^2, ϕ is a restriction of radial projection or a solution to* (4.5) *on a suitable domain U.*

REMARK. Conditions (2) and (3) can be replaced by the more general conditions: (2)′ for each $z \in N^2$, $\phi^{-1}(z)$ is contained in a single oriented line L_z of $\mathbb{R}^3$ with the correct orientation induced on each component, (3)′ no two lines L_z are parallel as oriented lines.

PROOF. Condition (2) (or (2)′) implies that ϕ is submersive, otherwise by [Bai-Woo-2], it would be locally of the form $\rho \circ \tilde{\phi}$ where $\tilde{\phi}$ is submersive and ρ of the form $z \mapsto z^k$. Such a composition clearly does not have connected fibres. Then, as in §3, we obtain a holomorphic map $\eta : N^2 \to \mathcal{G}_{\mathbf{R}^3}$ with $\eta(z) \cap U = \phi^{-1}(z)$ (or $\eta(z) = L_z$ in the case of condition (2)′). From the interpretation of g as giving, via stereographic projection σ, the direction of the fibres, condition (3) or (3)′ tells us that $\sigma^{-1} \circ g : N^2 \to S^2$ is injective and holomorphic, so bijective. Thus, up to composition with a conformal transformation, $N^2 = S^2$ and g is the identity map $g(z) = z$. Since h represents the holomorphic vector field c on S^2, $h(z)$ must be given by a quadratic polynomial in z. This can be written in the form

$$h(z) = \langle v(z), \bar{p} \rangle_{\mathbf{C}},$$

where $v(z) := (1 - z^2, i(1 + z^2), 2z)$ for some p of $\mathbf{C}^3$ where $\langle , \rangle_{\mathbf{C}}$ denotes the symmetric bilinear inner product on $\mathbf{C}^3$ given by $\langle z, w \rangle := z_1 w_1 + z_2 w_2 + z_3 w_3$. Then equation (4.1) can be written in the neat form

$$\langle v(z), x - \bar{p} \rangle_{\mathbf{C}} = 0. \tag{4.9}$$

Choose $A \in \mathrm{SO}(3)$, such that $A \cdot \mathrm{Im}(p) = (0, 0, r)$ for some $r \geqq 0$. Then, on applying A, equation (4.9) becomes:

$$\langle A \cdot v(z), A \cdot (x - \mathrm{Re}(p)) + (0, 0, ir) \rangle_{\mathbf{C}} = 0. \tag{4.10}$$

Now $\tilde{v} : \mathbf{C} \cup \{\infty\} = S^2 \to Q_1$ given by $\tilde{v} : z \mapsto [v(z)]$ is simply the standard identification of $S^2 = \mathbf{C} \cup \{\infty\}$ with the quadratic $Q_1 := \{[z_1, z_2, z_3] \in \mathbf{C}P^2 \mid z_1^2 + z_2^2 + z_3^2 = 0\}$ as in [Hof-Oss]. The linear map A defines an isometry $Q_1 \to Q_1$ which via the identification $\tilde{v}$ defines conformal map $B = \tilde{v}^{-1} \circ A \circ \tilde{v} : S^2 \to S^2$, which is in fact an isometry. Setting $B := \pi^{-1}(A)$ where $\mathrm{SU}(2) \xrightarrow{\pi} \mathrm{SO}(3)$ is the standard double cover, gives a matrix B of the form

$$B = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix},$$

so that $B : z \mapsto (az + b)/(-\bar{b}z + \bar{a})$. Thus $A \cdot \tilde{v}(z) = \tilde{v}(B(z))$. Set $w := B(z)$ and $y := A \cdot (x - \mathrm{Re}(p))$. Then equation (4.10) reads $\langle v(w), y + (0, 0, ir) \rangle_{\mathbf{C}} = 0$, which is equation (4.5) for $r > 0$, or (4.4) for $r = 0$. In the latter case it is clear that ϕ must be a restriction of radial projection or its negative. (For a more explicit proof see [Gud]).

REMARKS 4.7. (i) Whether a solution to (4.5) on a given domain U satisfies conditions (1), (2) and (3) of (4.8) depends on the domain. For example if we choose a different square root in Example 4.3, for instance the one defined on $\mathbf{C} \setminus \mathbf{R}_0^+$, then we obtain different harmonic morphisms

$$\hat{z}_r^+, \hat{z}_r^- : \mathbb{R}^3 \setminus C_r \to S^2$$

where $C_r := \{x \in \mathbb{R}^3 \mid r^2 \leqq x_1^2 + x_2^2 \text{ and } x_3 = 0\}$. We call $\{\hat{z}_r^+ : \mathbb{R}^3 \setminus C_r \to S^2 \mid r \in \mathbb{R}^+\}$ the *inner disk family*. $\hat{z}_1^+ : \mathbb{R}^3 \setminus C_1 \to S^2$ is the map described in Example 2.9 of [Ber-Cam-Dav], and in [Bai-1], [Bai-Woo-1] as the disk example. The image $\hat{z}_r^+(\mathbb{R}^3 \setminus C_r) \subset S^2$ of $\hat{z}_r^+$ is the upper hemisphere, so $\hat{z}_r^+$ is not surjective. Note that, in contrast to the outer disk family, the direction of the fibres of $\hat{z}_r^+$ changes continuously when crossing the disk D_r but discontinuously when crossing C_r, this being another possible Seifert surface for the branched covering $\pi : \tilde{\mathbb{R}}^3 \to \mathbb{R}^3$. More generally, we can find solutions to (4.8) on $\mathbb{R}^3 \setminus S_r$ for any Seifert surface S_r, i.e. any surface with boundary the envelope E, the resulting harmonic morphisms having very different properties according as S_r is bounded or not.

(ii) On the way we have shown that, if $N^2 = S^2$, $g(z) = z$ and $h(z)$ is a quadratic polynomial in z, then (3.3) has solutions satisfying conditions (4.8). This is not true, for example if $N^2 = S^2$, $g(z) = z$ and $h(z) = z^k$ $(k \geqq 3)$. In this case (3.3) can have no solution which is a surjective harmonic morphism to S^2, since h cannot represent a vector field at $z = \infty$; indeed only quadratic polynomials in z define vector fields globally on S^2.

EXAMPLE 4.8. Let $M^3 = \mathbb{R}^3, N^2 = \mathbb{C}$ and $g, h : \mathbb{C} \to \mathbb{C}$ be given by $g : z \mapsto z$ and $h : z \mapsto z^k/2$ for some $k \geqq 3$. Then $G : \mathbb{C} \times \mathbb{R}^3 \to \mathbb{C}$ is given by $G : (z, x) \mapsto z^k + (x_1 - ix_2)z^2 + 2x_3 z - (x_1 + ix_2)$. The projection π is therefore a k-fold covering of $\mathbb{R}^3$ except on the envelope $\tilde{E}$. As before one could use the map $(\gamma, c) : \mathbb{C} \to TS^2$ to get a parametrization of $\tilde{\mathbb{R}}^3$ which we know by Proposition 4.1 is homeomorphic to $\mathbb{C} \times \mathbb{R} = \mathbb{R}^3$.

We are mainly interested in the geometric envelope $E \subset \mathbb{R}^3$, for which we give a complete parametrization for any $k \geqq 3$. As before the envelope $\tilde{E}$ is given by solving two simultaneous equations:

$$G(z, x) \equiv -z^k - (x_1 - ix_2)z^2 - 2x_3 z + (x_1 + ix_2) = 0 \tag{4.10}$$

and

$$\frac{\partial G}{\partial z}(z, x) \equiv -kz^{k-1} - 2(x_1 - ix_2)z - 2x_3 = 0. \tag{4.11}$$

It is easily shown that the geometric envelope $E = \pi(\tilde{E})$ lies in the union of $(k-1)$ 2-planes in $\mathbb{C} \times \mathbb{R}$ given by $P_n := \operatorname{span}_{\mathbb{R}} \{(0, 1), e^{in(\pi/(k-1))}, 0)\}$, where $n \in \{0, 1, \ldots, k-2\}$. For $n \in \{0, 1, \ldots, 2k-3\}$ define $\tau_n : \mathbb{C} \times \mathbb{C} \times \mathbb{R} \to \mathbb{C} \times \mathbb{C} \times \mathbb{R}$ by

$$\tau_n : (z, w, x) \mapsto (e^{in(\pi/(k-1))} z, (-1)^n e^{in(\pi/(k-1))} w, (-1)^n x),$$

then $\{\tau_n \mid n \in \{0, 1, \ldots, 2k-3\}\}$ is a cyclic group of isometries of $\mathbb{C} \times \mathbb{C} \times \mathbb{R}$ of order $2(k-1)$. Easy calculations show that $p \in \tilde{E} \cap P_0$ if and only if $\tau_n(p) \in \tilde{E} \cap P_{(n \bmod (k-1))}$. This means that if one knows the part of the geometric envelope in the plane $P_0 \subset \mathbb{C} \times \mathbb{R}$, then one obtains the rest by applying the maps τ_n to $E \cap P_0$. One can show that $E \cap P_0$ is parametrized by $(x, y): \mathbb{R} \to \mathbb{R}^2$ with

$$\begin{pmatrix} x \\ y \end{pmatrix}(t) = \frac{1}{2(1+t^2)} \begin{pmatrix} (k-2)t^{k+1} - kt^{k-1} \\ 2(1-k)t^k \end{pmatrix}.$$

In particular, E consists of $2(k-1)$ such curves meeting at the origin, and so is not a manifold, not even topologically. It follows that $\tilde{E}$ cannot be a manifold at the origin. For the details of the above computations, see [Gud].

5. Multivalued harmonic morphisms from S^3 and H^3.

Recall from §3 that a surjective submersive harmonic morphism from a convex open subset U of S^3 to a Riemann surface determines a non-constant holomorphic map $\eta: N^2 \to \mathscr{G}_{S^3} = \tilde{G}_2(\mathbb{R}^4) \cong S^2 \times S^2$. The components of η are meromorphic functions f, g on N^2.

Conversely, such a pair f, g determines a map η so we can carry out the construction of Theorem 3.1 to obtain a 3-dimensional Riemannian manifold $\tilde{S}^3$ (or a subset thereof) and a harmonic morphism $\psi: \tilde{S}^3 \to N^2$ such that every local solution to (3.1) with $x \in S^3$ and $z \in N^2$ is covered by ψ. In terms of f and g, the condition $x \in \eta(z)$ reads

$$\langle \xi(z), x \rangle = 0 \tag{5.1}$$

where

$$\xi := s(f, g) := (1 + fg, i(1 - fg), f - g, -i(f + g)). \tag{5.2}$$

Here $\langle , \rangle_{\mathbb{C}}$ denotes the symmetric $\mathbb{C}$-bilinear inner product on $\mathbb{C}^4$ given by

$$\langle z, w \rangle := z_1 w_1 + z_2 w_2 + z_3 w_3 + z_4 w_4.$$

Note that if $P := \{\text{poles of } f\} \cup \{\text{poles of } g\}$ then setting $G(z, x) := \langle \xi(z), x \rangle$, $G: (N^2 \setminus P) \times S^3 \to \mathbb{C}$ is a harmonic morphism in each variable separately. The manifold $\tilde{S}^3$ is the submanifold of $N^2 \times S^3$ given by equation (5.1) away from poles of f or g. As before, dividing by f and/or g yields an equation valid near such poles.

PROPOSITION 5.1. *For any meromorphic functions f, g on a Riemann surface N^2, the covering harmonic morphism $\psi: (z, x) \mapsto z$ is a principal S^1-bundle over N^2 of degree $d = \deg f - \deg g$.*

PROOF. The fibres of ψ are closed geodesics of $\tilde{S}^3$ all of length 2π, indeed $\psi^{-1}(z) = \{z\} \times C_z$, where C_z is the great circle of S^3 given by equation (5.1). Such fibres have a natural orientation, so that, together with $\operatorname{Re}\xi(z)$ and $\operatorname{Im}\xi(z)$, they give an oriented basis for $\{z\} \times S^3$. We define the action of $e^{i\theta} \in S^1$ on each fibre of $\tilde{S}^3$ as rotation through $+\theta$. Thus $\psi : \tilde{S}^3 \to N^2$ is a principal bundle.

Regarding the degree, let us form the associated $\mathbb{R}^2$-bundle, $\tilde{\psi} : \tilde{\mathbb{R}}^4 \to N^2$, given as for $\tilde{S}^3$ by equation (5.1) but with $x \in \mathbb{R}^4$; this is naturally oriented and $\psi : \tilde{S}^3 \to N^2$ is its unit circle bundle. Regarding the degree of ψ, note that by Lemma 5.2, $\operatorname{Re}\xi^{\perp}/|\operatorname{Re}\xi^{\perp}|$ defines a section of $\psi : \tilde{S}^3 \to S^2$ away from zeros of f and g. We may calculate the degree of the bundle $\psi : \tilde{S}^3 \to S^2$ as the obstruction to extending $\operatorname{Re}\xi^{\perp}/|\operatorname{Re}\xi^{\perp}|$ over those zeros, see [Ste]. Checking orientation, it is seen that a zero of f of order k and a zero of g of order l (possibly at the same point) contributes k and $-l$, respectively, to this obstruction, hence the result.

LEMMA 5.2. *For the above situation, set*

$$\xi^{\perp} := \frac{1}{f} s\left(f, -\frac{1}{\bar{g}}\right) = \left(\frac{1}{f} - \frac{1}{\bar{g}}, i\left(\frac{1}{f} + \frac{1}{\bar{g}}\right), 1 + \frac{1}{f\bar{g}}, -i\left(1 - \frac{1}{f\bar{g}}\right)\right).$$

Then $\operatorname{Re}\xi^{\perp}$ *and* $\operatorname{Im}\xi^{\perp}$ *form an oriented basis for the associated* $\mathbb{R}^2$*-bundle* $\tilde{\psi} : \tilde{\mathbb{R}}^4 \to N^2$, *away from zeros of* f *and* g.

PROOF. First note that from equation (5.1) it follows that $\operatorname{Re}\xi$ and $\operatorname{Im}\xi$ form a basis for the horizontal space H_z^{ψ} of ψ at z. It is easily shown that this basis is oriented.

Next, a short calculation shows that $\langle \xi, \xi^{\perp} \rangle = \langle \xi, \overline{\xi^{\perp}} \rangle = 0$, so that $\{\operatorname{Re}\xi^{\perp}, \operatorname{Im}\xi^{\perp}\}$ is an orthogonal basis for the vertical space $V^{\tilde{\psi}}$ of $\tilde{\psi} : \tilde{\mathbb{R}}^4 \to N^2$.

Finally, a lengthy calculation (done using REDUCE) shows that the determinant of the matrix, made up of the four vectors $\operatorname{Re}\xi$, $\operatorname{Im}\xi$, $\operatorname{Re}\xi^{\perp}$ and $\operatorname{Im}\xi^{\perp}$ is positive, so that $\operatorname{Re}\xi^{\perp}$ and $\operatorname{Im}\xi^{\perp}$ is an *oriented* basis for the vertical space $V^{\tilde{\psi}}$ of $\tilde{\psi} : \mathbb{R}^4 \to N^2$ as claimed.

REMARK. The map $\tilde{\psi} : \tilde{\mathbb{R}}^4 \to N^2$ is itself a harmonic morphism. In fact it is the harmonic morphism covering the local solutions $\phi : U \subset R^4 \to N^2$ of (5.1), where now $x \in \mathbb{R}^4$ and $z \in N^2$. These are all harmonic morphisms with fibres which are parts of planes through the origin, see §7.

In the following examples it is convenient to write $w_1 := x_1 + ix_2$, $w_2 := x_3 + ix_4$, so that equation (5.1) reads:

$$w_1 + f(z)g(z)\bar{w}_1 - g(z)w_2 + f(z)\bar{w}_2 = 0. \tag{5.3}$$

EXAMPLE 5.3. Let $M^3 = S^3$, $N^2 = \mathbb{C} \cup \{\infty\} = S^2$ and let $f, g : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ be given by $g : z \mapsto z^k$ ($k \in \mathbb{N}^+$) and $f : z \mapsto 0$. Then equation (5.3) reads

$$w_1 - z^k w_2 = 0. \tag{5.4}$$

In the case $k > 1$ we can think of this as defining a multivalued harmonic morphism $z(w_1, w_2)$ on S^3 that is k-valued except on the circles $w_1 = 0$ and $w_2 = 0$ which thus form the geometric envelope E. By Proposition 5.1, the covering harmonic morphism $\psi : \tilde{S}^3 \to S^2$ is an S^1 bundle of degree $-k$. This can be described as the projection of a lens space $L(k, 1)$, see [Ste] §26. Indeed, we have the following diagram:

$$\begin{array}{ccc} & \tilde{S}^3 = L(k,1) & \\ {}^{\pi}\swarrow & & \searrow^{\psi} \\ S^3 \supset U & \xrightarrow{\phi} & S^2 \\ {}_{H}\searrow & & \swarrow_{z \mapsto z^k} \\ & S^2 & \end{array}$$

The projection π is a k-sheeted branched covering, branched over the circles $E : w_1 = 0,\ w_2 = 0$ of S^3. The outer square commutes. For any local solution $\phi : U \subset S^3 \to S^2 = \mathbf{C} \cup \{\infty\}, (w_1, w_2) \mapsto z$ the top triangle commutes on $\pi^{-1}(U)$. If we add in the map $z \mapsto z^k$ and the Hopf map H we see that the bottom triangle commutes on U and we are led to Baird's description of ψ as obtained from H by simultaneously cutting and pasting k copies of S^3 and S^2, see Example 2.5 of [Bai-2].

In the case $k = 1$, equation (5.4) has a unique solution $z = w_1/w_2$, defined globally on the whole of S^3. The geometric envelope E is empty and $\pi : \tilde{S}^3 \to S^3$ is a diffeomorphism, so that $\tilde{S}^3$ is diffeomorphic to the 3-sphere. It follows from equations (2.7) that the metric on $\tilde{S}^3$ is obtained from the standard one on S^3 by multiplying lengths by $\sqrt{5}$ on the horizontal spaces, thus $\tilde{S}^3$ is homothetic to a Berger sphere and $\psi : \tilde{S}^3 \to S^2$ is a Hopf map from this deformed sphere.

EXAMPLE 5.4. Let $M^3 = S^3$, $N^2 = \mathbf{C} \cup \{\infty\} = S^2$ and let $f, g : \mathbf{C} \cup \{\infty\} \to \mathbf{C} \cup \{\infty\}$ be given by $f : z \mapsto 1/z^k$ $(k \in \mathbf{N}^+)$ and $g : z \mapsto \infty$. Then equation (5.3) reads:

$$\frac{1}{z^k}\bar{w}_1 - w_2 = 0.$$

If $k = 1$, this has the unique solution $z = \bar{w}_1/w_2$, the conjugate Hopf map $\bar{H}$. For general k, the covering harmonic morphism $\psi : \tilde{S}^3 \to S^2$ has degree $+k$, so again $\tilde{S}^3$ is diffeomorphic to a lens space $L(k, 1)$ and we obtain a similar description to that of Example 5.3 replacing H by $\bar{H}$.

EXAMPLE 5.5. Let $M^3 = S^3$, $N^2 = \mathbf{C} \cup \{\infty\} = S^2$ and let $f, g : \mathbf{C} \cup \{\infty\} \to \mathbf{C} \cup \{\infty\}$ be given by $f : z \mapsto -z$, and $g : z \mapsto z$, then equation (5.3) becomes:

$$\bar{w}_1 z^2 + (w_2 + \bar{w}_2)z - w_1 = 0.$$

This can be thought of as defining a multivalued harmonic morphism $z(w_1, w_2)$ which is 2-valued except at the poles $(0,0,0,\pm 1)$, cf. Example 4.4. Solving this equation gives two solutions $\phi^{\pm}: S^3 \setminus \{(0,0,0,1), (0,0,0,-1)\} \to S^2$, with $\phi^{\pm}(x) = \sigma^{-1} \circ z^{\pm}(x) = \pm(x_1,x_2,x_3)/\sqrt{x_1^2 + x_2^2 + x_3^2}$. The harmonic morphism ϕ^+ can be described as orthogonal projection along geodesics through the poles $(0,0,0,\pm 1)$ to the equatorial great sphere. It is a sort of radial projection in the sense of [Bai-Woo-2]. It can be shown that $\tilde{S}^3$ is isometric to $S^2 \times S^1$, equipped with a warped product metric. The projection $\pi: \tilde{S}^3 = S^2 \times S^1 \to S^3$ collapses $S^2 \times \{\pm \pi/2\}$ to the poles, but is otherwise a 2:1 covering. The covering harmonic morphism $\psi: S^2 \times S^1 \to S^2$ is the natural projection confirming that it is an S^1-bundle of degree 0.

REMARK 5.6. Note that Examples 5.3, 5.4 and 5.5 give explicit examples of how any topological S^1-bundle over S^2 can be given a metric such that its projection map $\psi: \tilde{S}^3 \to S^2$ is a harmonic morphism. It is a general result of Baird and the second author that the natural projection of any S^1-bundle over a surface or, more generally, of any Seifert fibre space without reflections, is a harmonic morphism with respect to suitable metrics. Conversely, given any non-constant harmonic morphism from a closed 3-manifold to a surface the fibres of ϕ give M^3 the structure of a Seifert space without reflections. For this see [Bai-Woo-3].

EXAMPLE 5.7. Let $M^3 = S^3$ and $N^2 = \mathbf{C} \cup \{\infty\} = S^2$ and let $f, g: \mathbf{C} \cup \{\infty\} \to \mathbf{C} \cup \{\infty\}$ be given by $f: z \mapsto z^2$, and $g: z \mapsto z$. Then (5.3) defines a multivalued harmonic morphism, 3-valued outside E where two of the three values coincide. By Proposition 5.2, the covering harmonic morphism $\psi: \tilde{S}^3 \to S^2$ is an S^1-bundle of degree one, so $\tilde{S}^3$ is diffeomorphic to S^3. The envelope $\tilde{E}$ is given by solving simultaneously the equations

$$G(z,x) = 0 \quad \text{and} \quad \frac{\partial G}{\partial z}(z,x) = 0,$$

where $z \in N^2$ and $x \in S^3$ and $G(z,x) = \langle \xi(z), x \rangle$. Short calculations show that the geometric envelope $E = \pi(\tilde{E})$ is given by:

$$8\,|w_1|^4 + 20\,|w_1|^2 - 1 = 0 \quad \text{and} \quad \operatorname{Im} w_1 w_2^3 = 0.$$

It follows that E is a (2, 6)-link on the torus given by:

$$|w_1|^2 = \frac{\sqrt{27} - 5}{4} \in (0,1).$$

Thus $\pi: \tilde{S}^3 \to S^3$ exhibits an S^3 as a 3-sheeted branched cover of (another) S^3 branched over this (2, 6)-link.

Further investigation shows that this is a irregular branched cover (i.e. the group of covering transformations of the associated unbranched cover does not act transitively on the fibres) with each connected component of E covered by two circles, one of which belongs to $\tilde{E}$. Near a point of the latter, π exhibits branching of order 2 i.e. it has the form $(z, t) \mapsto (z^2, t)$ in some suitable coordinates (cf. §6). That such a branched cover must be homeomorphic to S^3 follows from the work of J. M. Montesinos [Mon]. Using his move D_2 one can reduce the (2, 6)-link to a pair of unlinked circles for which we know that the branched cover must be an S^3. We thank Joan Birman for instruction in this matter.

The case $M^3 = H^3$ can be treated similarly: we give no examples here.

6. The behaviour of π on the envelope, for $M^3 = \mathsf{R}^3$, S^3 and H^3.

Let M^3 be a 3-dimensional simply-connected space form R^3, S^3 or H^3 and let $\eta: N^2 \to \mathcal{G}_{M^3}$ be a non-constant holomorphic map from a Riemann surface to the space of oriented geodesics of M^3. Let Φ be the multivalued harmonic morphism from M^3 to N^2 defined by η (see Remark 3.3(i)) with covering manifold $\tilde{M}^3$ and projection $\pi: \tilde{M}^3 \to M^3$. In this section we discuss the form of the projection π near to a point on the envelope $\tilde{E}$. Recall from §3 that $\tilde{E}$ is the real analytic subset of $\tilde{M}^3$ where π fails to be a local diffeomorphism. As usual we write $E = d\pi(\tilde{E})$ for the geometric envelope.

DEFINITION 6.1. A non-constant holomorphic map $\eta: N^2 \to \mathcal{G}_{M^3}$ is called a *generalized radial projection* if all geodesics $\eta(z)$, where $z \in N^2$, pass through the same point $x_0 \in M^3$.

Note that: (i) by Theorem 4.7. of [Bai-Woo-2], any local solution of a generalized radial projection η will be a radial projection, or the restriction thereof, up to post-composition with conformal mappings, (ii) $\tilde{E}$ is always of (not necessarily pure) dimension 2 containing one or two spheres corresponding to the blow-ups of each point through which all the geodesics pass through together with the geodesics $\eta(z)$ corresponding to the zeros of $d\eta$ (if any).

The examples in §§4 and 5 exhibit three sorts of behaviour:

(A) Examples 4.4 and 5.5 are generalized radial projections, $\tilde{E}$ is 2-dimensional and π collapses each connected component of $\tilde{E}$ to a single point. Note that this is not a branched covering in the sense of Definition 4.2.

(B) In Examples 4.5, 5.3, 5.4 and 5.7 $\tilde{E}$ and E are 1-dimensional submanifolds, and near $\tilde{E}$, π has the simple branching form:

$$(z, t) \in \mathsf{C} \times \mathsf{R} \mapsto (z^k, t) \in \mathsf{C} \times \mathsf{R} \tag{6.1}$$

for some $k \in \{2, 3, \ldots\}$ with respect to suitable coordinates (z, t) on $\tilde{M}^3$ and (w, t) on M^3, with t a coordinate along $\tilde{E}$ or E and z (resp. w) transverse to $\tilde{E}$ (resp. E).

Indeed $k = 2$ in Examples 4.4 and 5.7 and in Examples 5.3 and 5.4 an arbitrary value of k can be obtained. In particular, these examples are branched coverings.

(C) In Example 4.8, $\tilde{E}$ and E are not submanifolds but consist of smooth arcs meeting at the origin. This isolated point will be called an *exceptional point* (see below); the canonical form (6.1) does not apply at such a point and π is not a branched cover in the sense of Definition 4.2.

We shall show that these three types of behaviour of π are the only ones possible: So let (z_0, x_0) be a point of $\tilde{M}^3$ thus $x_0 \in M^3$ is a point on the geodesic $\eta(z_0)$ of M^3. Let $\Sigma = \Sigma_{(z_0,x_0)}$ be a (small) smooth surface in M^3 which passes through x_0 and is perpendicular to the geodesic $\eta(z_0)$. We call Σ a *slice* for η at (z_0, x_0). Give Σ the orientation which together with that of $\eta(z_0)$ gives the orientation of M^3. Note that, in general, Σ cannot be chosen to cut all geodesics $\eta(z)$ for z near z_0 orthogonally but it does cut such geodesics transversally for z in some neighbourhood U_{z_0} of z_0. The metric on Σ induced from M^3 and its orientation define a canonical almost complex structure J_0^Σ on Σ given on each tangent space by rotation through $+\pi/2$. This is, of course, integrable. By construction, for $z \in U_{z_0}$, the geodesic $\eta(z)$ intersects Σ in a unique point $x(z)$ of M^3. We thus obtain a "slice map" $x^\Sigma : U_{z_0} \to \Sigma$, with $x^\Sigma(z) = \eta(z) \cap \Sigma$. The following makes precise Remark 3.3(ii), for justification see below.

REMARK 6.2. The slice map x^Σ is singular at a point $z \in U_{z_0}$ if and only if $x^\Sigma(z)$ lies in the envelope E.

Now set $\tilde{\Sigma} := \{(z, x(z)) \in N^2 \times M^3 \mid z \in U_{z_0}\}$. Clearly $\tilde{\Sigma}$ is a surface in $\tilde{M}^3$ and π maps $\tilde{\Sigma}$ to Σ. This map is a local diffeomorphism except at envelope points $(z,x) \in \tilde{E} \cap \tilde{\Sigma}$, where it has rank 0. Let $\tilde{\eta}(z) := \{(z,x) \in \tilde{M}^3 \mid x \in \eta(z)\}$. By Remark 2.8, π maps $\tilde{\eta}(z)$ to $\eta(z)$ isometrically. Note that, in contrast to $\{\eta(z) \mid z \in N^2\}$ which may not form a foliation at envelope points, the family of geodesics $\{\tilde{\eta}(z) \mid z \in N^2\}$ gives a conformal foliation of $\tilde{M}^3$ even at envelope points. The map $\beta : U_{z_0} \to \tilde{\Sigma}$ defined by $\beta(z) = (z, x^\Sigma(z))$ is diffeomorphic and defines a slice of the foliation $\{\tilde{\eta}(z) \mid z \in U_{z_0}\}$. Note that $\pi \circ \beta = x^\Sigma$ showing that dx^Σ is singular if and only if $d\pi$ is singular at $\beta(z)$ and this holds if and only if $x^\Sigma(z) = \pi(\beta(z)) \in E$, justifying Remark 6.2.

We now choose local coordinates for $\tilde{\Sigma}$ and Σ as follows: For Σ, let w be a local complex coordinate with respect to the almost complex structure J_0^Σ. For $\tilde{\Sigma}$ identify $U_{z_0} \subset N^2$ with an open subset U_0 of C via a complex chart on the Riemann surface N^2, such that $z_0 \in N^2$ corresponds to $0 \in$ C. By a slight abuse of notation we shall denote either a point of $U_{z_0} \subset N^2$ or its local coordinate in $U_0 \subset$ C by z. Then writing $z = u + iv$, the map $\beta : U_0 \to \tilde{\Sigma}$, defined above, defines local coordinates for $\tilde{\Sigma}$.

We now study the form of $\Pi := \pi|_{\tilde{\Sigma}} : \tilde{\Sigma} \to \Sigma$ in our coordinates on $\tilde{\Sigma}$ and Σ.

LEMMA 6.3. *The projection $\Pi : \tilde{\Sigma} \to \Sigma$ is, in the above local coordinates, given by*

$w \circ \Pi \circ \beta : U_0 \to \mathbf{C}$, $z \mapsto w(z)$, where $w(z)$ satisfies the following partial differential equation:

$$\frac{\partial w}{\partial \bar{z}} + A(z) \cdot \overline{\frac{\partial w}{\partial z}} = 0, \tag{6.2}$$

for some smooth complex-valued function A on $\tilde{\Sigma}$ *with $A(0) = 0$.*

PROOF. We must express holomorphicity of η in terms of the map $z \mapsto w(z)$. To do this let J_z^H denote the rotation by $+\pi/2$ in the normal space H_z to $\eta(z)$ at $\Pi(\beta(z))$. Then from the description of the complex structure on $\mathscr{G}_{M^3}$, see for example [Hit] or §2A of [Bai-Woo-2], holomorphicity of η implies that

$$\mathcal{N}\left(\frac{\partial x^\Sigma}{\partial v}\right) = J_z^H \circ \mathcal{N}\left(\frac{\partial x^\Sigma}{\partial u}\right) \text{ for all } z \in U_0, \tag{6.3}$$

where $\mathcal{N} : T_z\Sigma \to H_z$ is orthogonal projection (thus $\mathcal{N}$ gives the normal component of a tangent vector). Otherwise said, if we set $J_z^\Sigma = \mathcal{N}^{-1} \circ J_z^H \circ \mathcal{N}$ for $z \in U_{z_0}$, then J_z^Σ is an almost complex structure on $T_z\Sigma$ and (6.3) reads

$$\frac{\partial x^\Sigma}{\partial v} = J_z^\Sigma \frac{\partial x^\Sigma}{\partial u}. \tag{6.4}$$

Equivalently, $\partial x^\Sigma / \partial \bar{z}$ is of type (0, 1) in $(T_z\Sigma, J_z^\Sigma)$. To express this analytically, note that clearly, J_z^Σ is smoothly varying with z and coincides with the almost complex structure J_0^Σ at $z = 0$. Thus for small z, the (1, 0) co-vectors for J_z^Σ are the complex multiples of $dw + A(z)d\bar{w}$, where $A : U_0 \to \mathbf{C}$ is some smooth function with $A(0) = 0$. Then the holomorphicity condition (6.4) reads $(dw + A(z)d\tilde{w})\left(\frac{\partial}{\partial \bar{z}}\right) = 0$, which is just (6.2).

To study the envelope in our coordinates we need:

LEMMA 6.4. *For small z, the point of $\tilde{\Sigma}$ with coordinate z lies on the envelope $\tilde{E}$ if and only if $\partial w / \partial z = 0$.*

PROOF. Since π maps $\tilde{\eta}(z)$ to $\eta(z)$ isometrically, $d\pi$ is singular if and only if $d\Pi$ is. The Jacobian determinant of Π, with respect to our local coordinates, is

$$\left|\frac{\partial w}{\partial z}\right|^2 - \left|\frac{\partial w}{\partial \bar{z}}\right|^2 = (1 - |A(z)|^2) \cdot \left|\frac{\partial w}{\partial z}\right|^2.$$

Since $A(0) = 0$, $1 - |A(z)|^2$ remains non-zero near 0. This means that the Jacobian determinant is zero if and only if $\partial w / \partial z = 0$ as required.

We can now use the above lemmas to study the behaviour of π near a point on the envelope. First we find the form of $\Pi = \pi|_{\tilde{\Sigma}}$ in our local coordinates:

PROPOSITION 6.5. *Near a point* (z_0, x_0) *on the envelope* $\tilde{E}$, *the map* $w \circ \Pi \circ i : U_0 \to \mathbf{C}$ *is either*

(1) *constant, or*

(2) *a branched covering of the form* $z \mapsto w(z)$, *where*

$$w(z) = \alpha z^k + O(|z|^{k+1}), \text{ with } \alpha \in \mathbf{C}\setminus\{0\} \text{ and } k \geqq 2. \tag{6.5}$$

PROOF. From Lemma 6.3 it is clear that either $w \equiv 0$, or the leading term w_0 of the Taylor series of w in z and $\bar{z}$ satisfies $\partial w_0/\partial \bar{z} = 0$ and so is of the form αz^k.

We call the integer k in Proposition 6.5 the *branching order* of π at (z_0, x_0). It is clearly independent of the choice of Σ. We are now ready for our description of Π.

THEOREM 6.6. *Let* $\eta : N^2 \to \mathscr{G}_{M^3}$ *be a non-constant holomorphic map. Then either:*

(1) η *is a generalized radial projection, or*

(2) $\tilde{E}$ *is an analytic set of pure dimension* 1 *and there is a set S of isolated points of* $\tilde{E}$, *such that* $\tilde{E}\setminus S$ *consists of smooth arcs along which the branching order k of* π *is constant. Near a point of such an arc,* Π *is a branched cover with the simple branching form* (6.1).

PROOF. We shall show that the analytic set $\tilde{E}$ is of pure dimension 1 unless η is a generalized radial projection. First note that $\tilde{E}$ cannot contain any isolated points for, suppose that U is a small neighbourhood of such a point (z_0, x_0). Choose slices $\Sigma_{(z_0,x)} \subset U$ for $(z_0, x) \in \tilde{M}^3 \cap U$. Then $\Pi : \tilde{\Sigma}_{(z_0,x)} \to \Sigma_{(z_0,x)}$ has a branch point for $x = x_0$ but not for $x \neq x_0$. This is clearly impossible for topological reasons. First we show that $\tilde{E}$ cannot be of dimension 3. For in this case all slice maps x^Σ would have to be singular at all points of their domains and so constant. Thus η would be constant in contradiction to hypothesis.

Next, suppose that $\tilde{E}$ is of dimension 2. Then we shall show that $\tilde{E}$ must be perpendicular to all the geodesics $\tilde{\eta}(z)$ for $(z, x) \in \tilde{E}$. To see this consider a point $(z_0, x_0) \in \tilde{E}$. Note that by Lemmas 6.3 and 6.4, unless η is a generalized radial projection, (z_0, x_0) is an isolated point of $\tilde{\Sigma}_{(z_0,x_0)} \cap \tilde{E}$. Thus, if $\tilde{E}$ is of dimension 2, it must be tangent to $\tilde{\Sigma}_{(z_0,x_0)}$ at (z_0, x_0), otherwise it will intersect $\tilde{\Sigma}_{(z_0,x_0)}$ in more than one point. Thus $\tilde{E}$ must be horizontal at (z_0, x_0), and similarly at any of its points (z, x). But then, since by Remark 2.8, $\operatorname{Ker} d\pi_{(z,x)}$ is the horizontal space, $d(\pi|_{\tilde{E}}) = 0$ and so $\pi|_{\tilde{E}}$ is locally constant. This means that all the geodesics $\eta(z)$ for (z, x) in a connected component of $\tilde{E}$ go through the same point and so η is a generalized radial projection.

Hence, unless η is a generalized radial projection, $\tilde{E}$ is of pure dimension 1. It therefore consists of smooth arcs meeting in a set (possibly empty) of isolated points. Call a point of $\tilde{E}$ *exceptional* if *either* it is such a meeting point *or* (z, x) lies on a smooth arc of $\tilde{E}$ which is horizontal at (z, x). By real-analyticity, either the exception points are isolated or there is a whole smooth arc γ of them. In the latter case, $d(\pi|_\gamma) = 0$ and so each geodesic $\eta(z)$ for $(z, x) \in \gamma$ goes through the point $\pi(\gamma)$. By holomorphicity of η (see Lemma 4.4 of [Bai-Woo-2]) this implies that *all* geodesics go through $\pi(\gamma)$, so η is again a generalized radial projection.

Thus, unless η is a generalized radial projection, the exceptional points of $\tilde{E}$ are isolated. Let γ be an arc of $\tilde{E}$ without exceptional points and let $(z_0, x_0) \in \gamma$. Then since $\tilde{E}$ is not horizontal at (z_0, x_0), γ crosses the slice $\Sigma_{(z_0, x_0)}$ transversally. Then, for topological reasons, the branching order of π at (z, x) is constant along γ, and we clearly obtain the canonical form (6.1).

REMARK 6.7. In the case $M^3 = \mathbb{R}^3$ and away from the poles of g, $\tilde{E}$ is given by equations (4.1) and (4.2). (For notation and explanations see the proof of Theorem 3.2.) Then a short calculation shows that the exceptional points are a subset of the points of $\tilde{E}$ where $\partial^2 G/\partial z^2$ is zero. For instance, in Example 4.5,

$$G(z, x) = (1 - z^2)x_1 + i(1 + z^2)x_2 - 2zx_3 - 2h(z), \tag{6.6}$$

where $h(z) = 2irz$ $(r > 0)$, so that $\partial^2 G/\partial z^2 = 2(-x_1 + ix_2)$, which is never zero on $\tilde{E}$. Hence, there are no exceptional points and the branching along $\tilde{E}$ is everywhere of the form (6.1) with branching order $k = 2$. Similarly with $M^3 = S^3$, in Examples 5.3, 5.4, 5.7, there are no exceptional points, and π exhibits the branching (6.1). In Example 4.8 however, we have (6.6) with $h(z) = z^k$ $(k > 2)$ so that $\partial^2 G/\partial z^2 = 0$ at the origin. As we have seen, the origin is indeed an exceptional point, $\tilde{E}$ not being a submanifold at this point.

7. Multivalued harmonic morphisms from higher dimensional manifolds.

In §3 we described the correspondence of [Bai-Woo-2], between non-constant holomorphic maps $\eta : N^2 \to \mathscr{G}_{M^3}$ from Riemann surfaces to the space of oriented geodesics of a 3-dimensional space form and locally defined (or multivalued) harmonic morphisms from M^3 to N^2. For higher dimensions we have the following version:

Let M^m be a simply-connected m-dimensional space form and let $\mathscr{G}_{M^m}$ denote the space of oriented totally geodesic submanifolds of M^m of dimension $(m - 2)$. Then $\mathscr{G}_{M^m}$ is again naturally a complex manifold, with $\dim_{\mathbb{C}}(\mathscr{G}_{M^m}) = m - 1$. Given a non-constant holomorphic map $\eta : N^2 \to \mathscr{G}_{M^m}$, any locally defined map $\phi : U \subset M^m \to N^2$, $z = \phi(x)$ satisfying the condition $x \in \eta(z)$ is a submersive harmonic morphism, with totally geodesic fibres and every such harmonic morphism is locally given by such a map η. We can think of η as defining a multivalued harmonic morphism. So again we may carry out the construction of §3 to obtain a map $\pi : \tilde{M}^m \to M^m$, and a submersive harmonic morphism with totally geodesic fibres $\psi : M^m \to N^2$ covering all local solutions.

Now we consider the case $M^m = \mathbb{R}^m$. Let $\mathscr{G}^0_{\mathbb{R}^m}$ denote the submanifold of all

$(m-2)$-planes through the origin, then $\mathscr{G}_{\mathbb{R}^m} \backslash \mathscr{G}^0_{\mathbb{R}^m}$ can be identified with the tautological bundle of the complex quadric $Q_{m-2} := \{[z_1, \ldots, z_m] \in \mathbb{C}P^{m-1} \mid \sum z_i^2 = 0\}$ minus the zero section.

Indeed an $(m-2)$-dimensional affine subspace which does not go through the origin is given by an equation of the form (cf. [Bai-Woo-1] §2)

$$\langle \xi, x \rangle = 1 \tag{7.1}$$

where $x \in \mathbb{R}^m$ and $\xi \in \mathbb{C}^m$ is isotropic, i.e. $\sum \xi_i^2 = 0$. A holomorphic map $\eta : N^2 \to \mathscr{G}_{M^m}$ with $\eta(z) \in \mathscr{G}^0_{\mathbb{R}^m}$ for isolated points z, at most, can therefore be represented by a meromorphic map $\xi : N^2 \to (\mathbb{C} \cup \{\infty\})^m$ satisfying:

(1) ξ is never zero,

(2) $\sum \xi_i^2 = 0$, and

(3) no component of ξ is identically ∞.

If, on the other hand, $\eta(z) \in \mathscr{G}^0_{\mathbb{R}^m}$ for more than an isolated number of values of z, then, by meromorphicity, $\eta(z) \in \mathscr{G}^0_{\mathbb{R}^m}$ for all z. Now an $(m-2)$-plane through the origin is given by an equation of the form

$$\langle \xi, x \rangle = 0$$

where $x \in \mathbb{R}^m$ and ξ is isotropic, so a holomorphic map η can be represented by a holomorphic map $[\xi] : N^2 \to Q_{m-2}$, and the condition $x \in \eta(z)$ reads:

$$G(z, x) \equiv \langle \xi(z), x \rangle = 0 \tag{7.2}$$

where $z \in N^2$ and $x \in \mathbb{R}^m$. The construction of Theorem 3.1 in both cases gives a m-dimensional manifold $\tilde{\mathbb{R}}^m$, a map $\pi : \tilde{\mathbb{R}}^m \to \mathbb{R}^m$ and a submersive harmonic morphism $\psi : \tilde{\mathbb{R}}^m \to N^2$ with totally geodesic fibres which is a fibre bundle over N^2 with fibres isometric to $\mathbb{R}^{m-2}$.

In the case that $M^m = S^m$, we may identify a totally geodesic submanifold of dimension $(m-2)$ in S^m with an $(m-1)$-dimensional subspace of $\mathbb{R}^{m+1}$. Hence $\mathscr{G}_{S^m} \cong \tilde{G}_2(\mathbb{R}^{m+1}) = Q_{m-1}$ and η can again be represented by a holomorphic map $[\xi] : N^2 \to Q_{m-1}$. The condition $x \in \eta(z)$ again reads

$$G(z, x) \equiv \langle \xi(z), x \rangle = 0 \tag{7.2}$$

but with $z \in N^2$ and $x \in S^m$. Then the construction of Theorem 3.1 gives a fibre bundle $\psi : \tilde{S}^m \to N^2$ over N^2 with fibres isometric to S^{m-2}. Note that all holomorphic maps $[\xi] : N^2 \to Q_{m-1}$ can be written in terms of $(m-1)$ independent meromorphic functions on N^2, see [Hof-Oss].

We now discuss from our point of view a construction of harmonic morphisms from open subsets of $\mathbb{R}^4$ to surfaces, due to the second author [Woo]. (A version of this construction can be given for any anti-self-dual Einstein 4-manifold.)

Recall that an *orthogonal complex structure* on $\mathbb{R}^4$ is a choice of a constant complex structure J on each tangent space of $\mathbb{R}^4$ which is an isometry with respect to the Euclidean metric. Equivalently, an orthogonal complex structure is just a Kähler structure on $\mathbb{R}^4$ equipped with the standard metric. The set of such J's can be identified with S^2. Now for any Riemann surface P^2, let $G: S^2 \times \mathbb{R}^4 \to P^2$ be a smooth map such that:

(1) For each $x \in \mathbb{R}^4$, $G_x: S^2 \to P^2$ is holomorphic, and
(2) for each $J \in S^2$, $G_J: \mathbb{R}^4 \to P^2$ is holomorphic with respect to the orthogonal complex structure defined by J.

By [Fug] G is a harmonic morphism in each variable separately. Thus by Theorem 1.1. any local solution $J: U \subset \mathbb{R}^4 \to S^2$ to the equation

$$G(J, x) = \text{constant} \tag{7.3}$$

is a harmonic morphism. (Further, by [Woo], any submersive harmonic morphism arises this way, up to a post-composition with a conformal map.) We can therefore carry out the construction of Theorem 3.1 to obtain a map $\pi: \tilde{\mathbb{R}}^4 \to \mathbb{R}^4$ and a harmonic morphism $\psi: \tilde{\mathbb{R}}^4 \to S^2$ covering any local solution of (7.3).

To find examples note that if $\mu = \sigma(J) \in \mathbb{C} \cup \{\infty\}$ is not infinite ($\sigma: S^2 \to \mathbb{C} \cup \{\infty\}$ being stereographic projection as usual), complex coordinates on $(\mathbb{C}^2, J)$ are given by $(z_1 - \mu\bar{z}_2, z_2 + \mu\bar{z}_1)$, see §13.64 of [Bes]. It follows that any G satisfying (1) and (2) above is given by

$$G(\mu, (z_1, z_2)) = \Psi(\mu, z_1 - \mu\bar{z}_2, z_2 + \mu\bar{z}_1),$$

where Ψ is a holomorphic function of three complex variables.

We give some examples to show that the geometric envelope $E \subset \mathbb{R}^4$ can be of dimension 0, 1 or 2. For all our examples, the equation $G(\mu, (z_1, z_2)) = 0$ is a second order polynomial in μ with coefficients which are functions of z_1 and z_2. This means that E is given by the equation $d = 0$, where d is the discriminant of this polynomial.

EXAMPLE 7.1. If we choose $\Psi: \mathbb{C}^3 \to \mathbb{C}$, $\Psi: (\mu, (w_1, w_2)) \mapsto w_1^2 + w_2^2$, then $\tilde{E} = \{0\} \subset \mathbb{R}^4$, and $E = S^2 \times \{0\} \subset S^2 \times \mathbb{R}^4$, so E has dimension 0.

EXAMPLE 7.2. With the choice of $\Psi: \mathbb{C}^3 \to \mathbb{C}$, $\Psi: (\mu, (w_1, w_2)) \mapsto w_1 w_2 - \mu$, we have $E = \{(e^{i\theta}, 0) \in \mathbb{R}^4 \mid \theta \in \mathbb{R}\}$ and $\tilde{E} = S^2 \times E \cong S^2 \times S^1$, so $\dim E = 1$.

Example 7.3. For $\Psi: \mathbb{C}^3 \to \mathbb{C}$, $\Psi: (\mu, (w_1, w_2)) \mapsto w_1^2 + w_2^2 - \mu$, one easily sees that $E = \{(x_1, \ldots, x_4) \in \mathbb{R}^4 \mid |x|^2 = 1/2,\ x_1 x_4 = x_2 x_3\}$, which is a 2-dimensional submanifold of $\mathbb{R}^4$. One can furthermore show that $\tilde{E}$ is a 2-dimensional submanifold of $S^2 \times \mathbb{R}$.

Similar examples for S^4 and H^4 will arise from the work of P. Baird [Bai-3], where a more explicit version of [Woo] is given for these 4-manifolds.

REFERENCES

[Ahl-Sar] L. V. Ahlfors, L. Sario, *Riemann surfaces*, Princeton University Press, 1960.

[Bai-1] P. Baird, *Harmonic morphisms onto Riemann surfaces and generalized analytic functions*, Ann. Inst. Fourier (Grenoble) 37 (1987), 135–173.

[Bai-2] P. Baird, *Harmonic morphism and circle actions on 3- and 4-manifold*, Ann. Inst. Fourier (Grenoble) 40 (1990), 177–212.

[Bai-3] P. Baird, *Riemannian twistors and Hermitian structures on low-dimensional space forms*, J. Math. Phys. 33 (1992), 3340–3355.

[Bai-Eel] P. Baird, J. Eells, *A conservation law for harmonic maps*, in Geometry Symposium Utrecht 1980, Lecture Notes in Math. 894 (1981), 1–25.

[Bai-Woo-1] P. Baird, J. C. Wood, *Bernstein theorems for harmonic morphisms from* $\mathbb{R}^3$ *and* S^3, Math. Ann., 280 (1988), 579–603.

[Bai-Woo-2] P, Baird, J. C. Wood, *Harmonic morphisms and conformal foliation by geodesics of three-dimensional space forms*, J. Austr. Math. Soc. Ser. A 51 (1991), 118–153.

[Bai-Woo-3] P. Baird, J. C. Wood, *Harmonic morphisms, Seifert spaces and conformal foliations*, Proc. London Math. Soc. 64 (1992), 170–196.

[Bai-Woo-4] P. Baird, J. C. Wood, *The geometry of a pair of Riemannian foliations by geodesics and associated harmonic morphisms*, Bull. Soc. Math. Belg. Ser. B 44 (1992), 115–139.

[Ber-Cam-Dav] A. Bernard, E. A. Campbell, A. M. Davie, *Brownian motion and generalized analytic and inner functions*, Ann. Inst. Fourier(Grenoble) 29 (1979), 207–228.

[Bes] A. L. Besse, *Einstein Manifolds*, Ergeb. Math. Grenzgeb. (3) 10 (1987).

[Bru-Gib] J. W. Bruce, P. J. Giblin, *Curves and Singularities: A Geometrical Introduction to Singularity Theory*, Cambridge University Press, 1984.

[Eel-Lem] J. Eells, L. Lemaire, *A report on harmonic maps*, Bull. London Math. Soc. 10 (1978), 1–68.

[Eel-Sam] J. H. Sampson, *Harmonic mappings of Riemannian manifolds*, Amer. J. Math., 86 (1964), 109–160.

[Fug] B. Fuglede, *Harmonic morphisms between Riemannian manifolds*, Ann. Inst. Fourier (Grenoble) 28 (1978), 107–144.

[Gud] S. Gudmundsson, *The Geometry of Harmonic Morphisms*, Ph.D.-thesis, University of Leeds, 1992.

[Hit] N. Hitchin, *Monopoles and geodesics*, Comm. Math. Phys. 83 (1982), 579–602.

[Hof-Oss] D. A. Hoffman, R. Osserman, *The geometry of the generalized Gauss map*, Mem. Amer. Math. Soc. 28, No. 236, Amer. Math. Soc., Providence R.I. (1980).

[Ish] T. Ishihara, *A mapping of Riemannian manifolds which preserves harmonic functions*, J. Math. Kyoto Univ. 19 (1979), 215–229.

[Jac] C. G. J. Jacobi, *Über eine Lösung der partiellen Differentialgleichung* $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$, Crelle Journal für die reine und angewandten Mathematik 36, (1847), 113–134.

[Mon] M. Montesinos, *Reduccion de la conjectura de Poincaré a otras conjecturas geometrica*, Revista Mat. Hisp.-Americana Ser. (4), 32 (1972), 33–51.

[Rol] D. Rolfsen, *Knots and Links*, Publish or Perish, 1976.

[Ste] N. Steenrod, *The Topology of Fibre Bundles*, Princeton University Press, 1951.

[Woo] J. C. Wood, *Harmonic morphisms and Hermitian structures on Einstein 4-manifolds*, Internat. J. Math. Sci. 3(1992), 415–439.

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