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ZERO SETS OF FUNCTIONS IN HARMONIC HARDY SPACES

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1. Introduction.

For $0 < p < \infty$ we let h^p be the space of all *real-valued harmonic* functions in the open unit disc U which satisfy the growth condition

$$(1.1)\qquad \|u\|_p = \sup_{0 \leq r < 1} \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} |u(re^{i\theta})|^p \, d\theta \right\}^{1/p} < \infty,$$

and we let H^p, as usual, be the Hardy space of all *holomorphic* f in U which satisfy (1.1) with f in place of u.

It is well known that the zero sets $Z(f)$ of H^p-functions f are the same for all p: If $f \in H^p$ for some p and $f \not\equiv 0$ then $Z(f)$ satisfies the Blaschke condition; conversely, if a set $S \subset U$ satisfies the Blaschke condition then $S = Z(f)$ for some bounded holomorphic f. In this note we consider the analogous question for h^p and find the following answer:

THEOREM. *If $0 < p < q < \infty$ then there is a set $S \subset U$ such that*
(a) *$S = Z(u)$ for some $u \in h^p$, but*
(b) *if $v \in h^q$ and $Z(v) \supset S$ then $v \equiv 0$.*

This is reminiscent of the situation in several complex variables where the holomorphic H^p-spaces have different zero sets for different values of p [6; p. 145].

The case $p = 1$ can be settled right away, and with the same S for all $q > 1$. Let S be the circle of radius $1/2$ centered at $1/2$, but without the point 1. Then $S = Z(u)$ if

$$(1.2)\qquad u(\lambda) = \mathrm{Re}[\lambda/(1 - \lambda)].$$

Since $2u + 1 > 0$ in U, and positive harmonic functions are in h^1, we have $u \in h^1$. Assume now that $v \in h^q$ for some $q > 1$. Then $v = \mathrm{Re}\, f$ for some $f \in H^q$, $|f|^q$ has

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a harmonic majorant in U, hence also in the disc D bounded by $\bar{S}$. Therefore $f \in H^q(D)$, and if $\operatorname{Re} f = 0$ at every point of S, it follows from the Poisson integral in D that $\operatorname{Re} f \equiv 0$ in D, hence also in U.

The general case of the theorem is a corollary of the more precise Theorems A and B stated below in terms of the spaces $\operatorname{Re} H^p$ (the real parts of H^p-functions) and G^p, where $u \in G^p$, by definition, if u is harmonic in U and

$$\sup_{\lambda \in U}(1 - |\lambda|)^{1/p} |u(\lambda)| < \infty. \tag{1.3}$$

These spaces are related by the inclusions

$$\operatorname{Re} H^p \subset h^p \subset G^p \quad (0 < p < \infty). \tag{1.4}$$

That $\operatorname{Re} H^p \subset h^p$ is trivial. When $1 < p < \infty$, the classical theorem of Marcel Riesz shows that $\operatorname{Re} H^p = h^p$. But h^1 is larger than $\operatorname{Re} H^1$ (even though $h^1 \subset \operatorname{Re} H^p$ for all $p < 1$), and if $0 < p < 1$ there are functions in h^p which lie in no $\operatorname{Re} H^q$; see [2; Chap. 4], for example. Another difference between h^p and $\operatorname{Re} H^p$ (when $0 < p < 1$) is that $\operatorname{Re} H^p$ is separable whereas h^p is not. This, and other aspects of h^p, are described in [7].

The difficulty in proving $h^p \subset G^p$ occurs when $p < 1$, since the Poisson integral in then not available. The first proof, given by Hardy and Littlewood [5; Th. 1] was based on some elementary but rather complicated lemmas. Fefferman and Stein simplified this by finding a fairly easy proof of the inequality

$$|u(\lambda)|^p \leqq \frac{K}{m(D)} \int_D |u|^p \, dm \tag{1.5}$$

in which $K = K(p) < \infty$, m is plane Lebesgue measure, and D is any disc with center λ in which u is harmonic. (See [3; p. 172], [4; p. 121]. The inequality is also a consequence of [5; Th. 5].) To apply (1.5), pick $\lambda \in U$, $|\lambda| = 1 - \varepsilon > 1/2$, let D have center λ, radius ε, enlarge the domain of integration to the annulus $\{\lambda\colon 1 - 2\varepsilon < |\lambda| < 1\}$, and read off that

$$|u(\lambda)|^p \leqq \frac{4K}{\varepsilon} \|u\|_p^p. \tag{1.6}$$

On the other hand, G^p is larger than h^p for *all* p. This follows, for example from [1; Th. 5], which implies: If Ψ: $[0, 1) \to [1, \infty)$ satisfies $\psi(r) \uparrow \infty$ as $r \uparrow 1$, then there is a holomorphic f in U such that $|f(\lambda)| < \psi(|\lambda|)$ for all $\lambda \in U$, but

$$\min_\theta |f(r_j e^{i\theta})| \uparrow \infty \tag{1.7}$$

for some sequence $r_j \uparrow 1$. Take $\psi(r) = (1 - r)^{-1/p}$, let $f = u + iv$. Then u and v are in G^p, but (1.7) shows that at least one of them is not in h^p.

The preceding discussion shows that Theorems A and B really give more information than the one that we stated in this Introduction.

2. Main results.

From now on p and q are fixed, $0 < p < q < \infty$. Choose γ and δ so that

$$p/q < \gamma < \delta < 1. \tag{2.1}$$

Choose α, $0 < \alpha < \pi/2$, so that, setting $s = \sin\alpha$, $c = \cos\alpha$, we have

$$s^2 < \gamma/2p, \tag{2.2}$$

and then put

$$\beta = 1/pcs. \tag{2.3}$$

Let Ω be the strip consisting of all $z = x + iy$ such that

$$|cy - sx| < c\delta\pi/2. \tag{2.4}$$

For $n = 0, \pm 1, \pm 2, \ldots$, put

$$E_n = \{z \in \Omega\colon x = n\pi/\beta\},\ E = \textstyle\bigcup_{n=-\infty}^{\infty} E_n. \tag{2.5}$$

The function

$$\Phi(z) = \frac{\exp(z/c\delta e^{i\alpha}) - 1}{\exp(z/c\delta e^{i\alpha}) + 1} \tag{2.6}$$

maps Ω conformally onto U. We define

$$S = \Phi(E) \tag{2.7}$$

and can now state our results.

THEOREM A. *$S = Z(u_0)$ for some $u_0 \in \operatorname{Re} H^p$.*

THEOREM B. *If $v \in G^q$ and $Z(v) \supset S$ then $v \equiv 0$.*

3. Proof of Theorem A.

Define f in Ω by

$$f(z) = ie^{-i\beta z}, \tag{3.1}$$

put $u = \operatorname{Re} f$, $u_0 = u \circ \Phi^{-1}$. Then

$$u(z) = e^{\beta y} \sin \beta x \tag{3.2}$$

so that $E = Z(u)$, hence $S = Z(u \circ \Phi^{-1}) = Z(u_0)$.

We have to show that $u_0 \in \operatorname{Re} H^p$.

Let ψ be the harmonic function defined by

$$\psi(z) = \exp\left(x + \frac{s}{c}y\right) \cdot \cos\left(y - \frac{s}{c}x\right). \tag{3.3}$$

By (2.4), $\cos\left(y - \frac{s}{c}x\right) > \cos(\delta\pi/2) > 0$ in Ω, so that

$$\log \psi(z) > x + \frac{s}{c}y + \log\cos(\delta\pi/2) \tag{3.4}$$

in Ω. On the other hand, (3.1), (2.3), and (2.4) show that

$$p \log|f| = p\beta y = \left(\frac{s}{c} + \frac{c}{s}\right)y < \frac{sy}{c} + x + \frac{c\delta\pi}{2s}. \tag{3.5}$$

It follows from (3.4) and (3.5) that there is a constant $K < \infty$ such that

$$|f|^p < K\psi \quad \text{in} \quad \Omega. \tag{3.6}$$

If we now put $f_0 = f \circ \Phi^{-1}$, then $K\psi \circ \Phi^{-1}$ is a harmonic majorant of $|f_0|^p$ in U. Hence $f_0 \in H^p$. Since $u_0 = \operatorname{Re} f_0$, the proof is complete.

4. Proof of Theorem B.

Suppose now that $v \in G^q$ and $v(\lambda) = 0$ for all $\lambda \in S$. Put

$$w = v \circ \Phi \tag{4.1}$$

where $\Phi\colon \Omega \to U$ is given by (2.6). Then $Z(w)$ contains every segment E_n as in (2.5). We have to conclude that this forces $w \equiv 0$.

Define

$$\Omega_\gamma = \{z\colon |cy - sx| < c\gamma\pi/2\}. \tag{4.2}$$

This is a strip whose closure lies in Ω. We need an upper bound (namely (4.7)) for the growth of $|w(z)|$ as $z \to \infty$ within Ω_γ. This, followed by an application of the reflection principle and an argument of the Phragment-Lindelöf type, will lead to the desired conclusion.

The map $\Phi(z) = \lambda$ can be written in the form

$$\lambda = \frac{e^\tau - 1}{e^\tau + 1} \tag{4.3}$$

where

(4.4) $$\tau = e^{-i\alpha}z/c\delta = \frac{cx + sy}{c\delta} + i\frac{cy - sx}{c\delta}.$$

A simple calculation leads from (4.3) to

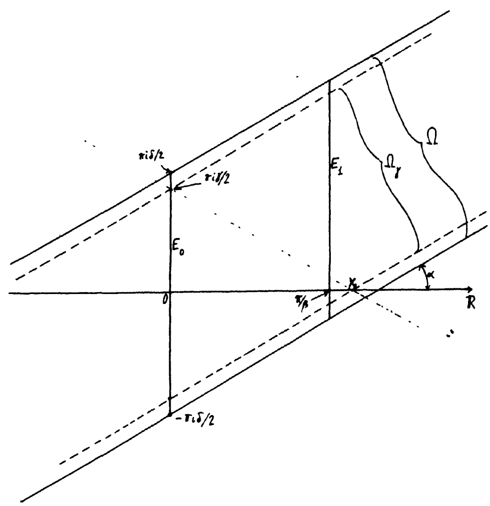
(4.5) $$\frac{1 + \lambda\bar{\lambda}}{1 - \lambda\bar{\lambda}} = \frac{\cosh(\operatorname{Re}\tau)}{\cos(\operatorname{Im}\tau)}.$$

In Ω_γ, $|\operatorname{Im}\tau| < \gamma\pi/2\delta$, so that $\cos(\operatorname{Im}\tau)$ is bounded below by a positive constant. Another calculation, using (4.4) and (2.4), shows that in Ω

(4.6) $$|\operatorname{Re}\tau| < \frac{|y|}{cs\delta} + \frac{c\pi}{2s}.$$

If we combine these estimates with the fact that $v \in G^q$, i.e., that $|v(\lambda)|^q = O((1 - \lambda\bar{\lambda})^{-1})$, we obtain

(4.7) $$|w(z)| < K\exp(|y|/cs\delta q) \quad \text{in} \quad \Omega_\gamma.$$



A look at Fig. 1 will clarify the next step.

The line $y = (\pi\gamma/2) - (sx/c)$ intersects the real axis at

(4.8) $$x = x_0 = \frac{c\pi\gamma}{2s} > cs\pi p = \pi/\beta,$$

using (2,2) and (2.3). The inequality $x_0 > \pi/\beta$ shows that the reflections of Ω_γ in the segments E_n cover the plane. Since $w(z) = 0$ for all $z \in E_n$ and for all n, it follows that w extends to a harmonic function in the whole plane, which we still denote by w, and that

$$w\left(\frac{k\pi}{\beta}, y\right) = 0 \tag{4.9}$$

for all integers k and all real y. Moreover, the extended function stil satisfies (4.7).

To finish, we apply the Phragmen-Lindelöf technique to the function w in the strip

$$\Sigma = \{z: 0 \leqq x \leqq \pi/\beta\}. \tag{4.10}$$

Note that $w = 0$ on the edges of Σ.

Since $q\delta > p$, there exists t such that

$$\frac{1}{cs\delta q} < t < \frac{1}{csp} = \beta. \tag{4.11}$$

For $\varepsilon > 0$ define

$$w_\varepsilon(z) = w(z) - \varepsilon(e^{ty} + e^{-ty})\cos\left(t\left(x - \frac{\pi}{2\beta}\right)\right). \tag{4.12}$$

The last cosine is $\geqq \cos(t\pi/2\beta) > 0$ *in* Σ, because $t < \beta$.

The first inequality in (4.11), combined with the estimate (4.7), shows now that $w_\varepsilon(z) < 0$ for all $z \in \Sigma$ for which $|y|$ is sufficiently large. It also follows from (4.12) that $w_\varepsilon(z) < 0$ on the edges of Σ. Since w_ε is harmonic, the maximum principle shows now that $w_\varepsilon(z) < 0$ for all $z \in \Sigma$. Hence, letting $\varepsilon \searrow 0$, $w(z) \leqq 0$.

The same argument, applied to $-w$ in place of w, gives $w(z) \geqq 0$. So $w(z) = 0$ for all $z \in \Sigma$, hence everywhere.

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