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ON THE ULTRABORNOLOGICAL PROPERTY OF THE VECTOR VALUED BOUNDED FUNCTION SPACE

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Abstract.

If Ω is a set, Σ a σ-algebra of subsets of Ω and X is a normed space, we show that the space $K(\Sigma, X)$ of all bounded X-countably valued Σ-measurable functions on Ω endowed with the supremum-norm is ultrabornological if and only if X is ultrabornological. As a consequence, the space $l_\infty(X)$ of all bounded sequences in X with the supremum-norm is ultrabornological if and only if X is ultrabornological.

In what follows Ω will be a set, Σ a σ-algebra of subsets of Ω and X a normed space. By $S(\Sigma, X)$ we shall denote the X-valued Σ-simple function linear space over the field K of the real or complex numbers. An X-valued function defined on Ω is Σ-measurable if it is the pointwise limit of a sequence of X-valued Σ-simple functions. By $l_\infty(\Sigma, X)$ we shall represent the linear space over K of all bounded X-valued Σ-measurable functions defined on Ω. Both linear spaces are supposed provided with the norm

$$\|f\| = \sup\{\|f(\omega)\|, \omega \in \Omega\}$$

On the other hand, $B(\Sigma, X)$ will stand for the closure of $S(\Sigma, X)$ in $l_\infty(\Sigma, X)$. By $K(\Sigma, X)$ we shall denote the (dense) subspace of $l_\infty(\Sigma, X)$ formed by all countably valued functions. If Σ is infinite, these two subspaces of $l_\infty(\Sigma, X)$ verify that $K(\Sigma, X) \subseteq B(\Sigma, X)$ only if X is finite-dimensional. Assuming that X is a Banach space, this is an easy consequence of Mazur's theorem, [1, p. 39] and Rosenthal's l_1-theorem, [1, p. 201]. Finally, by $l_\infty(X)$ we shall denote the linear space of all bounded sequences in X provided with the supremum-norm.

It was been shown in [3] that the space $S(\Sigma, X)$ is barrelled if and only if X is finite-dimensional, in [5] it has been proved that $B(\Sigma, X)$ is barrelled if and only if X is barrelled, in [6, p. 149] it is shown that $l_\infty(X)$ is barrelled if and only if X is barrelled and in [2] it has been proved that $l_\infty(\Sigma, X)$ is barrelled if and only if X is barrelled.

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It is not known whether or not some of these spaces are ultrabornological whenever X is an ultrabornological space. In what follows we shall prove that the space $K(\Sigma, X)$ is ultrabornological if and only if X is ultrabornological. Then, since $l_\infty(X)$ coincides with $K(2^{\mathsf{N}}, X)$, it follows that this space will be ultrabornological if and only if X is ultrabornological. But first of all, let us recall what is an ultrabornological space.

Assuming that E is a Hausdorff locally convex space over K and B is a Banach disk, we denote by E_B the Banach space constituted by the linear hull of B provided with the norm of the Minkowski functional of B, and by $\mathscr{B}$ we denote the familly of all Banach disks in E. The space E is said to be ultrabornological, [4, p. 70], if it is the locally convex hull of the familly $\{E_B, B \in \mathscr{B}\}$.

If $A \subseteq \Omega$, $e(A)$ will stand for the indicator function of A.

Before we start our discussion, we must take into account the two following observations:

a) $K(\Sigma, X)$ is a dense subspace of $l_\infty(\Sigma, X)$.

b) If $f \in K(\Sigma, X)$ and $\{x_n\}$ is the bounded sequence of its different values, then $f^{-1}(x_n) \in \Sigma$ for all $n \in \mathsf{N}$.

Indeed, if $f \in l_\infty(\Sigma, X)$, f is the pointwise limit of a sequence of Σ-simple functions and hence $f(\Omega)$ is norm separable. If (x_n) is a dense sequence in $f(\Omega)$, given $\varepsilon > 0$ and $n \in \mathsf{N}$, we set $E_n := \{\omega \in \Omega: \|f(\omega) - x_n\| < \varepsilon\}$. Since, as it can be easily proven, $\omega \to \|f(\omega) - x_n\|$ is a scalar Σ-measurable function, then $E_n \in \Sigma$ for each $n \in \mathsf{N}$. Defining $g(\omega) = x_n$ if $\omega \in E_n \setminus \cup \{E_i, 1 \leqq i \leqq n-1\}$ we have $g \in K(\Sigma, X)$ and $\|f - g\| < \varepsilon$, which shows a). On the other hand, if $f \in K(\Sigma, X) \setminus S(\Sigma, X)$, setting $Y := \mathrm{sp}\{f(\Omega)\}$ and choosing a weak*-total sequence $\{y_j^*, j \in \mathsf{N}\}$ in Y^*, then $A_{n,j} := \{\omega \in \Omega, y_j^* f(\omega) = y_j^* x_n\} \in \Sigma$, consequently, $f^{-1}(x_n) = \cap \{A_{n,j}, j \in \mathsf{N}\} \in \Sigma$. This shows b).

LEMMA 1. *Let $\{A_n, n \in \mathsf{N}\}$ be a sequence of non empty pairwise disjoint elements of Σ. If V is an absolutely convex subset of $K(\Sigma, X)$ which meets each Banach space E generated by a Banach disk of $K(\Sigma, X)$ in a neighbourhood of the origin in E, there exists an $m \in \mathsf{N}$ such that V absorbs the closed unit ball of $K(\Sigma / \cup \{A_n, n > m\}, X)$.*

PROOF. If the property is not true V does not absorb the closed unit ball of $K(\Sigma / \cup \{A_n, n > p\}, X)$ for each $p \in \mathsf{N}$. Hence, for each $p \in \mathsf{N}$ there is $f_p \in K(\Sigma / \cup \{A_n, n > p\}, X)$ such that $\|f_p\| = 1$ and $f_p \notin pV$.

As (f_p) is a bounded sequence in $K(\Sigma, X)$, the series $\sum_{i=1}^{\infty} \xi_i f_i$ converges in the completion $l_\infty(\Sigma, \hat{X})$ of $K(\Sigma, X)$ to some h_ξ for each $\xi \in l_1$. Clearly, h_ξ takes at most countably many values since for each $\omega \in \Omega$ the sum $\sum_{i=1}^{\infty} \xi_i f_i(\omega)$ is finite. Hence, $h_\xi(\omega) \in X$ for each $\omega \in \Omega$ and $h_\xi \in K(\Sigma, X)$. This proves that the Banach disk

$\left\{\sum_{i=1}^{\infty} \xi_i f_i, \xi \in B_{l_1}\right\}$ of $l_\infty(\Sigma, \hat{X})$ is contained in $K(\Sigma. X)$. From this we obtain a $k \in \mathrm{N}$ such that $f_k \in kV$, a contradiction.

THEOREM 1. *Let V be an absolutely convex subset of $K(\Sigma, X)$ which meets each Banach space E generated by a Banach disk of $K(\Sigma, X)$ in a neighbourhood of the origin in E. If X is ultrabornological, then V absorbs the closed unit ball of $S(\Sigma, X)$.*

PROOF. If V does not absorb the unit sphere of $S(\Sigma, X)$, there is some $f_1 \in S(\Sigma, X)$ with $\|f_1\| = 1$ such that $f_1 \notin 2V$. Let $\{\Omega_{1,1}, \Omega_{1,2}, \ldots, \Omega_{1,k(1)}\}$ be a partition of Ω by non-empty sets of Σ such that f_1 takes a different constant value in each $\Omega_{1,i}$ with $1 \leqq i \leqq k(1)$. Since $S(\Sigma, X)$ is the topological direct sum of the subspaces $S(\Sigma/\Omega_{1,i}, X)$, $1 \leqq i \leqq k(1)$, there is some $m(1) \in \{1, 2, \ldots, k(1)\}$ such that V does not absorb the unit sphere of $S(\Sigma/\Omega_{1,m(1)}, X)$. Hence, there is some $f_2 \in S(\Sigma/\Omega_{1,m(1)}, X)$ with $\|f_2\| = 1$ such that $f_2 \notin 4V$. Then, we choose a finite partition $\{\Omega_{2,1}, \Omega_{2,2}, \ldots, \Omega_{2,k(2)}\}$ of $\Omega_{1,m(1)}$ by non-empty sets of Σ such that f_2 is constant in each set $\Omega_{2,1}$, $1 \leqq i \leqq k(2)$ and takes a different value. This way we obtain a normalized sequence (f_n) of Σ-simple functions and a sequence $(\Omega_{n,m(n)})$ of sets in Σ such that, for each $n \in \mathrm{N}$.

(i) $\operatorname{supp} f_{n+1} \subseteq \Omega_{n,m(n)}$
(ii) f_n is constant in $\Omega_{n,m(n)}$
(iii) $\Omega_{n+1,m(n+1)} \subseteq \Omega_{n,m(n)}$
(iv) $f_n \notin 2nV$

Now set $E_n := \Omega_{n,m(n)}$ and define $P := \bigcap_{i=1}^{\infty} E_n$.

Suppose first that P is not empty. If x_n denotes the constant value of f_n in E_n, define $h_j(\omega) = f_j(\omega)$ if $\omega \notin P$ and $h_j(\omega) = x_j$ if $\omega \in P$ for each $j \in \mathrm{N}$. Then, we write $g_j := h_j - x_j e(P)$ for each $j \in \mathrm{N}$. Notice that $\operatorname{supp} g_n \subseteq E_{n-1} \backslash P$ for each $n \in \mathrm{N}$ and $\cap \{E_n \backslash P, n \in \mathrm{N}\} = \emptyset$. Since $x \to e(P)x$ is an isometry from X into $K(\Sigma, X)$, the ultrabornological property of X leads to the existence of some $r \in \mathrm{N}$ such that $x_i e(P) \in rV$ for each $i \in \mathrm{N}$. Hence, $x_n e(P) \in nV$ for each $n \geqq r$ and, consequently, $g_j \notin jV$ for each $j \geqq r$. If $P = \emptyset$, then for each $j \in \mathrm{N}$ define $g_j(\omega) = f_j(\omega)$ for all $\omega \in \Omega$. So $g_j \notin jV$ for each $j \in \mathrm{N}$.

As in both cases $\|g_j\| \leqq 1$ for each $j \in \mathrm{N}$ and each point $\omega \in \Omega$ belongs at most to finitely many supports of functions of the sequence (g_j), we proceed as in the end of the proof of Lemma 1 to show that $\left\{\sum_{j=1}^{\infty} \xi_j g_{r+j}, \xi \in B_{l_1}\right\}$ is a Banach disk of $l_\infty(\Sigma, \hat{X})$ contained in $K(\Sigma, X)$. Again this yields some integer $q > r$ such that $g_q \in qV$, a contradiction.

THEOREM 2. *$K(\Sigma, X)$ is ultrabornological if and only if X is an ultrabornological space.*

PROOF. Assume that X is ultrabornological but $K(\Sigma, X)$ is not and let V be an absolutely convex set in $K(\Sigma, X)$, meeting each Banach space E generated by a Banach disk of $K(\Sigma, X)$ in a neighbourhood of the origin in E, which is not a neighbourhood of the origin in $K(\Sigma, X)$. There is some $f_1 \in K(\Sigma, X)$ with $\|f_1\| = 1$ such that $f_1 \notin 2V$. We proceed by recurrence.

There is a partition $\{\Omega_{1,i}, i \in \mathrm{N}\}$ of Ω by non-empty sets of Σ such that f_1 is constant in each set $\Omega_{1,i}$. Then, by Lemma 1, there is an $n_i \in \mathrm{N}$ such that V does absorb the closed unit ball of $K(\Sigma / \cup \{\Omega_{1,n}, n > n_1\}, X)$. Consequently, V does not absorb the unit sphere of $K(\Sigma / \cup \{\Omega_{1,n}, n \leqq n_1\}, X)$. Let $\Omega_1 := \cup \{\Omega_{1,n}, n \leqq n_1\}$ and choose some $f_2 \in K(\Sigma / \Omega_1, X)$ with $\|f_2\| = 1$ such that $f_2 \notin 3V$. Then we choose a partition $\{\Omega_{2,i}, i \in \mathrm{N}\}$ of Ω_1 formed by non-empty sets of Σ such that f_2 is constant in each $\Omega_{2,i}$, and use Lemma 1 again to obtain an $n_2 \in \mathrm{N}$ such that V absorbs the closed unit ball of $K(\Sigma / \cup \{\Omega_{2,n}, n > n_2\}, X)$. Define $\Omega_2 := \cup \{\Omega_{2,i}, i \leqq n_2\}$. This way we obtain a normalized sequence (f_n) of functions of $K(\Sigma, X)$ and a sequence (Ω_n) of sets in Σ satisfying for each $n \in \mathrm{N}$ the following properties

(i) $\operatorname{supp} f_{n+1} \subseteq \Omega_n$

(ii) $e(\Omega_n) f_n \in S(\Sigma / \Omega_n, X)$

(iii) $\Omega_{n+1} \subseteq \Omega_n$

(iv) $f_n \notin (n+1)V$

For each $j \in \mathrm{N}$ we set $g_j := f_j - e(\Omega_j) f_j$. Clearly, $e(\Omega_j) f_j \in S(\Sigma, X)$ for each $j \in \mathrm{N}$ and taking into account the previous theorem, there is not loss of genrality in assuming that $e(\Omega_j) f_j \in V$ for each $j \in \mathrm{N}$. This implies that $g_j \notin jV$ for each $j \in \mathrm{N}$.

It is clear that $\operatorname{supp} g_i \cap \operatorname{supp} g_j = \emptyset$ if $i \neq j$, and it is not difficult to see from this fact that the closed linear span $[g_j]$ in $l_\infty(\Sigma, \hat{X})$ of the sequence (g_j) is a copy of c_0 which is contained in $K(\Sigma, X)$. Now this leads to the existence of some $k \in \mathrm{N}$ such that $g_k \in kV$, a contradiction.

If $K(\Sigma, X)$ is ultrabornological, X is ultrabornological, since the map $\delta_\omega : K(\Sigma, X) \to e(\Omega) X$ defined by $\delta_\omega(f) = f(\omega) e(\Omega)$ is a continuous projection for each $\omega \in \Omega$.

COROLLARY 1. *Suppose that every linear functional on $l_\infty(\Sigma, X)$ which is bounded on every Banach disk of $l_\infty(\Sigma, X)$ and vanishes on $K(\Sigma, X)$ is identically zero. Then the space $l_\infty(\Sigma, X)$ is ultrabornological if and only if X is ultrabornological.*

PROOF. If X is ultrabornological, this is a consequence of the previous theorem and of §35.7.(5) of [4] since, as we have noticed above, $K(\Sigma, X)$ is dense in $l_\infty(\Sigma, X)$. On the other hand, if $l_\infty(\Sigma, X)$ is ultrabornological, the same reasoning as above shows that X is ultrabornological.

COROLLARY 2. *$l_\infty(X)$ is ultrabornological if and only if X is an ultrabornological space.*

PROOF. This is also an obvious consequence of Theorem 2, since this space coincides with the space $K(2^{\mathbf{N}}, X)$.

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