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NORM DEPENDENCE OF THE COEFFICIENT MAP ON THE WINDOW SIZE

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Abstract.

For sparse exponent sequences $(\lambda_k)_{-\infty}^{\infty}$, satisfying a suitable "separation condition" defined by an auxiliary sequence ψ, one has a "coefficient map" $\mathbf{C}_\delta$ giving $(c_k)_{-\infty}^{\infty} =: \mathbf{c}$ from observation of $f = \sum_{k=-\infty}^{\infty} c_k e^{i\lambda_k t}$ on any arbitrarily small interval $[-\delta, \delta]$. In terms of ψ, we estimate the norm of $\mathbf{C}_\delta$: $L^2[-\delta,\delta] \to l^2$, asymptotically as $\delta \to 0$. In particular, for $(\lambda_k)_{-\infty}^{\infty} \sim k^p$ $(p > 1)$ we get a bound which is exponential in $(1/\delta)^{1/(p-1)}$, generalizing an earlier result for the case $p = 2$.

1. Introduction.

Let $\lambda = \{\lambda_k\}$ be a real "double sequence" $(k = 0, \pm 1, \pm 2, \ldots)$ and let $\mathring{\mathcal{M}} = \mathring{\mathcal{M}}(\lambda)$ denote the collection of all finite sums

$$f = \sum_k c_k e^{i\lambda_k t} \tag{1.1}$$

with complex coefficients c_k. We will here think of viewing such f through a *window* $(-\delta, \delta)$ and determining the coefficients $\{c_k\}$ from this, defining a coefficient map

$$\mathring{\mathbf{C}} = \mathring{\mathbf{C}}(\lambda): f = \sum_k c_k e^{i\lambda_k t} \mapsto (c_k) \tag{1.2}$$

for $f \in \mathring{\mathcal{M}}$. If we now let $\mathcal{M}_\delta = \mathcal{M}_\delta(\lambda)$ be the closure of $\mathring{\mathcal{M}}$ in $L^2(-\delta,\delta)$, then it is classical that $\mathring{\mathbf{C}}$ extends from $\mathring{\mathcal{M}}$ to $\mathcal{M}_\delta$ as a continuous linear operator

$$\mathbf{C}_\delta = \mathbf{C}_\delta(\lambda): \mathcal{M}_\delta \mapsto l^2: f = \sum_{k=-\infty}^{\infty} c_k e^{i\lambda_k t} \mapsto (c_k)_{-\infty}^{\infty} =: \mathbf{c} \tag{1.3}$$

provided the asymptotic density of λ is bounded by δ/π.

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In this paper, we consider sequences $\boldsymbol{\lambda}$ satisfying sparsity conditions of the form

$$|\lambda_{k+m} - \lambda_k| \geqq \psi_m \quad (m = 1, 2, \ldots) \tag{1.4}$$

for suitable $\boldsymbol{\psi} = \{\psi_m : m = 1, 2, \ldots\}$. Noting that $m/\psi_m \to 0$ ensures that $\mathbf{C}_\delta(\boldsymbol{\lambda})$ is well defined for all $\delta > 0$, we then investigate the rapidity with which $\|\mathbf{C}_\delta(\boldsymbol{\lambda})\| \to \infty$ as $\delta \to 0$. As a by-product of this analysis, we note that our estimates are uniform over the classes of exponent sequences $\Lambda = \Lambda(\boldsymbol{\psi})$ satisfying (1.4) for particular admissible sequences $\boldsymbol{\psi}$. Our results are new in this aspect as well as in the consideration of the asymptotics as $\delta \to 0$.

It is clear that $\mathbf{C}_\delta(\boldsymbol{\lambda})$ is made up of the coefficient functionals

$$\gamma_k : \mathcal{M}_\delta = \mathcal{M}_\delta(\boldsymbol{\lambda}) \to \mathbf{C} : f \mapsto c_k$$

and that each of these functionals can be represented as

$$\gamma_k : f \mapsto c_k = \langle f, g_k \rangle \tag{1.5}$$

for some $g_k \in L^2(-\delta, \delta)$. There is some arbitrariness in the determination of g_k since (1.5) constitutes an extension of γ_k from $\mathcal{M}_\delta$ to all of $L^2(-\delta, \delta)$; this also gives an extension $\tilde{\mathbf{C}}_\delta$ of $\mathbf{C}_\delta(\boldsymbol{\lambda})$ to $L^2(-\delta, \delta)$.

Since we are working with exponentials, it is then convenient to construct the Fourier transforms to obtain g_k and we actually will work with the adjoint of $\tilde{\mathbf{C}}_\delta$,

$$\tilde{\mathbf{C}}_\delta^* : (a_k) \mapsto \sum_k a_k g_k : l^2 \to L^2(-\delta, \delta), \tag{1.6}$$

to estimate $\|\mathbf{C}_\delta\| \leqq \|\tilde{\mathbf{C}}_\delta\| = \|\tilde{\mathbf{C}}_\delta^*\|$. (Note that one has the geometric characterization $\|\gamma_k\| = 1/[\text{distance from } e^{i\lambda_k t} \text{ to } \mathcal{M}_\delta(\boldsymbol{\lambda}^k)]$ where $\boldsymbol{\lambda}^k = \boldsymbol{\lambda} \backslash \{\lambda_k\}$. This gives $\|\gamma_k\| \geqq 1/\|e^{i\lambda_k t}\| = 1/\sqrt{2\delta}$, showing the uselessness of the crude estimate $\|\mathbf{C}_\delta f\|^2 = \sum_k |\langle \gamma_k, f \rangle|^2 \leqq \left(\sum_k \|\gamma_k\|^2\right) \|f\|^2$).

We will be able to treat conditions (1.4) for real sequences $\boldsymbol{\psi} = \{\psi_m : m = 1, 2, \ldots\}$ for which

$$0 < \psi_1 \leqq \psi_2 \leqq \ldots \quad \text{and} \quad \sum_1^\infty 1/\psi_k < \infty \tag{1.7}$$

Note that this already implies that $m/\psi_m \to 0$ which precisely corresponds to the condition that Λ have asymptotic density zero.

Our paper falls into three parts:

First, considering a sequence $\boldsymbol{\lambda}$ satisfying (1.4) subject to (1.7), we will apply an important theorem due to Korevaar and Luxemburg ([2]; Theorem 3.1) which we restate here in a relevant form:

THEOREM K–L. *Let* $\omega : \mathbf{R}_+ \to \mathbf{R}_+$ *be nondecreasing with* $\omega(t)/t^2$ *integrable at* ∞.

Then, for any $\delta > 0$, *there exists a number* $Q > 0$ *and an entire function* $P(\cdot)$ *such that:*

(i) P *is of exponential type* δ,

(ii) P *is normalized so* $P(0) = 1$,

(iii) $|P(s)| \leqq e^{Q}e^{-\omega(|s|)}$ *for* $s \in \mathsf{R}$.

Clearly, the constant Q in (iii) will depend on δ so $Q = Q(\delta) = Q(\delta; \omega)$. From this we then obtain an estimate:

$$\|\mathbf{C}_\delta\| \leqq Ae^{Q(\delta)} \quad \text{for all} \quad \delta > 0 \tag{1.8}$$

where A is independent of δ and we use a function ω depending only on ψ. So far, this is only slightly different from the treatment in [2].

Second, and this is the principal technical innovation of the paper, we extend the analysis of Theorem K–L from that of [2], specifically considering the estimation of $Q(\delta)$ in (1.8) so as to exhibit explicitly its asymptotics as $\delta \to 0$ as well as the dependence on ψ through $\omega(\cdot)$. From (1.8), this is precisely what is needed to investigate the asymptotic behavior of $\|\mathbf{C}_\delta\|$. For this estimation, we find it necessary to assume $\omega \in \Omega$ where

$$\Omega := \left\{\omega\colon \mathsf{R}_+ \to \mathsf{R}_+\colon \begin{array}{l}\omega(t) \text{ is nondecresing while} \\ \omega(\mathrm{t})/\mathrm{t}^2 \text{ is decreasing and integeable at } \infty\end{array}\right\}; \tag{1.9}$$

this strengthens very slightly the hypotheses above for Theorem K–L in requiring that $\omega(t)/t^2$ be decreasing.

Third, the combination of the above is applied to obtain specific growth estimates for a number of interesting special cases. In particular, we apply the general analysis to the case $\psi_m = am^p$ $(a > 0,\ p > 1)$ which corresponds to $\lambda_k \sim \pm ak^p$ and obtain for that case the estimate

$$\log\|\mathbf{C}_\delta\| = O([1/\delta]^{1/(p-1)}) \quad \text{as} \quad \delta \to 0 \tag{1.10}$$

(i.e., $Q(\delta) \leqq \mu[1/\delta]^{1/(p-1)}$ in (1.8) for small $\delta > 0$; we also have an estimate for μ). We recall that earlier investigation in [4] of the special case

$$\lambda_k = k^2 \quad (k \geqq 0), \quad \lambda_{-k} = -\lambda_k \tag{1.11}$$

resulted in an estimate $\log\|\mathbf{C}_\delta\| = O(1/\delta)$ which was there shown to be sharp (by an example due to Korevaar).

2. The Interpolation Family.

Assume that ψ satisfies (1.7) and that $\lambda = (\lambda_k)_{-\infty}^{\infty}$ is in $\Lambda(\psi)$. i.e., satisfies the separation condition (1.4). With ψ we associate the function Ψ given by

$$\Psi(s) := 2 \sum_{m=1}^{\infty} \log\left(1 + \frac{s^2}{\psi_m^2}\right). \tag{2.1}$$

We will show in the Appendix (Lemma A.1) that this function $\Psi(\cdot)$ is, indeed, in Ω. Given $\boldsymbol{\lambda} \in \Lambda(\psi)$, we next define a family of functions $(\eta_k)_{-\infty}^{\infty}$ by first defining

$$\mu_k(z) := \prod_{j \neq k} \left(\frac{z - \lambda_j}{\lambda_k - \lambda_j}\right) \quad (z \in \mathbf{C}) \tag{2.2}$$

and then setting

$$\eta_k(z) := \mu_k(z)\mu_k(2\lambda_k - z) = \prod_{j \neq k} \left[1 - \left(\frac{z - \lambda_k}{\lambda_k - \lambda_j}\right)^2\right]. \tag{2.3}$$

LEMMA 2.1. *Let ψ, Ψ be as above and define η_k for $k \in \mathbf{Z}$ as in* (2.3). *Then one has*

$$\eta_k(\lambda_j) = \delta_{j,k} \quad (j.\, k \in \mathbf{Z}) \tag{2.4}$$

and each $\eta_k(\cdot)$ is an entire function of exponential type 0 *with*

$$|\eta_k(\lambda_k + s)| \leqq e^{\Psi(|s|)} \quad \forall s \in \mathbf{R}. \tag{2.5}$$

PROOF. For each k and any $N > 0$, there is some $M = M_{k,N}$ such that

$$|\mu_k(z)| \leqq \left(\prod_{0 < |j-k| \leqq N} \left|\frac{z - \lambda_j}{\lambda_k - \lambda_j}\right|\right) \exp\left[(|z| + |\lambda_k|) \sum_{|j-k| > N} \frac{1}{|\lambda_j - \lambda_k|}\right]$$
$$\leqq M(1 + |z|)^{2N+1} \exp\left[|z| \sum_{m > N} \frac{1}{\psi_m}\right]$$

for all $z \in \mathbf{C}$. This estimate ensures suitable convergence of the product in (2.2) to have μ_k entire. Further, the sum in the exponential can be made arbitrarily small by taking N large since $\{1/\psi_m\}$ is summable by assumption when each μ_k (and so each η_k) is of exponential type 0. The property (2.4) is obvious from $\mu_k(\lambda_j) = \delta_{j,k}$. Finally, for real s we have

$$|\eta_k(\lambda_k + s)| \leqq \prod_{j \neq k} \left[1 + \left(\frac{s}{\lambda_k - \lambda_j}\right)^2\right]$$
$$\leqq \prod_m \left(1 + \frac{s^2}{\psi_m^2}\right)^2 = e^{\Psi(|s|)}$$

so one has (2.5) as desired.

Selecting any $\gamma \in \Omega$ such that $e^{-\gamma}$ is integrable, we take $\omega = \Psi + \gamma$ which is in Ω by Lemma A.1; then, fixing $\delta > 0$, we let $P(\cdot)$ and $Q = Q(\delta)$ be as in Theorem K–L. In terms of this P, we define the family of functions

$$G_k(z) := \eta_k(z)P(z - \lambda_k) \quad (z \in \mathbf{C}). \tag{2.6}$$

Our first principal result of this section is the following.

THEOREM 2.2. *We have:*
(i) *Each* G_k *is an entire analytic function of exponential type* δ,
(ii) *For* $j, k \in \mathbf{Z}$ *we have* $G_k(\lambda_j) = \delta_{j,k} := \{1 \text{ for } j = k; 0 \text{ else}\}$,
(iii) *Each* G_k, *considered on the reals, is in* $L^1(\mathbf{R})$ *with*

$$|G_k(\lambda_k + s)| \leqq e^{Q(\delta)} e^{-\gamma(|s|)}, \tag{2.7}$$

(iv) *Each* G_k *is in* $L^2(\mathbf{R})$ *and one has*

$$|\langle G_j, G_k \rangle| \leqq \left[4e^{2Q(\delta)} \int_0^\infty e^{-\gamma(s)}\, ds \right] e^{-\gamma(\psi_m/2)} \tag{2.8}$$

for any $j = k \pm m$ (i.e., $m = |k - j|$).

PROOF. The assertion (i) follows on combining Lemma 2.1 (for η_k) and Theorem K–L with $\omega = \Psi + \gamma$. As noted in Lemma 2.1, we have $\eta_k(\lambda_j) = \delta_{j,k}$; hence, since $P(0) = 1$, we have (ii). The estimate (2.7) is immediate from (2.5) combined with Theorem K–L (iii) so we have (iii).

Finally, to prove (iv) we assume, with no loss of generality, that $\lambda_j \leqq \lambda_k$ and set $\lambda := (\lambda_j + \lambda_k)/2$. Note that we then have $\lambda_j = \lambda - \tau$ and $\lambda_k = \lambda + \tau$ with $\tau := (\lambda_k - \lambda_j)/2 \geqq \psi_m/2$ by (1.4) so $\gamma(\tau) \geqq \gamma(\psi_m/2)$. Note also that for $t \leqq \lambda$ one has $t - \lambda_j =: s \leqq \tau$ so $2\tau - s \geqq \tau$ and $\gamma(|2\tau - s|) \geqq \gamma(\tau)$; for $t \geqq \lambda$ we set $s := t - \lambda_k \geqq -\tau$ and $\gamma(|2\tau + s|) \geqq \gamma(\tau)$. Thus, using (2.7),

$$\begin{aligned}
|\langle G_j, G_k \rangle| &\leqq \int_{-\infty}^{\infty} |G_j(t)|\,|G_k(t)|\, dt = \int_{-\infty}^{\lambda} + \int_{\lambda}^{\infty} \\
&= \int_{-\infty}^{\tau} |G_j(\lambda_j + s)|\,|G_k(\lambda_k - [2\tau - s])|\, ds \\
&\quad + \int_{-\tau}^{\infty} |G_j(\lambda_j + [2t + s])|\,|G_k(\lambda_k + s)|\, ds \\
&\leqq e^{2Q(\delta)} \left[\int_{\to\infty}^{\tau} e^{-\gamma(|s|)} e^{-\gamma(\tau)}\, ds + \int_{-\tau}^{\infty} e^{-\gamma(\tau)\varepsilon - \gamma(|s|)}\, ds \right] \\
&\leqq e^{2Q(\delta)} e^{-\gamma(\psi_m/2)} \left[2 \int_{-\infty}^{\infty} e^{-\gamma(|s|)}\, ds \right]
\end{aligned}$$

which is just (2.8). In particular, for $j = k$ this shows $G_k \in L^2(\mathbf{R})$.

Depending on the choice of $\gamma(\cdot)$, this construction will determine the "constant" $Q(\delta)$ of (2.7) as a function of $\delta > 0$. Also depending on the choice of $\gamma(\cdot)$, but now not on δ, we set

$$A^2 := \frac{2}{\pi}\int_0^\infty e^{-\gamma(s)}\,ds\left[e^{-\gamma(0)} + 2\sum_{m=1}^{\infty} e^{-\gamma(\psi_m/2)}\right]. \tag{2.9}$$

Noting that (1.7) gives $\psi_m \geqq cm$ for some $c > 0$ so $\gamma(\psi_m/2) \geqq \gamma(cm)$, we may compare the sum to the integral $\int_0^\infty e^{-\gamma(cs)}\,ds$ and observe that the integrability of $e^{-\gamma}$ ensures finiteness of A. Our other principal result of this section is the following.

THEOREM 2.3. *Let $\lambda \in \Lambda(\psi)$ for some sequence ψ satisfying* (1.7). *Then for any $\delta > 0$ the coefficient map $\mathbf{C}_\delta = \mathbf{C}_\delta(\lambda)$ defined by* (1.3) *satisfies*

$$\|\mathbf{C}_\delta\| \leqq A e^{Q(\delta)} \quad (\delta > 0) \tag{2.10}$$

with $Q(\delta)$ as in (2.7) *and A as in* (2.9), *independent of δ.*

PROOF. The argument is here quite similar to that in [4]. We use the Fourier transform $\mathscr{F}: L^2(\mathsf{R}) \to L^2(\mathsf{R})$ given by

$$\mathscr{F}: g \mapsto G \quad \text{with} \quad G(z) = \int_{-\infty}^{\infty} e^{-izt} g(t)\,dt \tag{2.11}$$

and note that, with a factor of 2π, this is an isometric isomorphism:

$$\langle g, \tilde{g}\rangle := \int_{-\infty}^{\infty} \overline{g(t)}\tilde{g}(t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{G(t)}\tilde{G}(t)\,dt = \frac{1}{2\pi}\langle G, \tilde{G}\rangle. \tag{2.12}$$

By Theorem 2.2 (iv), each G_k is in $L^2(\mathsf{R})$ so there exist functions $g_k \in L^2(\mathsf{R})$ with $G_k = \mathscr{F} g_k$ $(k \in \mathsf{Z})$. By the Paley-Wiener Theorem [3], since Theorem 2.2 (i) gives each G_k entire and of exponential type δ, the support of each g_k be in the "window" $[-\delta, \delta]$.

Initially, let us consider $f \in \mathring{\mathscr{M}}$ (so $f = \sum_k c_k e^{i\lambda_k t}$ is a finite sum) and view this through the window as $f \in L^2(-\delta, \delta)$. Then, for each $k \in \mathsf{Z}$, noting that $\operatorname{supp}(g_k) \subset [-\delta, \delta]$,

$$\begin{aligned}\langle f, g_k\rangle &:= \int_{-\delta}^{\delta} \overline{\left(\sum_j c_j e^{i\lambda_j t}\right)} g_k(t)\,dt \\ &= \sum_j \overline{c_j}\int_{-\infty}^{\infty} e^{-i\lambda_j t} g_k(t)\,dt = \sum_j G_k(\lambda_j)\overline{c_j}.\end{aligned}$$

By Theorem 2.2 (ii) we thus have, as in (1.5),

$$c_k = \overline{\langle f, g_k\rangle} \quad (k \in \mathsf{Z},\ f \in \mathring{\mathscr{M}}). \tag{2.13}$$

Now consider the Gramian matrix $\mathbf{G}$ with entries $\langle g_j, g_k\rangle$. Since we continue to consider the (fixed) function $f \in \mathring{\mathcal{M}}$ as a finite sum, we may take $\mathbf{G}$ to be a finite matrix, considering only the indices k for which $c_k \neq 0$; thus there are no convergence problems but we seek estimates independent of this restricted index set. As a Gramian matrix, $\mathbf{G}$ is positive definite so the l^2-induced matrix norm $\|\mathbf{G}\|_2$ is just the largest eigenvalue of $\mathbf{G}$. Hence,

$$\|\mathbf{G}\|_2 \leqq \|\mathbf{G}\|_\infty := \max_j \left\{ \sum_k |\langle g_j, g_k \rangle| \right\} \tag{2.14}$$

since $\|\mathbf{G}\|_\infty$ is itself the l^∞-induced matrix norm. Thus we have

$$\left\| \sum_k a_k g_k \right\|^2_{L^2(-\delta,\delta)} = \sum_{j,k} \langle g_j, g_k \rangle \overline{a_j} a_k \leqq \|\mathbf{G}\|_\infty \|\mathbf{a}\|^2$$

for (finite) vectors $\mathbf{a} = (a_k) \in l^2$. Hence, using (2.13),

$$\|\mathbf{c}\|^2 = \sum_j |c_k|^2 = \sum_k \langle f, g_k \rangle c_k = \langle f, \sum_k c_k g_k \rangle$$

$$\leqq \|f\| \left\| \sum_k c_k g_k \right\| \leqq \|f\| (\|\mathbf{G}\|_\infty \|\mathbf{c}\|^2)^{1/2}$$

so for $f \in \mathring{\mathcal{M}}$ we have the estimate

$$\|\mathbf{c}\|_{l^2} \leqq (\|\mathbf{G}\|_\infty)^{1/2} \|f\|_{L^2(-\delta,\delta)}. \tag{2.15}$$

We now use (2.12) and (2.8) to estimate $\|\mathbf{G}\|_\infty$ from (2.14). Fixing j, we consider $k \in \mathbf{Z}$ and set $m := |k - j|$ so

$$|\langle g_j, g_k \rangle| = \frac{1}{2\pi} \langle G_j, G_k \rangle \leqq 4e^{2Q(\delta)} \left[\frac{1}{2\pi} \int_0^\infty e^{-\gamma(s)}\, ds\, e^{-\gamma(\psi_m/2)} \right].$$

Summing over $k \in \mathbf{Z}$ then gives

$$\sum_k |\langle g_j, g_k \rangle| \leqq A^2 e^{2Q(\delta)}$$

for each j so $\|\mathbf{G}\|_\infty \leqq (Ae^{Q(\delta)})^2$. Combining this with (2.15) gives

$$\|\mathbf{C}_\delta f\| = \|\mathbf{c}\| \leqq Ae^{Q(\delta)} \|f\|$$

for all $f \in \mathring{\mathcal{M}}$. By the density of $\mathring{\mathcal{M}}$ in $\mathcal{M}_\delta$, this gives precisely the desired estimate (2.10).

3. The Mollifier.

Our next object is to re-examine Theorem K–L so as to introduce the "mollifier" $P(\cdot)$ with a reasonably explicit estimate for $Q = Q(\delta)$. To this end, given $\omega \in \Omega$ we set

$$(3.1) \qquad v(s) = \frac{\omega(s)}{s^2}, \quad dq = -s^2 dv.$$

Note that the definition (1.9) of Ω ensures that q is an unbounded increasing function of s and that $\omega(\alpha)/\alpha \to 0$ as $\alpha \to \infty$. For each α in $(0, \infty)$ we can then set

$$(3.2) \qquad \delta(\alpha) := \frac{1 + 2\omega(\alpha)}{\alpha} + 2\int_\alpha^\infty \frac{\omega(s)}{s^2} ds = \frac{1}{\alpha} + 2\int_\alpha^\infty \frac{dq}{s}.$$

LEMMA 3.1. *Fix $\omega \in \Omega$ and let $\delta(\cdot)$ be defined by* (3.2). *Then $\delta(\alpha)$ is nonincreasing on $(0, \infty)$ and $\delta(\alpha) \to 0$ as $\alpha \to \infty$ so for each $\delta > 0$ there exists an $a := \alpha(\delta)$ such that $\delta(\alpha) \leqq \delta$. Further, fixing $\delta > 0$, there is a sequence (a_j) such that*

$$(3.3) \qquad \sum_0^\infty a_j \leqq \delta(\alpha) \leqq \delta$$

$$(3.4) \qquad \sum_{a_j|s| \leqq 1} |a_j|^2 \geqq \frac{2\omega(|s|) - 1}{s^2} \quad \textit{for} \quad |s| > \alpha.$$

PROOF. Deferred to the Appendix.

We can now state our revised form of Theorem K–L, including the estimate of $Q(\delta)$.

THEOREM 3.2. *For any $\delta > 0$, define $P(z)$ by*

$$(3.5) \qquad P(z) := \prod_{j=1}^\infty \cos(a_j z) \quad (z \in \mathrm{C}),$$

using the sequence (a_j) of Lemma 3.1. *Then $P(\cdot)$ is an even entire function of exponential type δ with $P(0) = 1$. Further, one has*

$$(3.6) \qquad |P(s)| \leqq e^{Q(\delta)} e^{-\omega(|s|)}$$

for all $s \in \mathrm{R}$, where Lemma 3.1 *is used to define*

$$(3.7) \qquad Q(\delta) := 1/2 + \omega(\alpha(\delta)).$$

PROOF. We know that $\cos(\cdot)$ is even and entire of exponential type 1. By (3.3), it follows that P is a well-defined even, entire function of exponential type δ. Observing that

$$|\cos s| \leqq \exp\left[-\frac{s^2}{2}\right] \quad \text{for} \quad |s| \leqq 1,$$

it follows from (3.4) that for $|s| > \alpha$ we have

$$|P(s)| \leqq \prod \{|\cos(a_j s)|: a_j|s| \leqq 1\}$$
$$\leqq \exp\left[-\frac{s^2}{2} \sum_{a_j|s| \leqq 1} a_j^2\right] \leqq e^{1/2} e^{-\omega(|s|)}.$$

Since $|P(s)| \leqq 1 \leqq e^{\omega(\alpha)} e^{-\omega(|s|)}$ for any s, we have (3.6) for all $s \in \mathrm{R}$.

4. Examples.

We now specialize our work to treat some particular cases more explicitly. In each case, we take $\omega = (1 + \varepsilon)\Psi$, i.e., $\gamma := \varepsilon\Psi$. A principal point, here, is that the asymptotics of $Q(\delta)$ as $\delta \to 0$ are (almost) determined by the asymptotics of ψ_m as $m \to \infty$. In the first two examples, we also note the convenience of taking $\psi_m = \psi(m)$ for a suitable function $\psi_m = \psi(m)$ for a suitable function $\psi(\cdot)$, giving an integral version of (2.1) for the asymptotically correct determination of $\Psi(\cdot)$.

EXAMPLE 1. We first suppose $\psi(x) = ax^p$ $(a > 0, p > 1)$; when $p = 2$ this is the case considered in [4]. It is easily seen that $\psi := \{\psi_m\}$ satisfies (1.7). To simplify the explicit computation of various quantities, we deal with the integral version of (2.1), namely,

$$\psi(s) := 2\int_0^\infty \log\left(1 + \frac{s^2}{\psi(x)^2}\right) dx$$
$$= 2\int_0^\infty \log\left(1 + \frac{s^2}{a^2 x^{2p}}\right) dx$$
$$= 2s^{1/p}\left[\frac{1}{a^{1/p}} \int_0^\infty \log\left(1 + \frac{1}{u^{2p}}\right) du\right] =: \beta(p)s^{1/p}$$

where (cf., e.g., [1] p. 114) $\beta(p) = \dfrac{2\pi}{a^{1/p}\sin\dfrac{\pi}{2p}}$. Now, let $\varepsilon > 0$ and let $\omega(s) := (1 + \varepsilon)\Psi(s)$. From (3.2),

$$\delta(\alpha) = \frac{1 + 2\omega(\alpha)}{\alpha} + 2\int_\alpha^\infty \frac{\omega(s)}{s^2} ds$$
$$= \frac{1 + 2(1 + \varepsilon)\beta(p)\alpha^{1/p}}{\alpha} + \int_\alpha^\infty \frac{2(1 + \varepsilon)\beta(p)s^{1/p}}{s^2} ds$$
$$= \frac{1}{\alpha} + \frac{\vartheta}{\alpha^{1/q}}$$

where $\vartheta := \vartheta(\varepsilon, p) = 2(1 + q)(1 + \varepsilon)\beta(p)$ with $pq = p + q$. Since $\delta(\alpha) \leqq (1 + \varepsilon)\vartheta/\alpha^{1/q}$ for large α, we see that $\alpha(\delta) \leqq [(1 + \varepsilon)\vartheta/\delta]^q$ for large α, i.e., for small δ. By (3.7),

$$Q(\delta) := \tfrac{1}{2} + \omega(\alpha) \leqq \tfrac{1}{2} + (1+\varepsilon)\beta(p)\left(\frac{(1+\varepsilon)\vartheta}{\delta}\right)^{q/p}.$$

Thus, (2.10) becomes

$$\|\mathbf{C}_\delta\| \leqq Ae^{1/2}\exp\left[B(1/\delta)^{q/p}\right] \tag{4.1}$$

where, with a corresponding constant A, $B > B_0$ is arbitrary with

$$B_0 := \beta(p)\vartheta^{q/p} = 2^{2q-1}\left(\frac{1+q}{a}\right)^{q/p}\left(\frac{\pi}{\sin \pi/2p}\right)^q.$$

Since $\dfrac{q}{p} = \dfrac{1}{p-1}$, we have the promised estimate (1.10).

EXAMPLE 2. We next consider sequences which are even more sparse: $\psi(x) = ce^{\beta x}$ with $c, \beta > 0$, indicating how various quantities can be computed. We now have

$$\begin{aligned}
\Psi(s) &= 2\int_0^\infty \log\left(1 + \frac{s^2}{\psi(x)^2}\right)dx = \frac{2}{\beta}\int_0^\infty \log\left(1 + \frac{s^2}{c^2e^{2\beta x}}\right)\beta\,dx \\
&= \frac{1}{\beta}\int_{-\sigma}^\infty \log(1+e^{-r})\,dr \qquad \left(\text{where } \frac{s^2}{c^2} = e^\sigma,\ r = 2\beta x - \sigma\right) \\
&= \frac{1}{\beta}\int_0^\infty \log(1+e^{-r})\,dr + \frac{2}{\beta}\int_0^\sigma \log(1+e^r)\,dr \\
&\sim \frac{2}{\beta}|\log s|^2 \text{ as } s \to \infty.
\end{aligned}$$

Asymptotically, $\omega(s) \sim (1+\varepsilon)\dfrac{2}{\beta}|\log s|^2$, so one has $\delta(\alpha) \sim \dfrac{8(1+\varepsilon)}{\beta\alpha}[\log \alpha|^2$ by a simple computation and $\delta(\alpha) \leqq (1/\alpha)^{1/p}$ for arbitrary $p > 1$ and large α. Hence $\alpha(\delta) \leqq 1/\delta^p$ for small δ. Thus,

$$Q(\delta) := \tfrac{1}{2} + \omega(\alpha) \leqq \tfrac{1}{2} + \frac{2}{\beta}(1+\varepsilon)p^2[\log 1/\delta]^2$$

and one has, therefore,

$$\|C_\delta\| \leqq A\exp\left[B(\log 1/\delta)^2\right] \tag{4.2}$$

for any $B > B_0 := 2/\beta$ and a suitable constant A.

EXAMPLE 3. In this example, we consider the ultimate asymptotic sparsity: a finite sequence $\{\lambda_j\}$ of $L+1$ distinct real numbers. Taking these in increasing order and setting $c := \min\{|\lambda_k - \lambda_j| : j \neq k\} > 0$, we then automatically have the condition (1.4) with $\psi_m := mc$ for $m = 0, \ldots, L$ and (formally) $\psi_m := \infty$ for $m > L$, giving

$$\Psi(s) := 2 \sum_{m=1}^{L} \log\left(1 + \frac{s^2}{c^2 m^2}\right) = 4L \log s + O(1).$$

We now have $\omega(s) \sim 4L(1 + \varepsilon)\log s$ so $\delta(\alpha) \sim 16L(1 + \varepsilon)\frac{1}{\alpha}\log\alpha$ for large α. Hence, as in the previous example, $\alpha(\delta) \leqq 1/\delta^p$ for arbitrary $p > 1$ and small δ so

$$Q(\delta) := \tfrac{1}{2} + \omega(\alpha) \leqq \tfrac{1}{2} + 4p(1 + \varepsilon)L \log 1/\delta.$$

One therefore has algebraic growth in $1/\delta$ for the norm in this case:

$$\|C_\delta\| \leqq A\delta^{-\nu} \tag{4.3}$$

for any $\nu > \nu_0 := 4L$ with a corresponding constant A.

5. Appendix.

LEMMA A.1. *Let ψ be any increasing positive sequence satisfying* (1.7). *Then* (2.1) *defines a function Ψ on* R^+ *such that*

(1.5)
- (i) *Ψ is continuous and unbounded on $[0, \infty)$ with $\Psi(0) = 0$,*
- (ii) *Ψ is C^1 and (strictly) increasing on R^+,*
- (iii) *$\Psi(s)/s^2$ is decreasing on $(0, \infty)$,*
- (iv) *$\Psi(s)/s^2$ is integrable at ∞,*
- (v) *$e^{-\Psi}$ is integrable on R^+.*

PROOF. Since $\psi = \psi_k \to \infty$ as $k \to \infty$ so $\log(1 + 1/\psi) \sim 1/\psi$, we see from (1.7) that the sum in (2.1) is well defined and finite for each $s \geqq 0$. Further, each term in that sum is (strictly) increasing in s and continuous. By the Weierstrass M-test, the series converges uniformly on any closed and bounded interval in R^+ so $\Psi(s)$ is continuous. Similarly, $\Psi' = 4\sum_1^\infty \frac{s}{(s^2 + \psi_m^2)}$ which is finite by (1.7) and positive on R^+. Thus we have (5.1-i, ii). To see (iii), we observe that

$$\Psi(s)/s^2 = 2\sum_1^\infty \frac{\rho([s/\psi_m]^2)}{\psi_m^2}$$

with $\rho(u) := \dfrac{\log(1 + u)}{u}$ and note that ρ is strictly decreasing for $u > 0$.

To get (iv), we observe that $\Psi(s)/s^2$ will be integrable at ∞ if and only if the series $\{\int_1^\infty (1/s^2)\log(1 + s^2/\psi_k^2)\, ds\}$ is summable. From the identity

$$\int \frac{\log(1+u^2)}{u^2} = 2\tan^{-1} u - \frac{\log(1+u^2)}{u},$$

we get

$$\int_1^\infty (1/s^2)\log\left(1+\frac{s^2}{\psi^2}\right)ds = \frac{\pi}{\psi} - 2\frac{1}{\psi}\tan^{-1}\frac{1}{\psi} + \log\left(1+\frac{1}{\psi^2}\right).$$

Using (1.7) and (i), we get (iv). Statement (v) is obvious.

Finally, we provide the promised proof of Lemma 3.1.

PROOF OF LEMMA 3.1. As already noted, the definition (1.9) of Ω ensures that q is increasing, so the right hand side of (3.2) is (strictly) decreasing to 0 as $\alpha \to \infty$ by the integrability of ω/s^2. Thus, $\delta(\cdot)$ is invertible with $\alpha(\delta)$ defined for (small) $\delta > 0$.

Now, fixing $\delta > 0$ and so $\alpha = \alpha(\delta)$, we use (3.1) to define (independently of the choice of the integrating constant for q!)

(5.2) $$a_j := 1/q^{-1}(z_j) \text{ with } z_j := q(\alpha) + j/2$$

for $j = 0, 1, \ldots$. An integral comparison, noting that the function $1/q^{-1}(\cdot)$ is decreasing and that $z_{j+1} - z_j \equiv 1/2$, gives

$$\sum_0^\infty a_j = \frac{1}{\alpha} + \sum_1^\infty \frac{1}{q^{-1}(z_j)} \leqq \frac{1}{\alpha} + 2\int_{q(\alpha)}^\infty \frac{dz}{q^{-1}(z)}$$

which precisely gives (3.3) on using (3.2) for $z = q(s)$.

For $|s| > \alpha$, we now note that $j_* \geqq 1$ where $j_* = j_*(s, \alpha)$ is the smallest j for which $a_j|s| \leqq 1$; hence, $0 \leqq z_{j_*} - q(|s|) < \frac{1}{2}$. An argument similar to the above then gives

$$\begin{aligned}\sum_{a_j|s| \leqq 1} [a_j]^2 &= \sum_{j_*}^\infty \frac{1}{[q^{-1}(z_j)]^2} \geqq 2\int_{z_{j_*}}^\infty \frac{dz}{[q^{-1}(z)]^2} \\ &\geqq 2\int_{q(|s|)}^\infty \frac{dz}{[q^{-1}(z)]^2} - \frac{1}{s^2} \\ &\qquad \text{(since } q^{-1}(z) \geqq |s| \text{ for } s \leqq z \leqq z_{j_*}) \\ &= 2v(|s|) - 1/s^2\end{aligned}$$

which is just (3.4).

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