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ON THE APPROXIMATION OF CERTAIN INFINITE PRODUCTS

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1. Introduction.

We shall consider the infinite product

$$f(z) = \prod_{n=1}^{\infty}\left(1 + \frac{z}{q^n}\right) \tag{1}$$

with certain algebraic q such that $|q| > 1$. The q-exponential function f is an entire function

$$f(z) = 1 + \sum_{n=1}^{\infty}\left(z^n \Big/ \prod_{k=1}^{n}(q^k - 1)\right) \tag{2}$$

satisfying the functional equation

$$f(qz) = (1 + z)f(z). \tag{3}$$

Already Lototsky [8] proved the irrationality of $f(\alpha)$, if $q \geqq 2$ is a positive integer and $\alpha \neq 0, -q^n, n = 1, 2, \ldots$, is an element of an imaginary quadratic field. Since then there are many works considering the irrationality and irrationality measures of the values $f(\alpha)$, where q and α are certain elements of an imaginary quadratic field, see e.g. the papers [2], [3], [4], [5], [9], [11], [12], [13], [14], and [15]. The works [2], [9], and [10] consider also non-archimedian case. The most general qualitative results are those of Bézivin giving linear independence of the values of f and its derivatives at different points. In [6] linear independence questions of f and its derivatives at $z = \alpha$ with certain algebraic q and α are considered.

Our aim in the present work is to generalize the recent results of Popov [12], [13] and also to give non-archimedian analogues of these results. We shall use the Newton interpolation technique applied already in many of the above mentioned works. In particular, the Schnirelman integral is used in the non-archimedian case to replace the complex integral.

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2. Notations and results.

Let K be an algebraic number field of degree d over $\mathbf{Q}$. For every place v of K we denote $d_v = [K_v : \mathbf{Q}_v]$. Let $P = \{\text{primes } p\} \cup \{\infty\}$. If the finite place v of K lies over the prime p, we write $v \mid p$, for infinite place v of K we write $v \mid \infty$. We normalize the absolute value $|\ |_v$ of K so that

(i) if $v \mid p$, then $|p|_v = p^{-1}$,

(ii) if $v \mid \infty$, then $|x|_v = |x|$,

where $|\ |$ denotes the ordinary absolute value in $\mathbf{R}$ or in $\mathbf{C}$. Clearly we then have the product formula

$$\prod_v |\alpha|_v^{d_v} = 1$$

for all non-zero $\alpha \in K$. Further, for any vector $\underline{\alpha} = (\alpha_1, \ldots, \alpha_k)$, $\alpha_i \in K$, we define $|\underline{\alpha}|_v = \max(|\alpha_1|_v, \ldots, |\alpha_k|_v)$.

The absolute height $h(\alpha)$ of $\alpha \in K$ is defined by the formula

$$h(\alpha) = \prod_v \max(1, |\alpha|_v^{d_v/d}),$$

and the absolute height $h(\underline{\alpha})$ of the vector $\underline{\alpha}$ by

$$h(\underline{\alpha}) = \prod_v \max(1, |\underline{\alpha}|_v^{d_v/d}).$$

If v is a place of K and q is a non-zero element of K, we shall use the notation $\lambda = \lambda(v, q)$ to denote

$$\lambda = (d \log h(q))/(d_v \log |q|_v). \tag{4}$$

In particular we note that $\lambda = 1$ if $|q|_w \leqq 1$ for all $w \neq v$. *If* $|q|_v > 1$, then the values of f at non-zero points $\alpha \in K$, $\alpha \neq -q^n$, $n = 1, 2, \ldots$, are non-zero elements of K_v. Our first results consider the approximation of these numbers by the elements of K.

THEOREM 1. *Let $|\ |_v$ be any valuation of K, and suppose that $|q|_v > 1$ and $\lambda < 7/4$. Further, let $\alpha \neq -q^n$, $n = 1, 2, \ldots$, be a non-zero element of K. Then $f(\alpha) \notin K$, and there exist effectively computable positive constants $\gamma_1 = \gamma_1(q, \alpha, v)$ and $h_1 = h_1(q, \alpha, v)$ such that for any $\theta \in K$*

$$|f(\alpha) - \theta| > H^{-\frac{7d}{(7-4\lambda)d_v} - \gamma_1 (\log H)^{-1/2}},$$

where $H = \max(h(\theta), h_1)$.

We note that in the archimedian case Theorem 1 is analogous to the result of [13]. The following theorem considers simultaneous approximation of two values of f and generalizes [12].

THEOREM 2. *Let* $|\ |_v$ *be any valuation of* K, *and suppose that* $|q|_v > 1$ *and* $\lambda < 13/7$. *Suppose that* α_1 *and* α_2 *are non-zero elements of* K *satisfying*

$$\alpha_i \neq -q^n,\ \alpha_2 \neq \alpha_1 q^m,\ i = 1, 2;\ n = 1, 2, \ldots;\ m \in Z. \tag{5}$$

Then there exist effectively computable positive constants $\gamma_2 = \gamma_2(q, \alpha_i, v)$ *and* $h_2 = h_2(q, \alpha_i, v)$ *such that for any* $\underline{\theta} = (\theta_1, \theta_2) \in K^2$

$$\max_i |f(\alpha_i) - \theta_i|_v > H^{-\frac{13d}{(13-7\lambda)d_v} - \gamma_2(\log H)^{-1/2}},$$

where $H = \max(h(\underline{\theta}), h_2)$.

Next we generalize the result given in the archimedean case in the field $K = \mathbf{Q}$ as Theorem 1 of [12]. We shall prove the following

THEOREM 3. *Let* $|\ |_v$ *be any valuation of* K, *and suppose that* $|q|_v > 1$ *and* $\lambda < 6/5$. *Suppose that* α_1 *and* α_2 *are non-zero elements of* K *satisfying* (5). *Then there exist effectively computable positive constants* $\gamma_3 = \gamma_3(q, \alpha_i, v)$ *and* $h_3 = h_3(q, \alpha_i, v)$ *such that for any* $\theta \in K$

$$\left|\frac{f(\alpha_2)}{f(\alpha_1)} - \theta\right|_v > H^{-\frac{6d}{(6-5\lambda)d_v} - \gamma_3(\log H)^{-1/2}},$$

where $H = \max(h(\theta), h_3)$.

In the following result we consider the approximation of $f^{(i)}(\alpha)/f(\alpha)$, $i = 1, 2, \ldots, k-1$, where $k \geqq 2$ is a natural number. To formulate our result we define A and B by the formulae

$$A = k + 3(k-1)/\pi^2, \quad B = (k-1)(k/2 - 3/\pi^2).$$

THEOREM 4. *Let* $|\ |_v$ *be any valuation of* K, *and suppose that* $|q|_v > 1$ *and* $|q|_w \neq 1$ *for all* $w \mid \infty$. *Let also* $A + B - \lambda A > 0$. *If* α *satisfies the conditions of Theorem* 1, *then there exist effectively computable positive constants* $\gamma_4 = \gamma_4(q, \alpha, v, k)$ *and* $h_4 = h_4(q, \alpha, v, k)$ *such that for any* $\underline{\theta} = (\theta_1, \ldots, \theta_{k-1}) \in K^{k-1}$

$$\max_i \left|\frac{f^{(i)}(\alpha)}{f(\alpha)} - \theta_i\right|_v > H^{-\frac{(A+B)d}{(A+B-\lambda A)d_v} - \gamma_4\frac{\log\log H}{(\log H)^{1/2}}},$$

where $H = \max(h(\underline{\theta}), h_4)$.

REMARK. If $\alpha = -1$, then we can replace A and B above by

$$A = (k+1)/2 + 3(k-1)/\pi^2, \quad B = (k-1)((k+1)/2 - 3/\pi^2).$$

In the case $k = 2$ Theorem 4 is essentially Theorem 2 of [6]. Since $f'(z)/f(z) = \sum_{n=1}^{\infty} (q^n + z)^{-1}$, this case has some interesting corollaries on the approximation of certain numbers, e.g. the numbers $\sum_{n=1}^{\infty}(q^n-1)^{-1}$, $q=2,3,\ldots$, have an irrationality measure $\leqq(1/2-1/\pi^2)^{-1}\approx 2,51$, and the number $\sum_{n=1}^{\infty}F_n^{-1}$, where F_n denotes the nth Fibonacci number, has $6/(1-3/\pi^2)\approx 8,62$ as a measure of approximation by the elements of $\mathbf{Q}(\sqrt{5})$, see [6].

3. The asymptotic estimate for the remainder term.

Let $|\ |_v$ be a valuation of K. If $v|P$ we shall consider all the elements of K as elements of $\mathbf{C}_P$ ($\mathbf{C}_\infty=\mathbf{C}$) given by a corresponding embedding of K in $\mathbf{C}_P$. For $|q|_v>1$ the function $f(z)$ is then an entire function in $\mathbf{C}_P$.

All our resuts will be corollaries of one general result. To prove this we suppose in the following that for some $k\in N$ $\alpha_1,\ldots,\alpha_k$ are non-zero elements of K satisfying

$$\alpha_i\neq -q^n,\ \alpha_j\neq\alpha_i q^m,\ n=1,2,\ldots;\ m\in Z;\qquad i,j=1,2,\ldots,k;\ i\neq j. \tag{6}$$

If $v|\infty$, we define the complex integral

$$I_v(k,n)=\frac{1}{2\pi i}\int_{|z|=R}\frac{f(z)\,dz}{z^{\delta(n+1)}\prod_{\nu=0}^{n}(z+q^{\nu+1})\prod_{i=1}^{k}\prod_{\nu=0}^{n}(z-\alpha_i q^\nu)},$$

where $\delta=0$ or 1, and $R=|q|_v^{(k+1+\delta)n+1}$. We assume that $k\geqq 2$ if $\delta=0$. For finite v we use the Schnirelman integral

$$I_v(k,n)=\int_{0,R}\frac{f(z)z\,dz}{z^{\delta(n+1)}\prod_{\nu=0}^{n}(z+q^{\nu+1})\prod_{i=1}^{k}\prod_{\nu=0}^{n}(z-\alpha_i q^\nu)}.$$

For the basic properties of this integral we refer to [1].

In the following we shall need an asymptotic estimate for these integrals. We note that $c_1,c_2,\ldots$ denote effectively computable positive constants independent of n. Also the constants in O $(n\to\infty)$ are effectively computable.

LEMMA 1. *If* $|q|_v>1$, *then*

$$|I_v(k,n)|_v=|q|_v^{-(k+1+\delta)^2n^2/2+O(n)}.$$

PROOF. If $v|\infty$, then Lemma 1 follows from Lemma 2 of [12].
If $v|p$, then

$$\prod_{\nu=0}^{n}(z+q^{\nu+1})\prod_{i=1}^{k}\prod_{\nu=0}^{n}(z-\alpha_i q^\nu)=$$

$$z^{(k+1)(n+1)}\prod_{\nu=0}^{n}(1+q^{\nu+1}/z)\prod_{i=1}^{k}\prod_{\nu=0}^{n}(1-\alpha_i q^\nu/z)=$$

$$z^{(k+1)(n+1)}(1+w_n(z)),$$

where $|w_n(z)|_v \leqq |q|_v^{-(k+\delta-1)n}$ for all $|z|_v = R$ and $n \geqq c_1$. By using this notation we now have

$$I_v(k,n)=\int_{0,R}\frac{f(z)z\,dz}{z^{(k+1+\delta)(n+1)}}-\int_{0,R}\frac{f(z)w_n(z)z\,dz}{z^{(k+1+\delta)(n+1)}(1+w_n(z))}=I(1)+I(2),\text{ say.}$$

The results of [1] imply with (2) the formula

$$I(1)=\mathrm{Res}_{z=0}f(z)z^{-(k+1+\delta)(n+1)}=\prod_{\nu=1}^{(k+1+\delta)(n+1)-1}(q^\nu-1)^{-1}.$$

Thus

$$|I(1)|_v=|q|_v^{-(k+1+\delta)(n+1)((k+1+\delta)(n+1)-1)/2}. \tag{7}$$

On the other hand, since by (3)

$$f(z/q^{(k+1+\delta)n})=f(z)q^{(k+1+\delta)n((k+1+\delta)n+1)/2}\times\prod_{\nu=1}^{(k+1+\delta)n}(z+q^\nu)^{-1},$$

it follows from (7) that

$$\sup_{|z|_v=R}\left|\frac{f(z)w_n(z)z}{z^{(k+1+\delta)(n+1)}(1+w_n(z))}\right|_v\leqq|I(1)|_v|q|_v^{-n/2}\sup_{|z|_v=|q|_v}|f(z)|_v$$

for all $n \geqq c_2$. We thus have, for all $n \geqq c_3$,

$$|I_v(k,n)|_v=|I(1)|_v=|q|_v^{-(k+1+\delta)^2n^2/2+O(n)}.$$

This proves Lemma 1.

4. Approximation forms.

We first define for each $n \in N$ the polynomials $B_n^\nu(q)$ needed in the following consideration. Let

$$B_n^\nu(q)=\prod_{\mu=n-\nu+1}^{n}(q^\mu-1)\Big/\prod_{\mu=1}^{\nu}(q^\mu-1),\ \nu=1,2,\ldots,n,$$

$$B_n^0(q)=1.$$

In [7], p. 157, Gelfond proves that $B_n^\nu(q) \in Z[q]$. Further, we have an upper bound 2^n for the lengths of B_n^ν, see [13].

LEMMA 2. *There exists a denominator*

$$\Omega(k,n) = q^{(k+1+2\delta)n^2/2+O(n)}\left(\prod_{\mu=1}^{n}(q^\mu - 1)\right)\prod_{i=1}^{k}\left\{\alpha_i^{2n+1} \times \right.$$

$$\left.\left(\prod_{\substack{j=1\\ j\neq i}}^{k}(\alpha_i - \alpha_j)\right)\left(\prod_{\mu=0}^{n}(\alpha_i + q^{\mu+1})\right)\left(\prod_{j=1}^{i-1}\prod_{\mu=1}^{n}(\alpha_i q^\mu - \alpha_j)(\alpha_i - \alpha_j q^\mu)\right)\right\}$$

such that for $|q|_v > 1$ *we have*

$$\Omega(k,n) I_v(k,n) = \delta P_0 + \sum_{i=1}^{k} P_i f(\alpha_i),$$

where δP_0, $P_i \in Z[q, \alpha_1, \ldots, \alpha_k]$ *are polynomials satisfying*

$$\deg_q\{\delta P_0, P_i\} \leqq (k^2 + 1 + 2\delta)n^2/2 + O(n),$$

$$\deg_{\underline{\alpha}}\{\delta P_0, P_i\} \leqq O(n).$$

PROOF. At first we use (3) to replace $f(z)$ in $I_v(k,n)$ by

$$f(z) = q^{-(n+1)(n+2)/2} f(zq^{-n-1}) \prod_{\nu=0}^{n}(z + q^{\nu+1}).$$

The application of residue theorem or its analogue in $\mathbf{C}_p$, see Adams [1], then implies

$$q^{(n+1)(n+2)/2} I_v(k,n) = \delta\,\mathrm{Res}_{z=0} f(zq^{-n-1}) z^{-\delta(n+1)} \prod_{i=1}^{k}\prod_{\nu=0}^{n}(z - \alpha_i q^\nu)^{-1} +$$

$$\sum_{i=1}^{k}\sum_{\nu=0}^{n}\mathrm{Res}_{z=\alpha_i q^\nu} f(zq^{-n-1}) z^{-\delta(n+1)} \prod_{j=1}^{k}\prod_{\mu=0}^{n}(z - \alpha_j q^\mu)^{-1} =$$

$$\delta \sum_{\underline{\sigma}+\sigma=n} q^{-\sigma(n+1)}\left(\prod_{\nu=1}^{\sigma}(q^\nu - 1)^{-1}\right)\prod_{i=1}^{k}\prod_{\nu=0}^{n} -(\alpha_i q^\nu)^{-\sigma_{i,\nu}-1} +$$

$$\sum_{i=1}^{k}\sum_{\nu=0}^{n} f(\alpha_i q^{\nu-n-1})(\alpha_i q^\nu)^{-\delta(n+1)} \prod_{\substack{j=1\\ (j,\mu)\neq(i,\nu)}}^{k}\prod_{\mu=0}^{n}(\alpha_i q^\nu - \alpha_j q^\mu)^{-1} =$$

$$\delta \sum_{\underline{\sigma}+\sigma=n}(-1)^{k(n+1)} q^{\omega_1(n,\nu,\sigma)}\left(\prod_{\nu=1}^{\sigma}(q^\nu - 1)^{-1}\right)\prod_{i=1}^{k}\prod_{\nu=0}^{n}\alpha_i^{-\sigma_{i,\nu}-1} +$$

$$\sum_{i=1}^{k} f(\alpha_i)\alpha_i^{-\delta(n+1)-n}\sum_{\nu=0}^{n}(-1)^{n-\nu} q^{\omega_2(n,\nu)} B_n^\nu(q) \times$$

$$\left(\prod_{\mu=1}^{n}(q^{\mu}-1)^{-1}\right)\left(\prod_{\mu=0}^{n-\nu}(\alpha_i+q^{\mu+1})^{-1}\right)\prod_{\substack{j=1\\ j\neq i}}^{k}(\alpha_i-\alpha_j)^{-1}\times$$

$$\left\{\prod_{\mu=1}^{\nu}(\alpha_i q^{\mu}-\alpha_j)\prod_{\mu=1}^{n-\nu}(\alpha_i-\alpha_j q^{\mu})\right\}^{-1},$$

where

$$\omega_1(n,\nu,\sigma) = -\sigma(n+1) - kn(n+1)/2 - \sum_{i,\nu}\sigma_{i,\nu}\nu,$$

$$\omega_2(n,\nu) = n^2/2 - (k+1+\delta)n\nu + (k+1)\nu^2/2 + 3n/2 - (k+1+2\delta)\nu/2 + 1.$$

Here $\omega_1(n,\nu,\sigma) - (n+1)(n+2)/2 \geqq -(k+3)n^2/2 + O(n)$, and $\omega_2(n,\nu) - (n+1)(n+2)/2 \geqq -(k+1+2\delta)n^2/2 + O(n)$. Therefore it is clear that

$$\Omega(k,n)I_v(k,n) = \delta P_0 + \sum_{i=1}^{k} P_i f(\alpha_i),$$

where $P_i \in Z[q,\alpha_1,\dots,\alpha_k]$, $i = 0,1,\dots,k$.

The estimate for $\deg_{\underline{\alpha}}$ is obvious. All we need to do is to estimate $\deg_q$. The degree of $\Omega(k,n)$ in q satisfies

$$\deg_q \Omega(k,n) = (k^2+k+2+2\delta)n^2/2 + O(n). \tag{8}$$

Since $I_v(k,n)$ is a sum of rational expressions, where q appears only in the denominator polynomials having $\deg_q$ at least $(k+1)n^2/2 + O(n)$, it follows that the degrees of δP_0 and P_i, $i = 1,\dots,k$, satisfy the required bound. Thus our Lemma 2 is true.

Let us denote

$$R(k,n) = \Omega(k,n)I_v(k,n) = \delta P_0 + \sum_{i=1}^{k} P_i f(\alpha_i).$$

The above lemmas imply now the following result.

LEMMA 3. *If* $|q|_v > 1$, *then*

$$|R(k,n)|_v = |q|_v^{-bn^2+O(n)},$$

where $b = ((1+2\delta)k+\delta-1)/2$. *Further, for any place* w *of* K, *we have*

$$\max(|\delta P_0|_w, |P_i|_w) \leqq c_4^{\delta(w)n}\max(1,|q|_w)^{an^2+O(n)}\max(1,|\alpha_i|_w)^{O(n)},$$

where $a = (k^2+1+2\delta)/2$, *and* $\delta(w) = 1$ *for* $w\,|\,\infty$, $\delta(w) = 0$ *for* $w\,|\,p$.

Next, to prove our Theorem 4, we give a result obtained by Bundschuh and Väänänen [6]. There we have an approximation form

$$R_1(k,n) = \sum_{i=0}^{k-1} P_i f^{(i)}(\alpha), \quad P_i \in Z[q,\alpha],$$

with the properties given in

LEMMA 4. *If* $|q|_v > 1$ *and* $|q|_w \neq 1$ *for all* $w \mid \infty$, *then*

$$|R_1(k,n)|_v = |q|_v^{-b_1 n^2 + O(n \log n)},$$

where $b_1 = (k-1)(k/2 - 3\pi^{-2})$. *Further, for any place* w *of* K, *we have*

$$\max |P_i|_w \leqq c_5^{\delta(w) n \log n} \max(1, |q|_w)^{a_1 n^2 + O(n \log n)} \max(1, |\alpha|_w)^{O(n)},$$

where $a_1 = k + 3(k-1)\pi^{-2}$.

We note that the proof of this result uses considerations like above applied to the integral

$$\frac{1}{2\pi i} \int_{|z|=R} \frac{f(z)\, dz}{\prod_{\nu=0}^{n} ((z - \alpha q^\nu)^k (z + q^{\nu+1}))}$$

or its p-adic analogue.

5. Proof of the theorems.

We shall obtain all our results from a general theorem, Theorem A below, for which we make the following

Assumption A. v is a place of K, $q \in K$ satisfies $|q|_v > 1$, and $\underline{\alpha} = (\alpha_1, \ldots, \alpha_s)$ is an element of K^s. Further, for some $k \in N$, $f_1, \ldots, f_k$ are elements of K_v having the following properties: For each $n \geqq c_6$ there exists a linear form

$$r(n) = p_0 + \sum_{i=1}^{k} p_i f_i, \quad p_i \in Z[q, \underline{\alpha}],$$

such that

$$(9) \qquad |q|_v^{-(Bn^2 + c_7 n)} \leqq |r(n)|_v \leqq |q|_v^{-(Bn^2 - c_8 n)}$$

with some constant $B > 0$. For any place w of K the coefficients p_i, $i = 0, 1, \ldots, k$, satisfy the inequalities

$$(10) \qquad \max |p_i|_w \leqq c_9^{\delta(w) n} \max(1, |q|_w)^{An^2 + c_{10} n} \max(1, |\underline{\alpha}|_w)^{c_{11} n},$$

where a constant $A > 0$.

THEOREM A. *Let the conditions of Assumption A be valid. Suppose that*

$A+B-\lambda A>0$. *Then there exist positive constants* $\Gamma_1=\Gamma_1(q,\underline{\alpha},k,v)$ *and* $H_1=H_1(q,\underline{\alpha},k,v)$ *such that for any* $\underline{\theta}=(\theta_1,\ldots,\theta_k)\in K^k$

$$\max_i |f_i-\theta_i|_v > H^{-\frac{(A+B)d}{(A+B-\lambda A)d_v}-\Gamma_1(\log H)^{-1/2}},$$

where $H=\max(h(\underline{\theta}),H_1)$.

PROOF. By denoting $\Delta=p_0+\sum_{i=1}^{k}p_i\theta_i$, $f_i=\theta_i+\varepsilon_i$, $i=1,\ldots,k$, we obtain

$$\Delta=r(n)-\sum_{i=1}^{k}p_i\varepsilon_i.$$

We now assume that

$$\max|f_i-\theta_i|_v=\max|\varepsilon_i|_v<|q|_v^{-(A+B)n^2-\tau n} \tag{11}$$

with some $\tau\in N$, $n\geqq c_6$, which we specify later. By (9) and (10) we then have

$$\left|\sum_{i=1}^{k}p_i\varepsilon_i\right|_v\leqq c_{12}^n|q|^{-Bn^2+c_{10}n-\tau n}<|r(n)|_v.$$

for any $\tau\geqq c_{13}$. Therefore, by (9),

$$0<|\Delta|_v\leqq 2^{\delta(v)}|r(n)|_v\leqq 2^{\delta(v)}|q|_v^{-Bn^2+c_8n}. \tag{12}$$

Since $0\neq\Delta\in K$, we have

$$\prod_w|\Delta|_w^{d_w/d}=1$$

by the product formula. This together with (10) and (12) imply (for any $x\geqq 0$ we denote $\log_+x=\log\max(1,x)$)

$$\frac{d_v}{d}\left((-Bn^2+c_8n)\log|q|_v+\delta(v)\log 2\right)\geqq\frac{d_v}{d}\log|\Delta|_v=$$

$$-\sum_{w\neq v}\frac{d_w}{d}\log|\Delta|_w\geqq-\sum_{w\neq v}\frac{d_w}{d}(\delta(w)nc_{14}+(An^2+c_{10}n)\log_+|q|_w+$$

$$c_{11}n\log_+|\underline{\alpha}|_w+\log_+|\underline{\theta}|_w)\geqq-\log h(\underline{\theta})-An^2\log h(q)+$$

$$\frac{d_v}{d}An^2\log|q|_v-c_{15}n.$$

This leads to an inequality

$$-n^2\left(\frac{d_v}{d}(A+B)\log|q|_v-A\log h(q)\right)\geqq-\log h(\underline{\theta})-c_{16}n$$

or

$$-n^2(A + B - \lambda A)\frac{d_v}{d}\log|q|_v \geqq -\log h(\underline{\theta}) - c_{16}n. \tag{13}$$

We now assume that $\tau \geqq c_{13}$ fulfils the condition

$$\tau(A + B - \lambda A)\frac{d_v}{d}\log|q|_v > c_{16}.$$

Then we fix our parameter n in such a way that n is the smallest integer ($>\tau$) satisfying

$$\log H \leqq (n^2 - \tau n)(A + B - \lambda A)\frac{d_v}{d}\log|q|_v, \tag{14}$$

where $H = \max(h(\underline{\theta}), H_1)$ is large enough to make n satisfy $n \geqq c_6$ and $n > 2\tau$. By these choices we have a contradiction in (13). Therefore our assumption (11) is not valid and thus

$$\max|f_i - \theta_i|_v \geqq |q|_v^{-(A+B)n^2 - \tau n} \geqq |q|_v^{-(A+B)((n-1)^2 - \tau(n-1)) - c_{17}n}.$$

From our choice of n (see (14)) it now follows that

$$-(A + B)((n-1)^2 - \tau(n-1))\log|q|_v - c_{17}n\log|q|_v >$$
$$-\frac{(A + B)d\log H}{(A + B - \lambda A)d_v} - c_{18}(\log H)^{1/2}.$$

This estimate implies the truth of Theorem A.

If $\alpha_1, \ldots, \alpha_k$ are elements of K satisfying (6), then Lemma 3 implies Assumption A for the numbers $f_i = f(\alpha_i)$, $i = 1, \ldots, k$, where

$$A = (k^2 + 3)/2,\ B = 3k/2,\ \underline{\alpha} = (\alpha_1, \ldots, \alpha_k)$$

(we use Lemma 3 with $\delta = 1$). Suppose that $A + B - \lambda A > 0$. Then Theorem A gives

$$\max_i |f(\alpha_i) - \theta_i|_v > H^{-\frac{(k^2+3k+3)d}{(k^2+3k+3-(k^2+3)\lambda)d_v} - \Gamma_1(\log H)^{-1/2}}.$$

If we choose here $k = 1$ and $k = 2$, we have our Theorem 1 and 2, respectively.

REMARK. Of course one would expect that the exponent of H above becomes better with greater k. Unfortunately it is not so and therefore we give our results only in the cases $k = 1$ and $k = 2$. The reason for this situation is perhaps too strong growth of our denominator $\Omega(k, n)$.

To prove Theorem 3 we apply Lemma 3 with $\delta = 0$. This implies Assumption A for the numbers $f_i = f(\alpha_{i+1})/f(\alpha_1)$, $i = 1, \ldots, k-1$, where

$$A = (k^2 + 1)/2,\ B = (k - 1)/2,\ \underline{\alpha} = (\alpha_1, \ldots, \alpha_k).$$

We choose here $k = 2$ and apply Theorem A. This gives immediately Theorem 3.

Lemma 4 gives us Assumption A for the numbers $f_i = f^{(i)}(\alpha)/f(\alpha)$, $i = 1, \ldots, k - 1$, where

$$A = k + 3(k - 1)/\pi^2,\ B = (k - 1)(k/2 - 3/\pi^2),\ \underline{\alpha} = \alpha,$$

and where $c_i n$ are replaced by $c_i n \log n$. If $A + B - \lambda A > 0$, then Theorem A can be used to produce the estimate

$$\max_i \left| \frac{f^{(i)}(\alpha)}{f(\alpha)} - \theta_i \right|_v > H^{-\frac{k(k+1)d}{(k(k+1) - \lambda(2k + 6(k-1)/\pi^2))d_v} - \Gamma_4 \frac{\log\log H}{(\log H)^{1/2}}},$$

where $H = \max(h(\underline{\theta}), H_4)$, and Γ_4 and H_4 are positive constants independent of $\underline{\theta}$. We have to change only slightly the proof of Theorem A, since $c_i n$ are replaced by $c_i n \log n$. Thus Theorem 4 is true.

If $\alpha = -1$, then this consideration can be improved to give our Remark after Theorem 4. We need only to use the better values

$$A = (k + 1)/2 + 3(k - 1)/\pi^2,\ B = (k - 1)((k + 1)/2 - 3/\pi^2)$$

of the constants A and B, see [6].

REMARK. We note that in the case $\lambda = 1$ the main part of the power of H in these bounds is $-1 - t(k)/(k - 1)$, where $t(k)$ goes to $2 + 6/\pi^2$ in the general case, and to $1 + 6/\pi^2$ in the case $\alpha = -1$, when k grows.

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