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BOUNDED HOMEOMORPHISMS OVER HADAMARD MANIFOLDS

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Abstract.

Let F be a closed topological manifold and let H be a Hadamard manifold (that is, a simply connected, nonpositively curved, complete Riemannian manifold). We study the space of homeomorphisms on the product $F \times H$ which are bounded in the H-direction. The main result is that the homotopy type of this space of bounded homeomorphisms is independent of the metric on H. The proof is accomplished by relating bounded homeomorphisms to controlled homeomorphisms, a notion which depends only on the topology of H. The result is proved in the more general context of certain spaces which need not be products with H, namely, manifold approximate fibrations over H.

Introduction.

Given two spaces over a space B, say $p_1: E_1 \to B$ and $p_2: E_2 \to B$, we consider two notions of maps from E_1 to E_2 which are weaker than the traditional notion of a fibre map over B. The easiest to describe is bounded: if B is a metric space, a map $f: E_1 \to E_2$ is called *bounded* provided there exists a number c such that $d(p_1(e), f \circ p_2(e)) < c$ for all $e \in E_1$. The other type of map is controlled: given two spaces over B as above, a *controlled map* between them is a map $F: E_1 \times [0,1) \to E_2$ so that the map $\hat{F}: E_1 \times [0,1] \to B$ defined by $\hat{F}(e,t) = \begin{cases} p_2 F(e,t) & \text{if } t < 1 \\ p_1(e) & \text{if } t = 1 \end{cases}$ is continuous. Clearly, the notion of bounded depends on the metric of B, while just as clearly, controlled does not.

Bounded maps figure prominently in work by Anderson and Hsiang, [2], [3], whereas controlled maps figure prominently in work of Chapman, Ferry and Quinn (with the precise definition given above first appearing in [8]). A remarkable fact which has emerged from these two schools is that the obstructions to solving a bounded problem usually agree with the obstructions to solving the analogous controlled problem. For more information, consult the recent survey by Ferry, Hambleton, and Pedersen [6].

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This paper gives an explanation for this observation. In general, a bounded map is not controlled in any natural way, and a controlled map need not be bounded. We introduce a third sort of map, the *bounded, uniformly controlled maps*. These are both controlled and bounded, and so form an intermediate place in which to compare the two theories. We show that if we consider $B = \mathsf{R}^i$, the simplicial spaces of controlled homeomorphisms, bounded homeomorphisms and bounded, uniformly controlled homeomorphisms are all homotopy equivalent (in the case of manifold approximate fibrations).

Indeed, we will show somewhat more. Let H be any simply connected manifold with a complete metric of nonpositive curvature, a so-called *Hadamard manifold*. The Hadamard-Cartan theorem shows that H is diffeomorphic to R^i, but the metric properties can be rather different. We will show that the simplicial spaces of controlled homeomorphisms and bounded homeomorphisms are homotopy equivalent for any Hadamard manifold (again, in the manifold approximate fibration case). The equivalence of the bounded and the controlled theories is established by equating (up to homotopy equivalence) bounded homeomorphisms with those homeomorphisms which can be extended continuously to a sphere at infinity via the identity. This reduces the problem back to the euclidean case.

One of our motivations for writing this paper is that in an earlier paper [8] (see also [9]), we gave a classification of manifold fibrations over an i-dimensional manifold with given fibre germ $p: M \to \mathsf{R}^i$ (which is itself a manifold approximate fibration) in terms of a lifting problem through a bundle with fibre BTop^c $(p: M \to \mathsf{R}^i)$, the classifying space of the simplicial group of controlled homeomorphisms on the fibre germ. Therefore, succesful application of our classification theorem depends on an analysis of the homotopy type of the space of controlled homeomorphisms on a manifold approximate fibration over R^i. At present, the only hope for numerical information lies in the methods of Weiss and Williams ([14], [15], [16]). Their work, however, is concerned with bounded homeomorphisms. This partially explains our interest in relating controlled homeomorphisms to bounded homeomorphisms in the euclidean case.

We first began this work in response to a question of S. Weinberger, namely, are bounded and controlled homeomorphisms over hyperbolic space essentially the same? Of course, the positive answer to this question is a special case of our results. We exploit our philosophical finding that "bounded equals controlled over Hadamard manifolds" in forthcoming papers ([10], [11], [12]) to show that certain forget control maps are split injective thereby establishing results related to the Novikov conjecture. Finally, we mention the recent paper [1] by Anderson, Connolly, Ferry and Pedersen which uses one of our techniques to connect bounded topology with a topologically invariant theory.

Section 1. Statements of Results.

Let $p: M \to \mathsf{R}^i$ be a manifold approximate fibration. Thus, p is a proper approximate fibration and M is a manifold with $\partial M = \emptyset$ (see [8]). It is understood that R^i is endowed with the metric induced from the standard euclidean norm $\|\cdot\|$. Let $m = \dim M$.

A k-simplex of the simplicial set $\mathrm{Top}^{\mathrm{b}}(p: M \to \mathsf{R}^i)$ of *bounded homeomorphisms* is a homeomorphism

$$h: M \times \Delta^k \to M \times \Delta^k$$

such that

$$\begin{array}{ccc} M \times \Delta^k & \xrightarrow{h} & M \times \Delta^k \\ & {\scriptstyle p_2}\searrow \quad \swarrow{\scriptstyle p_2} & \\ & \Delta^k & \end{array}$$

commutes and such that there exists a $c > 0$ so that $\|p(x) - pp_1h(x,t)\| < c$ for each (x,t) in $M \times \Delta^k$ where $p_1: M \times \Delta^k \to M$ and $p_2: M \times \Delta^k \to \Delta^k$ are the projections.

Recall from [8] that a k-simplex of the simplicial set $\mathrm{Top}^{c}(p: \to \mathsf{R}^i)$ of *controlled homeomorphisms* is a homeomorphism

$$h: M \times \Delta^k \times [0,1) \to M \times \Delta^k \times [0,1)$$

such that

$$\begin{array}{ccc} M \times \Delta^k \times [0,1) & \xrightarrow{h} & M \times \Delta^k \times [0,1) \\ & {\scriptstyle p_2}\searrow \quad \swarrow{\scriptstyle p_2} & \\ & \Delta^k \times [0,1) & \end{array}$$

commutes, and the compositions

$$M \times \Delta^k \times [0,1) \xrightarrow{h} M \times \Delta^k \times [0,1) \xrightarrow{p \times \mathrm{id}} \mathsf{R}^i \times \Delta^k \times [0,1)$$

and

$$M \times \Delta^k \times [0,1) \xrightarrow{h^{-1}} M \times \Delta^k \times [0,1) \xrightarrow{p \times \mathrm{id}} \mathsf{R}^i \times \Delta^k \times [0,1)$$

extend continuously to maps $M \times \Delta^k \times [0,1] \to \mathsf{R}^i \times \Delta^k \times [0,1]$ via $p \times \mathrm{id}$: $M \times \Delta^k \times 1 \to \mathsf{R}^i \times \Delta^k \times 1$.

THEOREM 1.1. *If* $m \geqq 5$, *then* $\mathrm{Top}^{\mathrm{b}}(p: M \to \mathsf{R}^i)$ *and* $\mathrm{Top}^{\mathrm{c}}(p: M \to \mathsf{R}^i)$ *are homotopy equivalent.*

We remark that although in general there need not be a natural map between

$\mathrm{Top}^b(p\colon M \to \mathsf{R}^i)$ and $\mathrm{Top}^c(p\colon M \to \mathsf{R}^i)$, we will describe one which gives the homotopy equivalence in the special case that $M = \mathsf{R}^i \times F$ and p is the projection (see Remark 3.8).

We next want to generalize these results by replacing R^i by an arbitrary Hadamard manifold H^i. To accomplish this, we first generalize a result of Anderson and Hsiang which we now recall.

Let F be compact and let $\mathrm{Top}^b(\mathsf{R}^i \times F)$ denote the simplicial set of homeomorphisms of $\mathsf{R}^i \times F$ which are bounded in the R^i-direction; that is, Top^b (proj: $\mathsf{R}^i \times F \to \mathsf{R}^i$). Let $S^{i-1} * F$ denote the join of S^{i-1} and F and let $\mathrm{Top}(S^{i-1} * F, S^{i-1})$ denote the simplicial set of homeomorphisms of $S^{i-1} * F$ which restrict to the identity on S^{i-1}. There is a natural identification of $\mathsf{R}^i \times F$ with $(S^{i-1} * F)\backslash S^{i-1}$ and this induces a simplicial map $\mathrm{Top}^b(\mathsf{R}^i \times F) \to \mathrm{Top}(S^{i-1} * F, S^{i-1})$. The following result appears implicitly in [2].

THEOREM (Anderson-Hsiang). *The natural map* $\mathrm{Top}^b(\mathsf{R}^i \times F) \to \mathrm{Top}(S^{i-1} * F, S^{i-1})$ *is a homotopy equivalence.*

Recently, Madsen and Rothenberg have included a complete, elementary proof of this Theorem (and an equivariant version) in [13]. Basically their idea is to start with a homeomorphism in $\mathrm{Top}(S^{i-1} * F, S^{i-1})$ and then to do a geometric construction in the i different R-directions to obtain a homeomorphism which is bounded in each of the R-directions, hence bounded in the R^i-direction.

In this paper we give an alternative proof that applies to more general situations. Instead of working in the R-directions, we achieve boundedness in the radial direction and in the spherical direction. An advantage of this proof is that it works when R^i is replaced by an arbitrary Hadamard manifold H^i. Let $\mathrm{Top}^b(H \times F)$ denote the simplicial set of homeomorphisms of $H^i \times F$ which are bounded in the H-direction.

THEOREM 1.2. *The natural map* $\mathrm{Top}^b(H \times F) \to \mathrm{Top}(S^{i-1} * F, S^{i-1})$ *is a homotopy equivalence.*

We also consider more general situations than just products $H \times F$. More specifically, let $p\colon M \to H$ be a manifold approximate fibration and let $m = \dim M$. Let $H(\infty)$ denote the sphere at infinity of H. Then M can be compactified by adding $H(\infty)$ to get $\bar{M} = M \cup H(\infty)$. The topology on $\bar{M}$ is discussed in Section 4. Let $\mathrm{Top}^b(p\colon M \to H)$ and $\mathrm{Top}^e(p\colon M \to H)$ denote the simplicial sets of homeomorphisms of M which are bounded in the H-direction or, respectively, can be continuously extended to $H(\infty)$ via the identity. By using the controlled isotopy covering theorem for manifold approximate fibrations (see the appendix, Section 6) we get the following version of Theorem 1.2.

THEOREM 1.3. *If $m \geqq 5$, then the inclusion* $\mathrm{Top}^{\mathrm{b}}(p: M \to H) \to \mathrm{Top}^{\mathrm{e}}(p: M \to H)$ *is a homotopy equivalence.*

The homeomorphism type of the pair $(\bar{M}, H(\infty))$ is independent of the metric on H, as long as the metric is complete and of nonpositive curvature (see Section 4). Therefore, we have the following corollaries of Theorem 1.3.

COROLLARY 1.4. *The homotopy types of* $\mathrm{Top}^{\mathrm{b}}(H \times F)$ *and, more generally,* $\mathrm{Top}^{\mathrm{b}}(p: M \to H)$ *are independent of the metric on H (as long as the metric is complete and of nonpositive curvature) if $m \geqq 5$.*

COROLLARY 1.5. *If $m \geqq 5$,* $\mathrm{Top}^{\mathrm{b}}(p: M \to \mathsf{R}^i)$ *and* $\mathrm{Top}^{\mathrm{b}}(p: M \to H^i)$ *are homotopy equivalent for any i-dimensional Hadamard manifold H^i.*

The following result is a corollary of Theorem 1.1 and Corollary 1.4.

COROLLARY 1.6. *If $m \geqq 5$, then* $\mathrm{Top}^{\mathrm{b}}(p: M \to H)$ *is homotopy equivalent to* $\mathrm{Top}^{\mathrm{c}}(p: M \to H)$ *for any Hadamard manifold H.*

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Section 2. Controlled vs. uniformly controlled homeomorphisms.

In this section we show that controlled homeomorphisms can be assumed to be uniformly controlled. The main result (Theorem 2.2) holds over an arbitrary manifold B. Adopt the following data for this section.

Data. Let $p: M \to B$ be a manifold approximate fibration with $\dim M = m \geqq 5$ and $\partial M = \emptyset = \partial B$. Endow B with a fixed metric (i.e. a distance function d).

Let $\mathrm{Top}^{\mathrm{c}}(p: M \to B)$ be the simplicial set of controlled homeomorphisms (see [8] for a definition or replace R^i by B in the definition in Section 1). It is a Kan simplicial group. We will denote a typical k-simplex by either

$$h: M \times \Delta^k \times [0,1) \to M \times \Delta^k \times [0,1)$$

or

$$h_t: M \times \Delta^k \to M \times \Delta^k, 0 \leqq t < 1.$$

DEFINITION 2.1. $\mathrm{Top}^{\mathrm{c,u}}(p: M \to B)$ is the simplicial subgroup of $\mathrm{Top}^{\mathrm{c}}(p: M \to B)$ consisting of those simplices

$$h_t: M \times \Delta^k \to M \times \Delta^k, 0 \leqq t < 1,$$

such that

$$(p \times \mathrm{id}_{\Delta^k})h_t \text{ converges uniformly to } (p \times \mathrm{id}_{\Delta^k}) \text{ as } t \to 1^-,$$

i.e. for all $\varepsilon > 0$ there exists $t_0, 0 \leqq t_0 < 1$, such that $d((p \times \mathrm{id})h_t(x), (p \times \mathrm{id})(x)) < \varepsilon$

for all $t \geqq t_0$ and for all $x \in M \times \Delta^k$. (Here we are using d to denote the standard metric on $B \times \Delta^k$ induced by d on B and the standard metric on Δ^k.)

REMARK. The controlled homeomorphisms produced by the proof of the Controlled Straightening Theorem [8, Thm. 14.3] have the pleasant property that they are uniformly controlled; i.e., the homeomorphisms are simplices of $\mathrm{Top}^{c,u}(p\colon M \to B)$. This fact will be used in the proof of the Theorem below.

THEOREM 2.2. *The inclusion* $\mathrm{Top}^{c,u}(p\colon M \to B) \hookrightarrow \mathrm{Top}^{c}(p\colon M \to B)$ *is a homotopy equivalence.*

PROOF. Let $h_t\colon M \times \Delta^k \to M \times \Delta^k$, $0 \leqq t < 1$, denote a k-simplex of $\mathrm{Top}^c(p\colon M \to B)$ such that $h_t | M \times \partial\Delta^k$, $0 \leqq t < 1$, is a union of $k + 1 (k-1)$-simplices of $\mathrm{Top}^{c,u}(p\colon M \to B)$.

Define

$$\hat{p}\colon M \times \Delta^k \times [0,1] \to B \times \Delta^k \times [0,1]$$

by $\hat{p} | M \times \Delta^k \times [0,1) = (p \times \mathrm{id}_{\Delta^k}) \circ h$ and $\hat{p} | M \times \Delta^k \times \{1\} = (p \times \mathrm{id}_{\Delta^k})$. Then $\hat{p}$ is a $(\Delta^k \times [0,1])$-parametrized family of manifold approximate fibrations. It follows from the proof of the Controlled Straihtening Theorem [8, Thm. 14.3] that there exists a continuous family of homeomorphisms

$$H_s\colon M \times \Delta^k \times [0,1] \to M \times \Delta^k \times [0,1],\ 0 \leqq s < 1,$$

such that

1) H_s is fibre preserving over $\Delta^k \times [0,1]\ \forall s$,
2) $H_0 = \mathrm{id}$,
3) $H_s | M \times \Delta^k \times \{1\} = \mathrm{id}\ \forall s$,
4) $H_s | M \times \Delta^k \times \{t\} = h_t^{-1} \circ h_{\alpha_t(s)}\ \forall s,\ t \leqq 1$ where $\alpha_t\colon [0,1) \to [t,1)$ is defined by $\alpha_t(s) = \begin{cases} t & \text{if } s \leqq t \\ s & \text{if } s \geqq t \end{cases}$
5) $\hat{p}H_s$ converges uniformly to $p \times \mathrm{id}_{\Delta^k}$ as $s \to 1^-$.

Note that

$p \circ h_0 \circ H_s | M \times \Delta^k \times \{0\}$ converges uniformly to $p \times \mathrm{id}_{\Delta^k}$ as $s \to 1^-$ and that

$$H_s | M \times \partial\Delta^k \times \{0\} = h_0^{-1} h_s | M \times \partial\Delta^k \text{ for all } s.$$

It follows that $h_0 \circ H_s | M \times \Delta^k \times \{0\}$, $0 \leqq s < 1$, is a k-simplex of $\mathrm{Top}^{c,u}(p\colon M \to B)$ which agrees with h_s on $\partial\Delta^k$. Define $g_s\colon M \times \Delta^k \times [0,1] \to M \times \Delta^k \times [0,1]$, $0 \leqq s < 1$ by

$$g_s(x,y,t) = \begin{cases} (h_s(x,y),t) & \text{if } s \leqq t \\ (h_t(x,y),t) & \text{if } s \geqq t \end{cases}.$$

Then $g_s \circ H_s$ is a $(k+1)$-simplex of $\mathrm{Top}^c(p\colon M \to B)$ connecting $h_0 H_s$ and $h_s \operatorname{rel} \partial\Delta^k$.

Section 3. Bounded and uniformly controlled homeomorphisms.

In this section we complete the proof of Theorem 1.1. This is accomplished by introducing the group $\mathrm{Top}^{b,c,u}(p\colon M \to B)$ of bounded, uniformly controlled homeomorphisms and showing that it is homotopy equivalent to both the group of controlled homeomorphisms (Corollary 3.3) and, also to the group of bounded homeomorphisms in the special case $B = \mathrm{R}^i$ (Theorem 3.7). Adopt the following data.

Data. Let $p\colon M \to B$ be a manifold approximate fibration, where B is a topological manifold with a metric and $\partial M = \emptyset = \partial B$. Let $m = \dim M$. Mainly, but not exclusively, we are concerned with the case $B = \mathrm{R}^i$.

DEFINITION 3.1. $\mathrm{Top}^{b,c,u}(p\colon M \to B)$ is the simplicial subgroup of $\mathrm{Top}^{c,u}(p\colon M \to B)$ consisting of those k-simplices

$$h_t\colon M \times \Delta^k \to M \times \Delta^k,\ 0 \leqq t < 1$$

such that there exists a $\beta > 0$ so that $d((p \times \mathrm{id})h_t, p \times \mathrm{id}) < \beta$ for all t in $[0, 1)$.

LEMMA 3.2. *The inclusion* $\mathrm{Top}^{b,c,u}(p\colon M \to B) \hookrightarrow \mathrm{Top}^{c,u}(p\colon M \to B)$ *is a homotopy equivalence.*

PROOF. Let $h_t\colon M \times \Delta^k \to M \times \Delta^k$, $0 \leqq t < 1$, be a k-simplex in $\mathrm{Top}^{c,u}(p\colon M \to B)$ such that $h_t | M \times \partial\Delta^k$, $0 \leqq t < 1$, is in $\mathrm{Top}^{b,c,u}(p\colon M \to B)$, say with bound $\beta > 0$.

Note that h_t, $0 \leqq t < 1$, defines a family of homeomorphisms $h_{(s,t)}\colon M \to M$, $(s, t) \in \Delta^k \times [0, 1)$.

By uniform convergence, there exists $t_0 \in [0, 1)$ such that $d(ph_{(s,t)}, p) < \beta$ for each $(s, t) \in \Delta^k \times [t_0, 1)$.

Let $r\colon \Delta^k \times [0, 1) \to \Delta^k \times [t_0, 1) \cup \partial\Delta^k \times [0, 1)$ be a retraction and define

$$\tilde{h}_{(s,t)}\colon M \to M, (s, t) \in \Delta^k \times [0, 1), \text{ by } \tilde{h}_{(s,t)} = h_{r(s,t)}.$$

Then $\tilde{h}_{(s,t)}$ defines a k-simplex in $\mathrm{Top}^{b,c,u}(p\colon M \to \mathrm{R}^i)$ which equals h_t over $\partial\Delta^k$. Moreover, if $r_u\colon \mathrm{id} \simeq r$, $0 \leqq u \leqq 1$, is a homotopy rel $\Delta^k \times [t_0, 1)$, this can be used to construct a homotopy connecting h and $\tilde{h}$ rel $\partial\Delta^k$.

COROLLARY 3.3. *If* $m \geqq 5$, *then* $\mathrm{Top}^c(p\colon M \to B)$ *is homotopy equivalent to* $\mathrm{Top}^{b,c,u}(p\colon M \to B)$.

PROOF. Apply Lemma 3.2 and Theorem 2.2.

DEFINITION 3.4. The forgetful map $\alpha\colon \mathrm{Top}^{b,c,u}(p\colon M \to \mathrm{R}^i) \to \mathrm{Top}^b(p\colon M \to \mathrm{R}^i)$ is defined by

$$\alpha(h_t, 0 \leqq t < 1) = h_0$$

Our goal in Theorem 3.6 is to show that α is a homotopy equivalence.

REMARK 3.5. Here is an example which shows that controlled homeomorphisms need not be bounded.

Define $h_t\colon \mathsf{R}^i \to \mathsf{R}^i$, $0 \leqq t < 1$, by $h_t(x) = (2 - t)x$.
Then h_t, $0 \leqq t < 1$, is a controlled homeomorphism from $\mathrm{id}_{\mathsf{R}^i}$ to itself, but each h_t is unbounded.

LEMMA 3.6. *Let $m \geqq 5$. For every integer $k = 0, 1, 2, \ldots$ there exists $\varepsilon > 0$ so that if $p\colon M \to \mathsf{R}^i$ is a manifold approximate fibration,* $\dim M = m$, *and*

$$h\colon M \times \Delta^k \to M \times \Delta^k$$

is a k-simplex in $\mathrm{Top}^{\mathrm{b}}(p\colon M \to \mathsf{R}^i)$ *with bound ε, then there exists a k-simplex*

$$\tilde{h}_t\colon M \times \Delta^k \to M \times \Delta^k,\ 0 \leqq t < 1,$$

in $\mathrm{Top}^{\mathrm{b,c,u}}(p\colon M \to \mathsf{R}^i)$ *such that $\tilde{h}_0 = h$.*

Moreover, if we are additionally given

$$g_t\colon M \times \partial\Delta^k \to M \times \partial\Delta^k, 0 \leqq t < 1,$$

in $\mathrm{Top}^{\mathrm{b,c,u}}(p\colon M \to \mathsf{R}^i)$ *with $g_0 = h|M \times \partial\Delta^k$, then we can additionally require that*

$$\tilde{h}|M \times \partial\Delta^k = g_t \text{ for each } t.$$

PROOF. Note that $(p \times \mathrm{id})h^{-1}\colon M \times \Delta^k \to \mathsf{R}^i \times \Delta^k$ and $p \times \mathrm{id}\colon M \times \Delta^k \to \mathsf{R}^i \times \Delta^k$ are Δ^k-parametrized families of manifold approximate fibrations which are ε-close.

It follows from [7] that, if ε is small, there exists a manifold approximate fibration

$$q\colon M \times \Delta^k \times [0,1] \to \mathsf{R}^i \times \Delta^k \times [0,1]$$

fibred over $\Delta^k \times [0,1]$, with $q|M \times \Delta^k \times 0 = (p \times \mathrm{id})h^{-1}$ and $q|M \times \Delta^k \times 1 = p \times \mathrm{id}$.

Moreover, we can take $q|M \times \partial\Delta^k \times [0,1) = (p \times \mathrm{id})g^{-1}$.

By "controlled straightening" [8, Theorem 14.3], there exists a homeomorphism

$$H\colon M \times \Delta^k \times [0,1) \to M \times \Delta^k \times [0,1)$$

such that

i) H is fibre preserving over $\Delta^k \times [0,1)$,

ii) $H|M \times \Delta^k \times 0 = \mathrm{id}$,

iii) $(p \times \mathrm{id})H_s$ converges uniformly to $(p \times \mathrm{id})h^{-1}$ as $s \to 1^-$.

Moreover, by a relative version, we can take

$$H_s|M \times \partial\Delta^k = g_s h^{-1}|M \times \partial\Delta^k \text{ for each } s \text{ in } [0, 1).$$

Then $H_s h$, $0 \leqq s < 1$, *is a* k-simplex of $\mathrm{Top}^{b,c,u}(p\colon M \to \mathsf{R}^i)$ with initial level h and $H_s h|M \times \partial\Delta^k = g_s$.

THEOREM 3.7. *If* $\dim M \geqq 5$, *then the forgetful map* $\alpha\colon \mathrm{Top}^{b,c,u}(p\colon M \to \mathsf{R}^i) \to \mathrm{Top}^{b}(p\colon M \to \mathsf{R}^i)$ *is a homotopy equivalence.*

PROOF. Let $h\colon M \times \Delta^k \to M \times \Delta^k$ be a k-simplex in $\mathrm{Top}^b(p\colon M \to \mathsf{R}^i)$ and let $g_t\colon M \times \partial\Delta^k \to M \times \partial\Delta^k$, $0 \leqq t < 1$, be in $\mathrm{Top}^{b,c,u}(p\colon M \to \mathsf{R}^i)$ with initial level $g_0 = h|M \times \partial\Delta^k$. Let $\beta > 0$ be a bound for both h and g_t, $0 \leqq t < 1$. Let $\varepsilon > 0$ be given by Lemma 3.6. Choose $\delta > 0$ small.

Define an isotopy

$$\gamma\colon \mathsf{R}^i \times [0,1] \to \mathsf{R}^i \times [0,1]$$

by $\gamma_t(x) = (1 - t + t\delta)x$ for $0 \leqq t \leqq 1$. By the Controlled Isotopy Covering Theorem (see the appendix), there exists an isotopy

$$\Gamma\colon M \times [0,1] \to M \times [0,1]$$

such that $(p \times \mathrm{id})\Gamma$ is δ-close to $\gamma(p \times \mathrm{id})$.

Define a homeomorphism j from

$$[(M \times \Delta^k) \cup (M \times \partial\Delta^k \times [0,1))] \times [0,1]$$

to itself by

$$j|M \times \Delta^k \times [0,1] = \Gamma(h \times \mathrm{id})\Gamma^{-1}$$

and

$$j|M \times \partial\Delta^k \times t \times [0,1] = \Gamma(g_t \times \mathrm{id})\Gamma^{-1}|.$$

It follows that $(p \times \mathrm{id})j|M \times \Delta^k \times 1$ is $(2\delta + \delta\beta)$-close to $p \times \mathrm{id}$. If $\delta > 0$ was chosen so that $(2\delta + \delta\beta) < \varepsilon$, then Lemma 3.6 can be used to find a k-simplex

$$\tilde{j}_s\colon M \times \Delta^k \to M \times \Delta^k, 0 \leqq s < 1$$

in $\mathrm{Top}^{b,c,u}(p\colon M \to \mathsf{R}^i)$ such that $\tilde{j}_0 = j|M \times \Delta^k \times 1$ and

$$\tilde{j}_s\colon M \times \partial\Delta^k = \Gamma_1 g_s \Gamma_1^{-1},\ 0 \leqq s < 1.$$

Note that $j|M \times \partial\Delta^h \times [0,1) \times [0,1]$ is a homotopy in $\mathrm{Top}^{b,c,u}(p\colon M \to \mathsf{R}^i)$ from g_s to $\tilde{j}_s|M \times \partial\Delta^k$ and that $\alpha(j|)$ extends to a homotopy j from h to $\alpha(\tilde{j}_s)$. Thus α is surjective on homotopy groups. Moreover, since a null homotopy is also given by a simplex with certain properties on the boundary, the above argument shows that α is injective on homotopy groups and therefore is a homotopy equivalence.

PROOF OF THEOREM 1.1. We have the following homotopy equivalences:

$$\begin{array}{ccc} \mathrm{Top}^{b,c,u}(q\colon M \to \mathsf{R}^i) & \xrightarrow[(3.6)]{\alpha} & \mathrm{Top}^b(p\colon M \to \mathsf{R}^i) \\ \big\downarrow{\scriptstyle (3.2)} & & \\ \mathrm{Top}^{c,u}(p\colon M \to \mathsf{R}^i) & \xhookrightarrow[(2.3)]{} & \mathrm{Top}^c(p\colon M \to \mathsf{R}^i) \end{array}$$

REMARK 3.8. In the special case that $M = \mathsf{R}^i \times F$ and $p\colon \mathsf{R}^i \times F \to \mathsf{R}^i$ is the projection, we can describe a homotopy equivalence

$$\gamma\colon \mathrm{Top}^b(p\colon M \to \mathsf{R}^i) \to \mathrm{Top}^c(p\colon M \to \mathsf{R}^i)$$

explicitly as follows.

Define $\gamma_t\colon \mathsf{R}^i \to \mathsf{R}^i$, $0 \leqq t < 1$, by $\gamma_t(x) = (1 - t)x$. Let $\Gamma_t\colon \mathsf{R}^i \times F \to \mathsf{R}^i \times F$ be $\gamma_t \times \mathrm{id}_F$.

Then if

$$h\colon M \times \Delta^k \to M \times \Delta^k$$

is a k-simplex of $\mathrm{Top}^b(p\colon M \to \mathsf{R}^i)$, then

$$\gamma(h) = (\Gamma_t \times \mathrm{id}_{\Delta^k})h(\Gamma_t^{-1} \times \mathrm{id}_{\Delta^k}),\ 0 \leqq t < 1,$$

is a k-simplex of $\mathrm{Top}^c(p\colon M \to \mathsf{R}^i)$.

Moreover, if γ is used to fill-in the square in the proof of Theorem 1.1, then the resulting diagram commutes up to homotopy and we have a homotopy equivalence.

Section 4. Bounded homeomorphisms over Hadamard manifolds.

In this section we give proofs of Theorem 1.2 and Theorem 1.3. Throughout this section H will denote a Hadamard manifold of dimension $i > 1$. Most of our terminology concerning Hadamard manifolds comes from [5] and [4]. Fix a point x_0 in H and let $\exp\colon \mathsf{R}^i \to H$ denote the exponential map at x_0. Then exp is a homeomorphism and preserves radial distances. Let $d\colon H \times H \to [0, \infty)$ denote the distance function induced by the Riemannian structure on H.

Let $H(\infty)$ denote the sphere at infinity and let $\bar{H} = H \cup H(\infty)$ with the cone topology [5]. Extend d to $\bar{H}$ so that $d(x, y) = \infty$ if $x, y \in \bar{H}$, $x \neq y$, and x or y in $H(\infty)$. For each $x \in H$, $x \neq x_0$, let $\gamma_x\colon \mathsf{R} \cup \{+\infty\} \to \bar{H}$ be the unique (unit speed) geodesic such that $\gamma_x(0) = x_0$ and $\gamma_x(t) = x$ where $t = d(x_0, x)$.

Let $S(x_0)$ be the unit sphere in the tangent space H_{x_0}. For each v in $S(x_0)$ and $\varepsilon > 0$, the cone of axis v and angle ε is

$$C(v, \varepsilon) = \left\{ x \in \bar{H}\colon \begin{array}{l} \text{the angular distance between } v \text{ and } x \\ \text{measured from } x_0 \text{ is less than } \varepsilon \end{array} \right\}$$

(see [5, Definition 2.2]). If, in addition, $r > 0$ is given, the *truncated cone of axis* v *and radius* r is

$$T(v, \varepsilon, r) = C(v, \varepsilon) \setminus \{x \in H: d(x_0, x) \leqq r\}.$$

We will need the following result of Eberlein and O'Neill:

THEOREM 4.1. [5, Prop. 2.6]. *The set of truncated cones that contain x in $H(\infty)$ is a local basis for the cone topology at x.*

If $v \in S(x_0)$, $\varepsilon > 0$, and $r \geqq s > 0$, then we define an *annular sector*

$$A(v, \varepsilon, s, r) = T(v, \varepsilon, s) \setminus T(v, \varepsilon, r).$$

For $\varepsilon > 0$, $B(x_0, \varepsilon)$ denotes the ball of radius ε about x_0. Note that $B(x_0, \varepsilon) = \exp(B(0, \varepsilon))$ where $B(0, \varepsilon)$ is the ε-ball about 0 in R^i.

Throughout this section M will denote a manifold of dimension m (without boundary) and $p: M \to H$ will denote a proper map. We will assume that p is the projection of a fibre bundle or that p is an approximate fibration and $m \geqq 5$. This is because we will need p to have the controlled isotopy covering property.

The simplicial set $\mathrm{Top}^b(p\colon M \to H)$ of *bounded homeomorphisms on* M has k-simplices which are homeomorphisms $h: M \times \Delta^k \to M \times \Delta^k$ such that h is fibre preserving over Δ^k and there exists a constant $B > 0$ with $d(pp_1 h(x), pp_1(x)) < B$ for all x in $M \times \Delta^k$ where $p_1: M \times \Delta^k \to M$ is projection.

We now compactify M by adding $H(\infty)$ *to* M. That is, let $\bar{M} = M \cup H(\infty)$. Extend p to $\bar{p}: \bar{M} \to \bar{H}$ by setting $\bar{p}|H(\infty) = \mathrm{id}$. Now give $\bar{M}$ the coarsest topology which makes $\bar{p}$ continuous. Since p is a quotient map, M is a subspace of $\bar{M}$.

The topology of the pair $(\bar{M}, H(\infty))$ depends only on M, p and the topology of $(\bar{H}, H(\infty))$. Since [5] shows that the topology of $(\bar{H}, H(\infty))$ is independent of the metric on H (as long as the metric is complete and of nonpositive curvature), the topology on $(\bar{M}, H(\infty))$ is independent of the metric on H. For example, if $M = H \times F$ and p is projection, then $\bar{M}$ is the join $S^{i-1} * F$. This is because the pair $(\bar{H}, H(\infty))$ is homeomorphic to (B^i, S^{i-1}).

A homeomorphism $h: M \times \Delta^k \to M \times \Delta^k$ is *extendible* if h extends to a homeomorphism $\bar{h}: \bar{M} \times \Delta^k \to \bar{M} \times \Delta^k$ by setting $\bar{h}|H(\infty) \times \Delta^k = \mathrm{id}$. The simplicial set $\mathrm{Top}^e(p\colon M \to H)$ has k-simplices which are extendible homeomorphisms h: $M \times \Delta^k \to M \times \Delta^k$ which are fibre preserving over Δ^k.

LEMMA 4.2. $\mathrm{Top}^b(p\colon M \to H) \subset \mathrm{Top}^e(p\colon M \to H)$

PROOF. This follows immediately from the "intensive property" of [5, p. 50]. That the cone topology satisfies this condition follows from [5, Prop. 2.9].

Our main goal is to show that the inclusion $\mathrm{Top}^b(p\colon M \to H) \subset \mathrm{Top}^e(p$:

$M \to H$) is a homotopy equivalence. We will do this by factoring the inclusion through another simplicial set as follows.

If x, y are in H, define the *radial distance from x to y* to be

$$d_r(x, y) = |\|\exp^{-1}(x)\| - \|\exp^{-1}(y)\||$$

where $\|\cdot\|$ denotes the standard norm on R^i. A homeomorphism h: $M \times \Delta^k \to M \times \Delta^k$ *is radially bounded* if there exists a constant $B > 0$ such that

$$d_r(pp_1 h(x), pp_1(x)) < B \quad \text{for all } x \text{ in } M \times \Delta^k.$$

Let $\mathrm{Top}^{\mathrm{e,r}}(p\colon M \to H)$ denote the sub- simplicial set of $\mathrm{Top}^{\mathrm{e}}(p\colon M \to H)$ consisting of those extendible homeomorphisms which are, in addition, radially bounded. (Note that rotations about x_0 are radially bounded but not extendible.) Since $\exp^{-1}$ is distance nonincreasing, it follows that every bounded homeomorphism is radially bounded and we have inclusions

$$i\colon \mathrm{Top}^{\mathrm{b}}(p\colon M \to H) \to \mathrm{Top}^{\mathrm{e,r}}(p\colon M \to H)$$

and

$$j\colon \mathrm{Top}^{\mathrm{e,r}}(p\colon M \to H) \to \mathrm{Top}^{\mathrm{e}}(p\colon M \to H).$$

The first step is to achieve a spherical bound.

THEOREM 4.3. *The inclusion* i: $\mathrm{Top}^{\mathrm{b}}(p\colon M \to H) \to \mathrm{Top}^{\mathrm{e,r}}(p\colon M \to H)$ *is a homotopy equivalence.*

PROOF. Since $\mathrm{Top}^{\mathrm{b}}(p\colon M \to H)$ and $\mathrm{Top}^{\mathrm{e,r}}(p\colon M \to H)$ are Kan simplicial sets, it suffices to show that i induces an isomorphism on homotopy groups. To this end let h: $M \times \Delta^k \to M \times \Delta^k$ be a k-simplex in $\mathrm{Top}^{\mathrm{e,r}}(p\colon M \to H)$ such that $h|M \times \partial\Delta^k$ represents the union of $k + 1$ $(k - 1)$-simplices of $\mathrm{Top}^{\mathrm{b}}(p\colon M \to H)$, say with bound B. By using a collar neighborhood of $\partial\Delta^k$ in Δ^k, we may assume that there is a neighborhood N of $\partial\Delta^k$ in Δ^k such that $h|M \times N$ is bounded by B. For notational convenience we will assume that the radial bound on h is less than $1/2$ and that $B < 1$. Extend h to $\bar{h}$: $\bar{M} \times \Delta^k \to \bar{M} \times \Delta^k$.

Our goal is to construct an isotopy $\tilde{g}_u$: id $\simeq \tilde{g}$, $0 \leqq u \leqq 1$, on $M \times \Delta^k$ such that

i) $\tilde{g}_u$ is fibre preserving over Δ^k for each u,

ii) $\tilde{g}_u|M \times \partial\Delta^k = \mathrm{id}$ for each u,

iii) $\tilde{g}_u^{-1} h \tilde{g}_u$ is a k-simplex of $\mathrm{Top}^{\mathrm{e,r}}(p\colon M \to H)$ for each u (with radial bound $< 1/2$),

iv) $\tilde{g}^{-1} h \tilde{g}$ is bounded.

Conditions i) and ii) will follow from the construction. Conduction iii) is achieved by constructing each $\tilde{g}_u$ so that it is a radial expansion. We'll describe where condition iv) comes from below.

Choosing ε_j. For each $j = 1, 2, 3, \ldots$ choose $\varepsilon_j > 0$ such that for every vector in the unit tangent sphere $S(x_0)$,

$$\operatorname{diam}[C(v, 2\varepsilon_j) \cap \partial B(x_0, j+1)] < 1.$$

Choosing V_j. For each $j = 1, 2, 3, \ldots$ choose a finite set $V_j \subset S(x_0)$ such that the collection $\{C(v, \varepsilon_j) | v \in V_j\}$ covers $\bar{H}$.

Now we will achieve condition iv) above by constructing $\tilde{g}$ to satisfy the following.

For every x in $M \times \Delta^k$ there exists a j in $\{1, 2, 3, \ldots\}$ and a v in V_j such that $pp_1(x)$ and $pp_1\tilde{g}^{-1}h\tilde{g}(x)$ are both in $A(v, 2\varepsilon_j, j-1, j+1)$. Note that the triangle inequality implies that

$$\operatorname{diam}[A(v, 2\varepsilon_j, j-1, j+1)] \leqq 5.$$

Choosing r_j. We claim that for every $j = 1, 2, 3, \ldots$ there exists $r_j > 0$ such that for each $v \in V_j$,

$$\bar{h}(\bar{p}^{-1}(T(v, \varepsilon_j, r_j)) \times \Delta^k) \subset \bar{p}^{-1}(T(v, 2\varepsilon_j, r_j - 1)) \times \Delta^k.$$

Moreover, we can define the r_j inductively so that $r_j > \max\{j, r_{j-1} + 2\}$. To verify this claim, select for each v in V_j a neighborhood U_v of $C(v, \varepsilon_j) \cap H(\infty)$ in $\bar{H}$ such that

$$\bar{h}(\bar{p}^{-1}(U_v) \times \Delta^k \subset \bar{p}^{-1}(C(v, 2\varepsilon_j) \times \Delta^k).$$

Using compactness and the fact that $C(v, \varepsilon_j) \cap H(\infty) = \bigcap_{r>0} T(v, \varepsilon_j, r)$, there exists $r_v > 0$ such that $T(v, \varepsilon_j, r_v) \subset U_v$. Now let $r_j > \max\{j, r_{j-1} + 2, r_v | v \in V_k\}$ and use the fact that h has radial bound $\frac{1}{2}$.

We now proceed to construct $\tilde{g}_u$. First let α: $[0, \infty) \to [0, \infty)$ be the PL homeomorphism such that for each $j = 1, 2, 3, \ldots, \alpha$ takes $[j-1, j]$ linearly onto $[r_{j-1}, r_j]$ (where $r_0 = 0$).

Let μ: $\Delta^k \to [0, 1]$ be a map such that $\mu^{-1}(0) = \partial\Delta^k$ and $\Delta^k \backslash N \subset \mu^{-1}(1)$. We represent points in $\mathrm{R}^i \times \Delta^k$ by (sy, t) where $s \in [0, +\infty)$, $y \in S^{i-1}$, and $t \in \Delta^k$.

Define g: $\mathrm{R}^i \times \Delta^k \to \mathrm{R}^i \times \Delta^k$ by $g(sy, t) = ([\mu(t)\alpha(s) + (1 - \mu(t))s]y, t)$. Then g induces $\hat{g}$: $H \times \Delta^k \to H \times \Delta^k$ defined by

$$\hat{g} = (\exp \times \mathrm{id}_{\Delta^k}) \circ g \circ (\exp^{-1} \times \mathrm{id}_{\Delta^k}).$$

Also set $g_u(sy, t) = (((1-u)s + u[\mu(t)\alpha(s) + (1 - \mu(t))s])y, t)$, $0 \leqq u \leqq 1$. Then g_u: id $\simeq g$ is an isotopy and induces an isotopy $\hat{g}_u$: id $\simeq \hat{g}$.

We can now finish the construction of $\tilde{g}_u$. In the case that p: $M \to H$ is a fibre bundle projection (trivial of course), simply choose an isotopy $\tilde{g}_u$: id $\simeq \tilde{g}$ of $M \times \Delta^k$ such that $(p \times \mathrm{id}_{\Delta^k})\tilde{g}_u = \hat{g}_u(p \times \mathrm{id}_{\Delta^k})$ and $\tilde{g}_u | M \times \partial\Delta^k = \mathrm{id}$ for each u. In the case that $p: M \to H$ is a manifold approximate fibration, simply use the Controlled Isotopy Covering Theorem (Section 6) to get an isotopy with $(p \times \mathrm{id}_{\Delta^k})\tilde{g}_u$ close to $\hat{g}_u(p \times \mathrm{id}_{\Delta^k})$.

The next step is to achieve a radial bound. The proof of the following theorem is the argument used in [13] to achieve boundedness in an R-direction.

THEOREM 4.4. *The inclusion* j: $\mathrm{Top}^{e,r}(p: M \to H) \to \mathrm{Top}^e(p: M \to H)$ *is a homotopy equivalence.*

PROOF. We show that j induces an isomorphism on homotopy groups.

Let $h: M \times \Delta^k \to M \times \Delta^k$ be a k-simplex of $\mathrm{Top}^e(p: M \to H)$ such that for some neighborhood N of $\partial\Delta^k$ in Δ^k, $h|M \times N$ is radially bounded, say by 1.

Choose $1 = r_1 < r_2 < r_3 < \ldots$ inductively so that $r_i \to \infty$ as $i \to \infty$ and so that

$$p^{-1}(B(x_0, r_{k-1})) \times \Delta^k \subset h[p^{-1}(B(x_0, r_k)) \times \Delta^k] \subset p^{-1}(B(x_0, r_{k+1})) \times \Delta^k$$

for each $k = 1, 2, 3, \ldots$. Moreover, assume $k \leqq r_k$.

Let g: $H \times \Delta^k \to H \times \Delta^k$ be a homeomorphism, constructed in a similar fashion as $\hat{g}$ in the proof of Theorem 4.3, so that $g(B(x_0, k) \times N)$ for each $k, g|H \times \partial\Delta^k = \mathrm{id}$ and g only moves points in the radial direction. Again there will be an obvious isotopy g_u: $\mathrm{id} \simeq g$ which can be covered (or approximately covered) by an isotopy $\tilde{g}_u$: $\mathrm{id} \simeq \tilde{g}$, $0 \leqq u \leqq 1$, of $M \times \Delta^k$.

If this is done correctly, then the following conditions will be satisfied:

i) $\tilde{g}_u$ if fibre preserving over Δ^k for each u,

ii) $\tilde{g}_u|M \times \partial\Delta^k = \mathrm{id}$ for each u,

iii) $\tilde{g}_u^{-1} h \tilde{g}_u$ is a k-simplex of $\mathrm{Top}^e(p: M \to H)$ for each u,

iv) $\tilde{g}^{-1} h \tilde{g}$ is radially bounded (by the number 3).

Section 5. An Example.

We give an elementary example which shows that $\mathrm{Top}^b(p: X \to \mathsf{R}^1)$ need not be homotopy equivalent to $\mathrm{Top}^c(p: X \to \mathsf{R}^1)$ if $p: X \to \mathsf{R}^1$ is not a manifold approximate fibration.

Let $X = \{(x, y) \in \mathsf{R}^2 | y = 0, \text{ or } x \in Z \text{ and } 0 \leqq y \leqq 1\}$. Define p: $X \to \mathsf{R}^1$ by $p(x, y) = x$. Then it is straightforward to see that $\mathrm{Top}^b(p: X \to \mathsf{R}^1)$ is not connected, whereas $\mathrm{Top}^c(p: X \to \mathsf{R}^1)$ is contractible.

Section 6. Appendix: The controlled isotopy covering theorem.

THEOREM 6.1. *Let* q: $M \to B$ *be a manifold approximate fibration,* $\dim M \geqq 5$, $\partial M = \emptyset = \partial B$. *Let* h: $B \times [0,1] \to B \times [0,1]$ *be an isotopy. Then there exists a continuous family of isotopies*

$$\tilde{h}_s: M \times [0,1] \to M \times [0,1], 0 \leqq s < 1, \textit{ such that}$$

1) $\tilde{h}_0 = \mathrm{id}$
2) $(q \times \mathrm{id}_{[0,1]})\tilde{h}_s$ *converges to* $h(q \times \mathrm{id}_{[0,1]})$ *as* $s \to 1^-$.

REMARK. By an isotopy we mean in particular that $h|B \times 0 = \mathrm{id}$ and $\tilde{h}_s|M \times 0 = \mathrm{id}$ for all s.

PROOF. Consider $f\colon M \times [0,1] \to B \times [0,1]$ defined by

$$f = h(q \times \mathrm{id}_{[0,1]}).$$

Then f is a $[0,1]$-parametrized family of manifold approximate fibrations.

By the proof of [8, Cor. 14.4] there exists a homeomorphism

$$H\colon M \times [0,1] \times [0,1) \to M \times [0,1] \times [0,1)$$

such that

i) H is fibre preserving over $[0,1] \times [0,1)$,
ii) $H|M \times [0,1] \times 0 = \mathrm{id}$,
iii) $H|M \times 0 \times [0,1) = \mathrm{id}$,
iv) if $H_s = H|M \times [0,1] \times s$, then $(q \times \mathrm{id}_{[0,1]})H_s$ converges uniformly to f.

Simply set $\tilde{h}_s = H_s$.

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