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REMARKS ON THE SYMMETRIC SPACE $\mathscr{Sp}_2(\mathscr{H})/\mathscr{U}(\mathscr{H})$ AND ITS QUANTIZATION

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0. Introduction.

Consider a Bose-Fock space. Exponentials $\exp h_L$ and $\exp h_K^*$ of quadratic forms of creations and annihilations respectively play an important role in both mathematics and physics. In mathematics they help to write an explicit form of Shale-Weil representations and in physics they are used to describe squeezing of bosons.

One of the main technical problems in this connection consists of changing the order of operators in the superposition $(\exp h_K^*)$ $(\exp h_L)$. It is obvious that something like Campbell-Baker-Hausdorff formulas would be welcome. However, they are not available! Instead one uses in [2] a formula for $(\exp h_K^*)$ $(\exp h_L)$ which is sufficient to handle an explicit form of the Shale-Weil representations which we call here a quantization of the restricted symplectic group $\mathscr{Sp}_2(\mathscr{H})$. The only handicap is that the formula is verified in [2] only in a special case and that the proof requires long and tedious integration (left to the reader!).

In this paper we present a short proof of the formula for $(\exp h_K^*)$ $(\exp h_L)$ in whole generality. We give a thorough exposure of the relation of this formula to the quantized $\mathscr{Sp}_2(\mathscr{H})$. Moreover we show that the formula can be used to design a kind of a product of uniquelly selected representatives of the cosets of the symmetric space $\mathscr{Sp}_2(\mathscr{H})/\mathscr{U}(\mathscr{H})$, where $\mathscr{U}(\mathscr{H})$ denotes the subgroup of all unitary operators.

At the end of the paper we point out that the formula for $(\exp h_K^*)$ $(\exp h_L)$ goes further than the application to $\mathscr{Sp}_2(\mathscr{H})$ representation and squeezing of bosons.

1. Preliminaries.

In the sequal $\mathscr{H}$, $\langle , \rangle$ will denote a complex Hilbert space. Consider the symplectic form $\sigma(x, y) = \frac{1}{2}i(\langle x, y\rangle - \langle y, x\rangle)$. It is well known that a symplectic

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operator M can uniquely be represented in the form $M = NU$, where N is symplectic real selfadjoint and non negative and U is unitary [2]. Let $\mathscr{S}\!p_2(\mathscr{H})$ be the subgroup of the symplectic group consisting of the operators M with $I - (M'M)^{\frac{1}{2}}$ Hilbert-Schmidt. In [3] $\mathscr{S}\!p_2(\mathscr{H})$ is called the restricted symplectic group. Given a contraction L we call $\kappa L = (I - L)(I + L)^{-1}$ the Cayley transform of L and we define $(\kappa L)^{\frac{1}{2}} = (I - L)(I - L^2)^{-\frac{1}{2}}$ taking for $(I - L^2)^{-\frac{1}{2}}$ the positive square root of $(I - L^2)^{-1}$. The restricted symplectic group consist of the operators $M = (\kappa L)^{\frac{1}{2}}U$, where and $L \in \pounds_1(\mathscr{H})$ (= conjugate linear real-selfadjoint Hilbert-Schmidt contractions) and U is unitary. We have the identity $(\kappa \tanh - M)^{\frac{1}{2}} = \exp M$ for $M \in \pounds_{\mathrm{HS}}(\mathscr{H})$ (= conjugate linear real-self-adjoint Hilbert-Schmidt operators). The operator $\exp M$ has only positive eigenvalues and $\tanh - M$ is obviously in $\pounds_1(\mathscr{H})$. Consider the symmetric space $\mathscr{S}\!p_2(\mathscr{H})/\mathscr{U}(\mathscr{H})$ of the cosets $M\mathscr{U}(\mathscr{H})$, where $M \in \mathscr{S}\!p_2(\mathscr{H})$. To each $M\mathscr{U}(\mathscr{H})$ there corresponds a unique $L \in \pounds_1(\mathscr{H})$ such that $(\kappa L)^{\frac{1}{2}} \in M\mathscr{U}(\mathscr{H})$. The product $(\kappa L)^{\frac{1}{2}}(\kappa K)^{\frac{1}{2}}$ belongs to a coset of $M\mathscr{U}(\mathscr{H})$ with the unique representant of the form $(\kappa N)^{\frac{1}{2}}$, $N \in \pounds_1(\mathscr{H})$. Hence $(\kappa L)^{\frac{1}{2}}(\kappa K)^{\frac{1}{2}} = (\kappa N)^{\frac{1}{2}}U$ and then $\kappa N = (\kappa L)^{\frac{1}{2}}(\kappa K)(\kappa L)^{\frac{1}{2}}$. We define $L \# K = (I - L^2)^{-\frac{1}{2}}(L + K)(I + LK)^{-1}(I - L^2)^{\frac{1}{2}}$ and find that,

$$\kappa(L \# K) = (\kappa L)^{\frac{1}{2}}(\kappa K)(\kappa L)^{\frac{1}{2}} = \kappa N.$$

Hence $L \# K \in \pounds_1(\mathscr{H})$. If $[L, K] = 0$ then $L \# K = (L + K)(I + LK)^{-1}$. We have proved that $K \# L$ is allways a contraction which does not seem obvious for non commuting $K, L \in \pounds_1(\mathscr{H})$.

We shall use an axiomated version of Bose-Fock space as described in [4]. We recall main definitions. We consider the free commutative algebra generated by the Hilbert space $\mathscr{H}$ and the unit element ϕ called the vacuum. We denote this algebra by $\Gamma_0\mathscr{H}$ and call it the Bose algebra of $\mathscr{H}$. We extend the scalar product of $\mathscr{H}$ to $\Gamma_0\mathscr{H}$ in such a way that the vacuum is a unit vector, and the adjoint $\mathbf{a}(x)$ to the operator $\mathbf{a}^+(x)$ of multiplication by $x \in \mathscr{H}$ is defined on the whole $\Gamma_0\mathscr{H}$ and fulfils the Leibnitz rule. The operators $\mathbf{a}^+(x)$, $\mathbf{a}(y)$ shall be called the creation and the annihilation by x, y respectively.

We set $\Gamma_1\mathscr{H} = \{f \cdot \exp x \mid f \in \Gamma_0\mathscr{H}, x \in \mathscr{H}\}$ and write $\Gamma\mathscr{H}$ for the completion of $\Gamma_0\mathscr{H}, \langle , \rangle$ and $\mathscr{H}^n$ for the closure in $\Gamma\mathscr{H}$ of the linear span $\mathscr{H}_0^n$ of all the n-fold products of elements of $\mathscr{H}$.

The field operator $i(\mathbf{a}^+(x) - \mathbf{a}(x))$ is known to be essentially selfadjoint. The unitary operators $W(x) = \exp(\mathbf{a}^+(x) - \mathbf{a}(x))$, $x \in \mathscr{H}$, are called the Weyl operators.

Given $L \in \pounds_1(\mathscr{H})$, we define the operator $\mathbf{a}^+(h_L)$ on $\Gamma_0\mathscr{H}$ setting

$$\mathbf{a}^+(h_L) = \sum \mathbf{a}^+(e_n)\mathbf{a}^+(Le_n),$$

where $\{e_n\}$ is an ortonormal basis in $\mathscr{H}$. Furthermore, we define $\mathbf{a}^+(\delta_L)$ setting

$$\mathbf{a}^+(\delta_L) = \exp - \tfrac{1}{2}\mathbf{a}^+(h_L).$$

Observe that the dual $\mathbf{a}(\delta_L)$ of $\mathbf{a}^+(\delta_L)$ is defined on the whole $\Gamma_0\mathscr{H}$. Let Γ be the homomorphism lifting the linear operators on $\mathscr{H}$ to operators on $\Gamma\mathscr{H}$ i.e.

$$\Gamma A(x_1 x_2 \dots x_n) = (Ax_1)(Ax_2)\dots(Ax_n)$$

for all $x_1, x_2, \dots, x_n \in \mathscr{H}$. We introduce a unitary operator $\theta_L : \Gamma\mathscr{H} \to \Gamma\mathscr{H}$, setting

$$\theta_L = \det(I - L^2)^{\frac{1}{4}}\mathbf{a}^+(\delta_L)\Gamma(I - L^2)^{\frac{1}{2}}\mathbf{a}(\delta_{-L})$$

Elementary computation ascertains that

$$\theta_L W(x) = W((\kappa L)^{\frac{1}{2}} x)\theta_L$$

Remembering that the representation $M = (\kappa L)^{\frac{1}{2}} U \in \mathscr{Sp}_2(\mathscr{H})$ is unique, we define $\theta(M) = \theta_L \Gamma U$ getting a projective representation $\mathscr{Sp}_2(\mathscr{H}) \ni M \to \theta(M) \in \mathscr{U}(\Gamma\mathscr{H})$ of $\mathscr{Sp}_2(\mathscr{H})$ known as the Shale-Weil representation [3, 8]. Hence for M_1, $M_2 \in \mathscr{Sp}_2(\mathscr{H})$, we have

$$\theta(M_1)\theta(M_2) = c(M_1, M_2)\theta(M_1 M_2).$$

where $c(M_1, M_2) \in S^1$. Consider a special case,

$$\theta((\kappa L)^{\frac{1}{2}})\theta((\kappa K)^{\frac{1}{2}}) = c((\kappa L)^{\frac{1}{2}}, (\kappa K)^{\frac{1}{2}})\theta((\kappa L)^{\frac{1}{2}}(\kappa K)^{\frac{1}{2}}) = c((\kappa L)^{\frac{1}{2}}, (\kappa K)^{\frac{1}{2}})\theta_N \Gamma U$$

where $(\kappa N)^{\frac{1}{2}} U = (\kappa L)^{\frac{1}{2}}(\kappa K)^{\frac{1}{2}} = (\kappa L \# K)^{\frac{1}{2}} U$ as it was shown earlier.

2. The representation of the symmetric space $\mathscr{Sp}(\mathscr{H})/\mathscr{U}(\mathscr{H})$.

In connection with the Shale-Weil representation emerges an operator-identity which is of an independent interest (cf. [2]). In this section we will prove this identity using a method which is much simpler than those applied in [2] and [5] and which provides more general results. Afterward we shall show the connections of this identity with the Shale-Weil representation of $\mathscr{Sp}_2(\mathscr{H})$ and the symmetric space $\mathscr{Sp}_2(\mathscr{H})/\mathscr{U}(\mathscr{H})$.

LEMMA 1. *For $L, K \in \mathfrak{L}_1(\mathscr{H})$*

$$\langle \delta_L, \delta_K \rangle = \det(I - KL)^{-\frac{1}{2}}.$$

PROOF. Set $a_n = (n!)^{-2}\langle(\tfrac{1}{2}h_L)^n, (\tfrac{1}{2}h_K)^n\rangle$ and $t_n = \tfrac{1}{2}\mathrm{Tr}(KL)^n$. We first verify that

$$(n!)^{-2}\langle(\tfrac{1}{2}h_N)(\tfrac{1}{2}h_L)^{n-1}, (\tfrac{1}{2}h_K)^n\rangle$$

$$= \tfrac{1}{2}n^{-1}a_{n-1}\,\mathrm{Tr}(KN) + ((n-1)!(n-2)!n)^{-1}\langle(\tfrac{1}{2}h_L)^{n-1}, (\tfrac{1}{2}h_{KNK})(\tfrac{1}{2}h_K)^{n-2}\rangle$$

and a short calculation gives $a_n = \sum_{j=1}^{n} n^{-1} t_j a_{n-j}$. Since $\sum_{n=0}^{\infty} a_n x^n = \exp\left(\sum_{n=0}^{\infty} b_n x^n\right)$ for $x \in [-1, 1]$ if and only if $a_n = \sum_{j=1}^{n} n^{-1} j b_j a_{n-j}$ we get that

$$\langle \delta_L, \delta_K \rangle = \sum_{n=0}^{\infty} a_n = \exp\left(\sum_{n=1}^{\infty} n^{-1} t_n\right)$$

$$= \exp(\mathrm{Tr}(\log(I - KL)^{-\frac{1}{2}})) = \det(I - KL)^{-\frac{1}{2}}$$

For alternative treatment see [1] or [6]. The case where $K = L$ appears already in [7].

LEMMA 2. *Given $x \in \mathscr{H}$ and $L \in \pounds_1(\mathscr{H})$, we have on $\Gamma_1 \mathscr{H}$ the relations*

$$\mathbf{a}(\delta_L) \exp \mathbf{a}^+(x) = \exp - \tfrac{1}{2}\langle Lx, x\rangle (\exp \mathbf{a}^+(x))(\exp - \mathbf{a}(Lx))\mathbf{a}(\delta_L)$$

and

$$(\exp \mathbf{a}(x))\mathbf{a}^+(\delta_L) = \exp - \tfrac{1}{2}\langle x, Lx\rangle \mathbf{a}^+(\delta_L)(\exp - \mathbf{a}^+(Lx))(\exp \mathbf{a}(x)).$$

PROOF. It follows easily from the identity $\mathbf{a}(\delta_L)\mathbf{a}^+(x) = (\mathbf{a}^+(x) - \mathbf{a}(Lx))\mathbf{a}(\delta_L)$ and its dual.

LEMMA 3. *Given $x \in \mathscr{H}$ and $K, L \in \pounds_1(\mathscr{H})$, we have the relation*

$$\langle \delta_L \exp x, \delta_K \rangle = \det(I - KL)^{-\frac{1}{2}} \exp - \tfrac{1}{2}\langle x, (I - KL)^{-1} Kx \rangle.$$

PROOF. Using induction we get

$$\langle \delta_L \exp x, \delta_K \rangle = \langle \delta_L \exp(LK)^p x, \delta_K \rangle \exp - \tfrac{1}{2}\left\langle x, \left(\sum_{n=0}^{2p-1} (KL)^n\right) Kx \right\rangle$$

for all $p \in \mathrm{N}$. Since LK is a contraction the limit for $p \to \infty$ exists and the relation follows.

Using these Lemmas we can show the desired identity.

THEOREM 4. *For $L, K, (I - KL)^{-1} K \in \pounds_1(\mathscr{H})$ we have $\delta_K \Gamma_1 \mathscr{H} \subset D(\delta_L^*)$ and the identity,*

$$\mathbf{a}(\delta_L)\mathbf{a}^+(\delta_K) = \det(I - KL)^{-\frac{1}{2}} \mathbf{a}^+(\delta_{(I-KL)^{-1}K}) \Gamma(I - KL)^{-1} \mathbf{a}(\delta_{L(I-KL)^{-1}})$$

PROOF. We show the weak identity

$$\langle f, \mathbf{a}(\delta_L)\mathbf{a}^+(\delta_K) g\rangle$$

$$= \langle f, \det(I - KL)^{-\frac{1}{2}} \mathbf{a}^+(\delta_{(I-KL)^{-1}K}) \Gamma(I - KL)^{-1} \mathbf{a}(\delta_{L(I-KL)^{-1}}) g\rangle$$

for $f, g \in \Gamma_1 \mathscr{H}$. The identity is easily rewritten using the usual notation,

$$\langle \delta_L f, \delta_K g\rangle = \det(I - KL)^{-\frac{1}{2}}\langle \delta_K^* \Gamma(I - LK)^{-\frac{1}{2}} f, \delta_L^* \Gamma(I - KL)^{-\frac{1}{2}} g\rangle$$

Since the nth derivative of $\exp(x + ta)$ in 0 is equal to $a^n \exp x$, it is sufficient to show the identity for $f = \exp x$ and $g = \exp y$. The right-hand side of the identity can be rewritten using Lemma 2,

$$\begin{aligned}\det(I - KL)^{-\frac{1}{2}}&\langle \delta_K^* \exp(I - LK)^{-\frac{1}{2}} x, \delta_L^* \exp(I - KL)^{-\frac{1}{2}} y\rangle \\ &= \det(I - KL)^{-\frac{1}{2}} \exp - \tfrac{1}{2}(\langle x, (I - KL)^{-1} Kx\rangle \\ &+ \langle Ly, (I - KL)^{-1} y\rangle - 2\langle x, (I - KL)^{-1} y\rangle).\end{aligned}$$

The left-hand side of the identity can be rewritten as follows. Using $(\exp y^*)(\exp x) = (\exp \langle y, x\rangle)(\exp x)(\exp y^*)$ (cf. [4]) and lemma 2 we get

$$\begin{aligned}\langle \delta_L \exp x, \delta_K \exp y\rangle &= \exp(-\tfrac{1}{2}\langle Ly, y\rangle + \langle x, y\rangle)\langle \delta_L \exp(x - Ly), \delta_K\rangle \\ &= \det(I - KL)^{-\frac{1}{2}} \exp(-\tfrac{1}{2}\langle Ly, y\rangle + \langle x, y\rangle - \tfrac{1}{2}\langle (x - Ly), \\ &\qquad (I - KL)^{-1} K(x - Ly)\rangle),\end{aligned}$$

where the last identity follows from lemma 3. Since $K(I - LK)^{-1} = (I - KL)^{-1} K$ the Theorem holds.

An operator ϕ is said to have the standard form if $\phi = \text{constant} \cdot \mathbf{a}^+(\delta_K)\Gamma A \mathbf{a}(\delta_L)$, where $K, L \in \pounds_1(\mathscr{H})$ and A is bounded operator. We observe that operators in the Shale-Weil representation have all the standard form (in fact if ϕ is unitary it implements a restricted symplectic operator). The following lemma establishes the uniqueness of the standard forms.

LEMMA 5. *Let $K, L, M, N \in \pounds_1(\mathscr{H})$, and let A, B be bounded linear operators and c a constant. If*

$$\mathbf{a}^+(\delta_K)\Gamma A \mathbf{a}(\delta_L) = c\mathbf{a}^+(\delta_M)\Gamma B \mathbf{a}(\delta_N)$$

on the coherent vectors $\exp x$, $x \in \mathscr{H}$, *then*

$$K = M \quad L = N \quad A = B, \quad \textit{and} \quad c = 1.$$

PROOF. We have the weak identity for $x, y \in \mathscr{H}$,

$$\langle \exp x, \delta_K \Gamma A \delta_L^* \exp y\rangle = \langle \exp x, c\delta_M \Gamma B \delta_N^* \exp y\rangle$$

and

$$\langle \delta_K^* \exp x, \Gamma A \delta_L^* \exp y\rangle = c\langle \delta_M^* \exp x, \Gamma B \delta_N^* \exp y\rangle$$

Using the calculation from theorem 4 we find,

$$\exp \tfrac{1}{2}(\langle x, (K - M)x\rangle + \langle (L - N)y, y\rangle + 2\langle x, (B - A)y\rangle) = c.$$

Setting $t^{\frac{1}{2}}x$ for x and $t^{\frac{1}{2}}y$ for y and differentiating at 0 we get,

$$\langle x,(K-M)x\rangle + \langle (L-N)y,y\rangle + 2\langle x,(B-A)y\rangle = 0.$$

Since x and y are free to choose the lemma follows.

It is now possible to find the product of two arbitrary operators in the metaplectic representation of the symmetric space by use of direct calculations [9].

THEOREM 6. *Let $L, K, (I+KL)^{-1}K, L(I+KL)^{-1} \in \pounds_1(\mathscr{H})$. Then $L \# K \in \pounds_1(\mathscr{H})$ and on $\Gamma_1\mathscr{H}$ we have the identity,*

$$\theta_L\theta_K = (\det U)^{\frac{1}{2}}\theta_N \Gamma U$$

where,

$$N = L \# K$$

and

$$U = (I-N^2)^{-\frac{1}{2}}(I-L^2)^{\frac{1}{2}}(I+KL)^{-1}(I-K^2)^{\frac{1}{2}}.$$

is unitary.

PROOF. We compute,

$$\begin{aligned}\theta_L\theta_K &= \det(I-L^2)^{\frac{1}{4}}\delta_L\Gamma(I-L^2)^{\frac{1}{2}}\delta^*_{-L}\det(I-K^2)^{\frac{1}{4}}\delta_K\Gamma(I-K^2)^{\frac{1}{2}}\delta^*_{-K}\\ &= \det(I-L^2)^{\frac{1}{4}}(I-K^2)^{\frac{1}{4}}\delta_L\Gamma(I-L^2)^{\frac{1}{2}}\delta^*_{-L}\delta_K\Gamma(I-K^2)^{\frac{1}{2}}\delta^*_{-K}\end{aligned}$$

Using the identity in theorem 4 we find

$$\delta^*_{-L}\delta_K = \det(I+KL)^{-\frac{1}{2}}\delta_{(I+KL)^{-1}K}\Gamma(I+KL)^{-1}\sigma^*_{-L(I+KL)^{-1}}.$$

Let us set $c = \det(I-L^2)^{\frac{1}{4}}(I+KL)^{-\frac{1}{2}}(I-K^2)^{\frac{1}{4}}$. Then

$$\theta_L\theta_K = c\delta_{L+(I-L^2)^{\frac{1}{2}}(I+KL)^{-1}K(I-L^2)^{\frac{1}{2}}}\Gamma(I-L^2)^{\frac{1}{2}}(I+KL)^{-1}(I-K^2)^{\frac{1}{2}}$$
$$\delta^*_{-(K+(I-K^2)^{\frac{1}{2}}L(I+KL)^{-1}(I-K^2)^{\frac{1}{2}})},$$

where we have used that $\delta^*_L\Gamma A = \Gamma A\delta^*_{A^*LA}$ and $\Gamma A\delta_L = \delta_{A^*LA}\Gamma A$. Using

$$L \# K = L + (I-L^2)^{\frac{1}{2}}(I+KL)^{-1}K(I-L^2)^{\frac{1}{2}} \in \pounds_1(\mathscr{H})$$

we get

$$\theta_L\theta_K = c\delta_{L\#K}\Gamma(I-L^2)^{\frac{1}{2}}(I+KL)^{-1}(I-K^2)^{\frac{1}{2}}\delta^*_{-K\#L}.$$

But we know that $\theta_L\theta_K = k\theta_N\Gamma U$ for some constant k, a unitary U and $N \in \pounds_1(\mathscr{H})$ which means that $\theta_N = k^{-1}\theta_L\theta_K\Gamma U^{-1}$ i.e.

$$\det(I - N^2)^{\frac{1}{2}}\delta_N \Gamma(I - N^2)^{\frac{1}{2}}\delta^*_{-N}$$
$$= k^{-1}c\delta_{L\# K}\Gamma(I - L^2)^{\frac{1}{2}}(I + KL)^{-1}(I - K^2)^{\frac{1}{2}}U^{-1}\delta^*_{-U(K\# L)U^{-1}}.$$

From lemma 5 we find the identities

$$(I - N^2)^{\frac{1}{2}} = (I - L^2)^{\frac{1}{2}}(I + KL)^{-1}(I - K^2)^{\frac{1}{2}}U^{-1}$$
$$N = L\# K$$
$$k = c\det(I - N^2)^{-\frac{1}{2}} = (\det(I - N^2)^{-\frac{1}{2}}(I - L^2)^{\frac{1}{2}}(I + KL)^{-1}(I - K^2)^{\frac{1}{2}})^{\frac{1}{2}} = \det U^{\frac{1}{2}}$$

We have used theorem 4 to find the composition of θ_L's. However, the identity of theorem 4 can be also useful in a quite different capacity. Consider the complex wave representation of $\Gamma\mathscr{H}$ (cf. [4]). Then elements of $\Gamma\mathscr{H}$ are represented by functions. Take δ_L and δ_K for commuting $K, L \in \pounds_1(\mathscr{H})$ such that $\|K + L\| < 2$. Then for $f, g \in \Gamma_0\mathscr{H}$ we formally have

$$\langle \delta_K f, \delta_L g\rangle = \int \overline{f(z)}g(z)\exp - \tfrac{1}{2}(\langle z, Lz\rangle + \langle Kz, z\rangle)\gamma^{\frac{1}{2}}(dz),$$

where the left hand side is defined only for $\|K\|$, $\|L\| < 1$ while the right hand side exists for $\|K + L\| < 2$. Hence the right hand side

$$\det(I - KL)^{-\frac{1}{2}}\mathbf{a}^+(\delta_{(I-KL)^{-1}K})\Gamma(I - KL)^{-1}\mathbf{a}(\delta_{L(I-KL)^{-1}})$$

of the identity of theorem 4 defines $\delta_L^*\delta_K$ for $\|KL\| < 1$ (for K, L commuting, $\|K + L\| < 2$ implies that $\|KL\| < 1$).

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