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DAMPING OSCILLATORY INTEGRALS WITH POLYNOMIAL PHASE

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1. Introduction.

Let $\mathscr{P}_N$ denote the space of real valued polynomials on R of degree at most N. Oberlin [O2] has proved the following two theorems.

THEOREM 1. *Given a positive integer N, there exists a constant C_N such that if j is a positive integer, then*

$$\left|\int_a^b e^{ip(x)}|p^{(j)}(x)|^{1/j}\,dx\right| \leqq C_N \quad \text{if } p\in\mathscr{P}_N \text{ and } a<b.$$

THEOREM 2. *Given a positive integer N, there exists a constant C_N such that if $j = 1$ or 2, then*

$$\left|\int_a^b e^{ip(x)}|p^{(j)}(x)|^{1/j+is}\,dx\right| \leqq C_N(1+|s|)^{1/j} \quad \text{if } p\in\mathscr{P}_N,\ a<b, \text{ and } s\in\mathbf{R}.$$

Oberlin also conjectured that Theorem 2 holds when j is any positive integer. In this paper we show that the conjecture is correct. In fact, we prove the following result which generalizes both Theorem 1 and Theorem 2.

THEOREM 3. *Given a positive integer N, there exists a constant C_N such that if $\alpha_1,\dots,\alpha_N$ are nonnegative real numbers with $\sum_{j=1}^N j\alpha_j = 1$, then*

$$\left|\int_a^b e^{ip(x)}\left(\prod_{j=1}^N |p^{(j)}(x)|^{\alpha_j}\right)^{1+is}dx\right| \leqq C_N(1+|s|)^{\sigma}$$

if $p\in\mathscr{P}_N$, $a<b$, and $s\in\mathbf{R}$, where $\sigma = \sum_{j=1}^N \alpha_j$.

As an application of Theorem 3, we obtain sharp $L^p - L^{p'}$ estimates for singular convolution operators of the form

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$$Tf(x) = \int_0^1 f(x - \gamma(t))dt, \quad x \in \mathrm{R}^n,$$

where γ is the curve $\gamma(t) = (t, t^2, \ldots, t^n)$, $t \in \mathrm{R}$, $n \geqq 2$.

THEOREM 4. *Let T be the operator defined above. Then there is a positive constant C such that $\|Tf\|_{L^{p'}} \leqq C\|f\|_{L^p}$ if and only if $\dfrac{2n(n+1)}{n^2+n+2} \leqq p \leqq 2$.*

Notice that Theorems 1–3 may be interpreted as decay estimates for oscillatory integrals. For example, Theorem 2 shows that

$$\left|\int_a^b e^{i\lambda p(x)}|p''(x)|^{1/2+is}\,dx\right| \leqq C_N(1+|s|)^{1/2}|\lambda|^{-1/2}$$

is satisfied uniformly for all polynomials of degree at most N. In their work on regularity for dispersive equations, Kenig, Ponce, and Vega [KPV] have proved a similar result which holds for a larger class of phase functions, but for which the constant is not uniform over $\mathscr{P}_N$. The interested reader should also consult the papers [CDMM, O1, S2] for further results on oscillatory integrals and singular convolution operators.

The author would like to thank Dan Oberlin for several helpful comments and suggestions. It should be noted that the outline of the proof of Theorem 3 is essentially the same as that of Theorem 2 in [O2]. For completeness, we repeat the proofs of the results from [O2] that we need.

2. Lemma A.

The key step in the proof of Theorem 3 is establishing Lemma A, which generalizes a lemma appearing in [O2]. The proof of the lemma proceeds via three steps. The first and third steps are as in [O2], while the second step generalizes a result in Oberlin's paper.

LEMMA A. *Given a positive integer N, there is a constant K_N such that if*

$$r(x) = \prod_{j=1}^{J_1}(x - a_j) \prod_{j=J_1+1}^{J_2}[(x-a_j)^2 + b_j] =: \prod_{j=1}^{J_2} g_j(x)$$

is a monic polynomial of degree at most N with the a_j's distinct and each $b_j > 0$, then there is a pairwise disjoint collection $\{I_l\}_1^L$ of at most K_N disjoint subintervals of R *such that*

$$\text{(A-1)} \qquad \int_{\mathrm{R}\setminus\cup I_l} \left|\frac{r^{(m)}}{r}\right|^{1/m} \leqq K_N, \quad 1 \leqq m \leqq N$$

and so that for each l there exists $C = C_l \in (0, \infty)$, $j = j_l \in \{1, 2, \dots, J_2\}$, and a non-negative integer $t = t_l$ with

(A-2) $$\frac{C}{K_N}|x - a_j|^t \leqq |r(x)| \leqq K_N C|x - a_j|^t,$$

(A-3) $$\frac{C}{K_N}|x - a_j|^{t-1} \leqq |r'(x)| \leqq K_N C|x - a_j|^{t-1}, \text{ and}$$

(A-4) $$|r^{(m)}(x)| \leqq K_N C|x - a_j|^{t-m}$$

for $x \in I_l$ and $2 \leqq m \leqq N$.

Before proving Lemma A, we give two elementary results we will need for the second step of the proof.

LEMMA B. *Given a positive integer N, there is a constant $K = K(N)$ so that if f is a smooth function on $\mathbb{R}$ then*

$$\left(\frac{f'}{f}\right)^{(m-1)} = \frac{f^{(m)}}{f} + E_m \quad \text{for } 1 \leqq m \leqq N,$$

where E_m may be written as the sum of at most K terms, each having the form

$$\gamma\left(\frac{f^{(\alpha_1)}}{f}\right)^{\beta_1}\left(\frac{f^{(\alpha_2)}}{f}\right)^{\beta_2}\cdots\left(\frac{f^{(\alpha_l)}}{f}\right)^{\beta_l},$$

where the α_j's and β_j's are positive integers, $1 \leqq \alpha_j \leqq m - 1$, $l \leqq K$, $|\gamma| \leqq K$, and $\alpha_1\beta_1 + \dots + \alpha_l\beta_l = m$.

PROOF OF LEMMA B. Proceeding by induction on m, we differentiate the induction hypothesis to obtain

$$\left(\frac{f'}{f}\right)^{(m)} = \frac{f^{(m+1)}}{f} - \frac{f^{(m)}}{f}\frac{f'}{f} + E_m'.$$

A typical term of E_m' has the form

$$\gamma\beta_j\left(\frac{f^{(\alpha_j)}}{f}\right)^{\beta_j - 1}\left(\frac{f^{(\alpha_j+1)}}{f}\right)\prod_{i \neq j}\left(\frac{f^{(\alpha_i)}}{f}\right)^{\beta_i}$$
$$-\gamma\beta_j\left(\frac{f^{(\alpha_j)}}{f}\right)^{\beta_j - 1}\left(\frac{f^{(\alpha_j)}}{f}\frac{f'}{f}\right)\prod_{i \neq j}\left(\frac{f^{(\alpha_i)}}{f}\right)^{\beta_i}$$

so we need only observe that $\alpha_j(\beta_j - 1) + \alpha_j + 1 + \sum_{i \neq j}\alpha_i\beta_i = m + 1$.

LEMMA C. *Given a positive integer N, there is a constant $K = K(N)$ so that if*

$$f(x) = \frac{x - a}{(x - a)^2 + b}, \quad x \in \mathbb{R}$$

with $a \in \mathbb{R}$ and $b > 0$, then

$$|f^{(m)}(x)| \leqq K|x-a|^{-(m+1)}, \quad x \in \mathrm{R} \ \textit{and}$$
$$|f^{(m)}(x)| \leqq Kb^{-(m+1)/2}, \quad |x-a|^2 \leqq b$$

for $m = 0, 1, \ldots, N$.

PROOF OF LEMMA C. An induction argument shows that there is a constant $C = C(m)$ so that $f^{(m)}$ may be written as the sum of at most C terms, each having the form

$$\frac{\gamma(x-a)^\beta}{[(x-a)^2+b]^{(\beta+m+1)/2}},$$

where β is a nonnegative integer no larger than C and $|\gamma| \leqq C$. The desired estimates follow easily from this representation.

PROOF OF LEMMA A. Fix a positive integer N. We will let K denote a quantity depending only on N, the exact value of which may increase at each occurrence.

Given r as in the statement of the lemma, write $\frac{r'}{r} = \sum_1^{J_2} f_j$, where either $f_j(x) = \frac{1}{x-a_j}$ (in which case we say that f_j is type 1) or $f_j(x) = \frac{2(x-a_j)}{(x-a_j)^2+b_j}$ (when we say f_j is type 2).

STEP 1. *There is a constant K such that given r, we can write* R *as the disjoint union of that most K subintervals $\{I_l\}$ such that for each l there is an integer $j(l)$ with*

$$|f_j(x)| \leqq |f_{j(l)}(x)|, \quad x \in I_l, \quad 1 \leqq j \leqq J_2.$$

PROOF OF STEP 1. This follows from the fact that there are at most N of the f_j's, and that the equation $|f_{j_1}(x)| = |f_{j_2}(x)|$ $(j_1 \neq j_2)$ can have at most six solutions.

STEP 2. *There is a constant K such that given an interval I and an index j_0 with*

$$|f_j(x)| \leqq |f_{j_0}|, \quad x \in I, \quad 1 \leqq j \leqq J_2,$$

there is a subset $\tilde{I}$ of I such that $I \backslash \tilde{I}$ is the disjoint union of at most K intervals,

$$\frac{1}{K|x-a_{j_0}|} \leqq \left|\frac{r'}{r}(x)\right| \leqq \frac{K}{|x-a_{j_0}|}, \quad x \in I\backslash\tilde{I} \tag{1}$$

$$\left|\frac{r^{(m)}}{r}(x)\right| \leqq \frac{K}{|x-a_{j_0}|^m}, \quad x \in I\backslash\tilde{I}, \quad 1 \leqq m \leqq N \ \textit{and} \tag{2}$$

$$\int_{\tilde{I}} \left|\frac{r^{(m)}}{r}\right|^{1/m} \leqq K, \quad 1 \leqq m \leqq N. \tag{3}$$

PROOF OF STEP 2. For ease of notation assume $j_0 = 1$. Define

$$T = \{x \in I\colon \text{for each } j \neq 1, \text{ either } |f_j(x)| \leqq |f_1(x)|/2N \text{ or } f_1(x)f_j(x) \geqq 0\}.$$

Define sets S_j, $1 \leqq j \leqq J_2$ as follows: If f_j is type 1, then $S_j = \emptyset$. Otherwise, set

$$S_j = \{x \in I\colon |x - a_j|^2 < b_j \text{ and } |x - a_1|^2 > b_j\} \quad \text{for } j \neq 1$$

and

$$S_1 = \{x \in I\colon |x - a_1|^2 < b_1\}.$$

Now define $\tilde{I} = (I \setminus T) \cup (\cup_1^{J_2} S_j)$, so that $I \setminus \tilde{I} = T \setminus (\cup S_j)$.

For $x \in I$ we have

$$|r'/r(x)| = |\textstyle\sum f_j(x)| \leqq N|f_1(x)| \leqq 2N|x - a_1|^{-1}.$$

Also, $x \in T$ implies

$$\begin{aligned}|r'/r(x)| &\geqq \Bigg|\sum_{j:f_1f_j(x) \geqq 0} f_j(x)\Bigg| - \Bigg|\sum_{j:f_1f_j(x)<0}{}' f_j(x)\Bigg| \\ &\geqq |f_1(x)| - N|f_1(x)|/2N = |f_1(x)|/2.\end{aligned}$$

Since $|f_1(x)| \geqq |x - a_1|^{-1}$ if $x \in I \setminus S_1$, this shows that if $x \in T \setminus S_1 \supseteq T \setminus (\cup S_j) = I \setminus \tilde{I}$, then $|r'/r(x)| \geqq |x - a_1|^{-1}/2$, which finishes the proof of (1).

We prove (2) by induction on m. So assume $2 \leqq m \leqq N$ and that (2) is true for $1, 2, \ldots, m - 1$. By Lemma B,

$$\textstyle\sum f_j^{(m-1)} = (r'/r)^{(m-1)} = r^{(m)}/r + E_m,$$

where E_m is the sum of at most K terms, each having the form $\gamma \prod_{j=1}^{l} (r^{(\alpha_j)}/r)^{\beta_j}$, with $\gamma, l, \alpha_j, \beta_j$ as described in the lemma. Since $1 \leqq \alpha_j \leqq m - 1$, the induction hypothesis gives

$$\left|\prod_1^l (r^{(\alpha_j)}/r)^{\beta_j}\right| \leqq \prod_1^l (K|x - a_1)|^{-\alpha_j})^{\beta_j}, \quad x \in I \setminus \tilde{I},$$

and hence $\sum \alpha_j \beta_j = m$ implies $|E_m| \leqq K|x - a_1|^{-m}$. Therefore (2) will be proved once we establish

$$|f_j^{(m-1)}(x)| \leqq K|x - a_1|^{-m}, \quad x \in I \setminus \tilde{I}, \quad 1 \leqq j \leqq J_2.$$

First observe that $|f_1^{(m-1)}(x)| \leqq K|x - a_1|^{-m}$ follows immediately if f_1 is type 1, or by Lemma C if f_1 is type 2. If $j \neq 1$ and f_j is type 1, then

$$|f_j^{(m-1)}(x)| \leqq K|f_j(x)|^m \leqq K|f_1(x)|^m \leqq K|x - a_1|^{-m}$$

for $x \in I$. If f_j is type 2, then $x \in I \setminus S_j$ implies $|x - a_j|^2 \geqq b_j$ or $|x - a_1|^2 \leqq b_j$. In the case that $|x - a_j|^2 \geqq b_j$, we have $|f_j(x)| \geqq |x - a_j|^{-1}$, and hence by Lemma C

$$|f_j^{(m-1)}(x)| \leqq K|x - a_j|^{-m} \leqq K|f_j(x)|^m \leqq K|f_1(x)|^m \leqq K|x - a_1|^{-m}.$$

In the remaining case of $|x-a_j|^2 < b_j$ and $|x-a_1|^2 \leqq b_j$, Lemma C gives

$$|f_j^{(m-1)}(x)| \leqq K b_j^{-m/2} \leqq K|x-a_1|^{-m}.$$

Hence $x \in I \setminus S_j \supseteq I \setminus \tilde{I}$ implies $|f_j^{(m-1)}(x)| \leqq K|x-a_1|^{-m}$.

The proof of (3) also in inductive, and so we begin with the case $m = 1$. (Recall $\tilde{I} = (\cup S_j) \cup (I \setminus T)$. Since $|r'/r| \leqq N|f_1|$ for $x \in I$, it suffices to prove $\int_{\tilde{I}} |f_1| \leqq K$.

First observe that $\int_{S_1} |f_1| \leqq K$, and in fact $\int_{S_j} |f_j| \leqq K$ for $1 \leqq j \leqq J_2$, since either $S_j = \emptyset$ or $\int_{S_j} |f_j| \leqq \int_{\{|x-a_j|^2 < b_j\}} |f_j| = 2\log 2$.

Now suppose $j \neq 1$ and $S_j \neq \emptyset$. Then $|f_1(x)| \leqq 2|x-a_1|^{-1}$ implies $\int_{S_j} |f_1| \leqq \int_{\{|x-a_j|^2 < b_j\}} 2b_j^{-1/2} = 4$. Thus

$$(*) \qquad \int_{S_j} |f_1| \leqq K, \quad 1 \leqq j \leqq J_2.$$

Finally, $I \setminus T \subseteq \cup_{j \neq 1} U_j$, where

$$U_j := \{x \in I : |f_j(x)| > |f_1(x)|/2N \text{ and } f_1 f_j(x) < 0\}.$$

Hence it suffices to prove

$$(**) \qquad \int_{U_j} |f_1| \leqq K, \quad j \neq 1.$$

Write $\int_{U_j} |f_1| = \int_{U_j \cap S_1} |f_1| + \int_{U_j \setminus S_1} |f_1|$. We may use (*) to see that the first integral is bounded by K. For the second, it suffices to bound $\int_{U_j \setminus S_1} |x-a_1|^{-1} dx$. We now assume that $a_1 < a_j$, the case $a_1 > a_j$ being similar.

If $x \in U_j \setminus S_1$, then

$$|x-a_1|^{-1} \leqq |f_1(x)| \leqq 2N|f_j(x)| \leqq 4N|x-a_j|^{-1}.$$

Also, $x \in U_j$ implies $f_1(x) f_j(x) < 0$, and hence $U_j \subseteq (a_1, a_j)$. Thus $a_j - a_1 = (a_j - x) + (x - a_1) \leqq (4N+1)(x-a_1)$ for $x \in U_j \setminus S_1$. Hence $\int_{U_j \setminus S_1} |x-a_1|^{-1} dx \leqq \int_{a_1}^{a_j} (4N+1)/(a_j - a_1) = 4N+1$. This finishes the proof of (3) in the case $m = 1$.

Let us now suppose that $m \geqq 2$ and that (3) holds for $1, 2, \ldots, m-1$. By Lemma B, it suffices to prove

$$(4) \qquad \int_{\tilde{I}} \prod_{j=1}^{l} \left| \frac{r^{(\alpha_j)}}{r} \right|^{\beta_j/m} \leqq K$$

and

$$(5) \qquad \int_{\tilde{I}} |f_j^{(m-1)}|^{1/m} \leqq K,$$

where α_j, β_j, l are as in Lemma B.

If we define $p_j = m/\alpha_j\beta_j$, then $\sum 1/p_j = 1$. Therefore the generalized Hölder inequality and the induction hypothesis, together with the facts that $1 \leqq \alpha_j \leqq m-1$ and $l \leqq K$, imply

$$\int_{\tilde{I}} \prod_1^l |r^{(\alpha_j)}/r|^{\beta_j/m} \leqq \prod_1^l \left(\int_{\tilde{I}} (r^{(\alpha_j)}/r|^{1/\alpha_j} \right)^{1/p_j} \leqq K.$$

To prove (5), we first consider (recall again $\tilde{I} = (\cup S_i) \cup (I \setminus T))$

$$\int_{S_i} |f_j^{(m-1)}|^{1/m} = \int_{S_i \setminus S_j} |f_j^{(m-1)}|^{1/m} + \int_{S_i \cap S_j} |f_j^{(m-1)}|^{1/m} =: A + B,$$

where f_i is type 2. Since $x \in I \setminus S_j$ implies $|f_j(x)| \geqq |x - a_j|^{-1}$, and $|f_j^{(m-1)}(x)| \leqq K|x - a_j|^{-m}$ follows from Lemma C, we have

$$|f_j^{(m-1)}(x)| \leqq |f_j|^m(x) \leqq |f_1|^m(x), \quad x \in I \setminus S_j.$$

Thus $A \leqq \int_{S_i \setminus S_j} |f_1| \leqq K$ by (*).

Also,

$$B \leqq \int_{S_j} |f_j^{(m-1)}|^{1/m} \leqq K \int_{\{|x-a_j|^2 < b_j\}} |f_j^{(m-1)}(x)|^{1/m} \, dx,$$

and hence Lemma C gives $B \leqq K \int_{\{|x-a_j|^2 < b_j\}} b_j^{-1/2} \leqq K$. Thus

$$(***) \qquad \int_{S_i} |f_j^{(m-1)}|^{1/m} \leqq K.$$

It remains to estimate $\int_{I \setminus T} |f_j^{(m-1)}|^{1/m}$. Since $I \setminus T \subseteq \cup_{i \neq 1} U_i$ and $|f_j^{(m-1)}(x)| \leqq |f_1|^m(x)$ on $I \setminus S_j$, it suffices to consider

$$\int_{U_i} |f_j^{(m-1)}|^{1/m} = \int_{U_i \setminus S_j} |f_j^{(m-1)}|^{1/m} + \int_{U_i \cap S_j} |f_j^{(m-1)}|^{1/m}$$
$$\leqq \int_{U_i} |f_1| + \int_{S_j} |f_j^{(m-1)}|^{1/m}.$$

The first integral is appropriately bounded by (**), while the second integral has already been estimated by (***). This concludes the proof of Step 2.

STEP 3. *There exists a constant K such that given an interval I and an index j_0 such that $|f_j(x)| \leqq |f_{j_0}(x)|$ for $x \in I$, $1 \leqq j \leqq J_2$, we may write I as the disjoint union of at most K intervals $\{I_l\}$ such that for each l there are $C = C(l) \in (0, \infty)$ and $t = t(l) \in \mathbf{N}$ with*

$$\frac{C}{K} |x - a_{j_0}|^t \leqq |r(x)| \leqq KC|x - a_{j_0}|^t, \quad x \in I_l.$$

PROOF OF STEP 3. Assume $j_0 = 1$. Recall $r(x) = \prod_1^{J_2} g_j(x)$. Since $J_2 \leqq N$, it is enough to show the following: there are absolute constants P and B such that given g_j we can write I as the union of at most P subintervals I_p and on each I_p either

$$\text{(3-1)} \qquad \text{there exists } C > 0 \text{ with } C/B \leqq |g_j| \leqq BC$$

or

$$\text{(3-2)} \qquad |x - a_1|/B \leqq |g_j| \leqq B|x - a_1|$$

or

$$\text{(3-3)} \qquad |x - a_1|^2/B \leqq |g_j| \leqq B|x - a_1|^2.$$

The proof of the following lemma is elementary.

LEMMA. *Suppose* $x, a_1, a_j \in \mathrm{R}$ *and* $|x - a_1| \leqq 2|x - a_j|$. *If* $|a_1 - a_j|/2 \leqq |x - a_1|$, *then* $|x - a_1|/2 \leqq |x - a_j| \leqq 3|x - a_1|$. *If* $|a_1 - a_j|/2 > |x - a_1|$, *then* $|a_1 - a_j|/3 \leqq |x - a_j| \leqq 3|a_1 - a_j|/2$.

Suppose first that f_j is type 1. Then $|f_j| \leqq |f_1|$ on I implies $|x - a_1| \leqq 2|x - a_j| = 2|g_j(x)|$ on I. Thus the above lemma gives subintervals of I on which (3-1) or (3-2) hold.

Now if f_j is type 2, then $b_j \leqq g_j \leqq 2b_j$ on the interval $(a_j - \sqrt{b_j}, a_j + \sqrt{b_j})$. On $I \backslash (a_j - \sqrt{b_j}, a_j + \sqrt{b_j})$, we have

$$|x - a_j|^{-1} \leqq |f_j(x)| \leqq |f_1(x)| \leqq 2|x - a_1|^{-1},$$

and hence $|x - a_1| \leqq 2|x - a_j|$. Since $(x - a_j)^2 \leqq g_j(x) \leqq 2(x - a_j)^2$ on this set, the lemma gives

$$(x - a_1)^2/4 \leqq (x - a_j)^2 \leqq g_j(x) \leqq 2(x - a_j)^2 \leqq 18(x - a_1)^2$$

if $|a_1 - a_j|/2 \leqq |x - a_1|$, while

$$(a_1 - a_j)^2/9 \leqq (x - a_j)^2 \leqq g_j(x) \leqq 2(x - a_j)^2 \leqq 9(a_1 - a_j)^2/2$$

if $|a_1 - a_j|/2 > |x - a_1|$. This completes the proof of Step 3 and Lemma A.

3. Theorem 3.

Following [O2], we begin the proof with some reductions. Again K denotes a constant depending only on N. We may assume that $\sigma < 1$, since the case $\sigma = 1$ (which implies $\alpha_1 = 1$ and $\alpha_j = 0$ for $j \neq 1$) reduces to the $j = 1$ case of Theorem 2. A scaling argument shows that we may assume p' to be monic. Then an approximation argument shows that it is enough to prove Theorem 3 under the additional assumption that $r(x) := p'(x)$ satisfies the other hypotheses of Lemma A. Finally, let

$$F(x) := \frac{\prod_1^N |p^{(j)}(x)|^{\alpha_j}}{|p'(x)|}.$$

Note that

$$F' = F\left(\sum \alpha_j \frac{p^{(j+1)}}{p^{(j)}} - \frac{p''}{p'}\right),$$

from which it is clear that $p \in \mathscr{P}_N$ implies that F has as most K critical points. Thus is suffices to consider the case where p', $F(x) - (1 + |s|)^{\sigma - 1}$, and $|p''/(p')^2| - 1/(1 + |s|)$ are of constant sign on $I := (a, b)$.

Case 1. $\left|\dfrac{p''}{(p')^2}\right| \leqq \dfrac{1}{1 + |s|}$ *on* I.

After the change of variable $u = p(x)$, the integral in Theorem 2 becomes

$$\int_J e^{i(u + s\log|p'(p^{-1}(u))|)} (F(p^{-1}(u))^{1+is}\, du.$$

Assume first that $F(x) \leqq (1 + |s|)^{\sigma - 1}$ on I. Van der Corput's Lemma (see [S2]) the gives

$$\left|\int_J e^{i(u + s\log|p'(p^{-1}(u))|)} (F(p^{-1}(u))^{1+is} du\right|$$

$$\leqq K\left(\|F\|_{L^\infty(I)} + \int_J \left|\frac{d}{du}(F(p^{-1}(u)))^{1+is}\right| du\right)$$

$$\leqq K + K(1 + |s|) \int_J \left|\frac{d}{du} F(p^{-1}(u))\right| du$$

$$\leqq K + K(1 + |s|)\|F\|_{L^\infty(I)} \leqq K(1 + |s|)^\sigma.$$

Now assume that $F(x) \geqq (1 + |s|)^{\sigma - 1}$ on I. Then

$$\prod |p^{(j)}|^{\alpha_j} = \frac{(\prod |p^{(j)}|^{\alpha_j})^{1/(1-\sigma)}}{(\prod |p^{(j)}|^{\alpha_j})^{1/(1-\sigma)-1}}$$

$$\leqq (\prod |p^{(j)}|^{\alpha_j})^{1/(1-\sigma)} \left(\frac{(1 + |s|)^{1-\sigma}}{|p'|}\right)^{\sigma/(1-\sigma)} = (1 + |s|)^\sigma \left(\frac{\prod |p^{(j)}|^{\alpha_j}}{|p'|^\sigma}\right)^{1/1(1-\sigma)}$$

$$= (1 + |s|)^\sigma \left(\prod \left|\frac{p^{(j)}}{p'}\right|^{\alpha_j}\right)^{1/(1-\sigma)} = (1 + |s|)^\sigma \prod \left|\frac{p^{(j)}}{p'}\right|^{\alpha_j/(1-\sigma)}.$$

Choose intervals $\{I_l\}$ as in Lemma A with $r = p'$. Then

$$\left|\int_{I\setminus\cup I_l} e^{ip(x)}\left(\prod |p^{(j)}(x)|^{\alpha_j}\right)^{1+is} dx\right|$$

$$\leqq (1+|s|)^\sigma \int_{I\setminus\cup I_l} \prod \left|\frac{p^{(j)}}{p'}\right|^{\alpha_j/(1-\sigma)} \leqq (1+|s|)^\sigma \prod \left\| \left|\frac{p^{(j)}}{p'}\right|^{\alpha_j/(1-\sigma)} \right\|_{L^{q_j}(I\setminus\cup I_l)},$$

where $1/q_j = (j-1)\alpha_j/(1-\sigma)$, since $\sum 1/q_j = 1$. Now $\left\| \left|\frac{p'}{p'}\right|^{\alpha_1/(1-\sigma)} \right\|_{L^\infty} = 1$, and for $2 \leqq j \leqq N$ Lemma A gives $\int_{I\setminus\cup I_l} \left|\frac{p^{(j)}}{p'}\right|^{1/(j-1)} \leqq K$, which shows that

$$\left|\int_{I\setminus\cup I_l} e^{ip(x)}\left(\prod |p^{(j)}(x)|^{\alpha_j}\right)^{1+is} dx\right| \leqq K(1+|s|)^\sigma.$$

Fix l. By Lemma A there exists $C > 0$, $t \in \mathbf{N}$, and $a \in \mathbf{R}$ such that

$$\frac{1}{1+|s|} \geqq \left|\frac{p''(x)}{(p'(x)^2)}\right| \geqq \frac{C|x-a|^{t-1}}{K(C|x-a|^t)^2} = \frac{1}{KC|x-a|^{t+1}}$$

for $x \in I \cap I_l$. Hence by (A-4),

$$|F(x)| = \frac{\prod |p^{(j)}(x)|^{\alpha_j}}{|p'(x)|} \leqq K \frac{\prod (C|x-a|^{t-j+1})^{\alpha_j}}{C|x-a|^t}$$

$$= \frac{K}{(C|x-a|^{t+1})^{1-\sigma}} \leqq \frac{K}{(1+|s|)^{1-\sigma}}.$$

Another appeal to van der Corput's Lemma then shows that

$$\left|\int_{I\cap I_l} e^{ip(x)}\left(\prod |p^{(j)}(x)|^{\alpha_j}\right)^{1+is} dx\right| \leqq K(1+|s|)^\sigma,$$

which finishes the proof of Case 1.

Case 2. $\left|\frac{p''}{(p')^2}\right| \geqq \frac{1}{1+|s|}$ *on* I.

Choose intervals $\{I_l\}$ as in Lemma A with $r = p'$. If $F(x) \geqq (1+|s|)^{\sigma-1}$ on I, then as in Case 1 we have

$$\prod |p^{(j)}|^{\alpha_j} \leqq (1+|s|)^\sigma \prod \left|\frac{p^{(j)}}{p'}\right|^{\alpha_j/(1-\sigma)},$$

while if $F(x) \leqq (1+|s|)^{\sigma-1}$, then

$$\prod |p^{(j)}|^{\alpha_j} \leqq \frac{(1+|s|)^\sigma}{1+|s|}|p'| \leqq (1+|s|)^\sigma \left|\frac{p''}{(p')^2}\right| |p'| = (1+|s|)^\sigma \left|\frac{p''}{p'}\right|.$$

Therefore Lemma A gives

$$\left|\int_{I\setminus\cup I_l} e^{ip(x)}\left(\prod|p^{(j)}(x)|^{\alpha_j}\right)^{1+is} dx\right| \leqq \int_{I\setminus\cup I_l} \prod |p^{(j)}|^{\alpha_j} \leqq K(1+|s|)^{\sigma}.$$

Now we consider $I \cap I_l$. The Case 2 hypothesis and Lemma A give

$$\frac{1}{1+|s|} \leqq \left|\frac{p''}{(p')^2}\right| \leqq \frac{KC|x-a|^{t-1}}{(C|x-a|^t)^2} = \frac{K}{C|x-a|^{t+1}},$$

or

$$|x-a| \leqq K\left(\frac{1+|s|}{C}\right)^{1/(t+1)}$$

for some $a \in \mathrm{R}$, $t \in \mathrm{N}$, $C > 0$. Since

$$\prod |p^{(j)}|^{\alpha_j} \leqq K \prod (C|x-a|^{t-j+1})^{\alpha_j} = KC^{\sigma}|x-a|^{(t+1)\sigma-1},$$

we have

$$\int_{I\cap I_l} \prod |p^{(j)}|^{\alpha_j} \leqq \int_{\{|x-a| \leqq K(\frac{1+|s|}{C})^{1/(t+1)}\}} KC^{\sigma}|x-a|^{(t+1)\sigma-1}\,dx \leqq K(1+|s|)^{\sigma}.$$

This finishes the proof of Case 2 and Theorem 3.

4. Theorem 4.

For $n \geqq 2$, let $\gamma(t) = (t, t^2, \ldots, t^n)$ for $t \in \mathrm{R}$, and define the operator T by

$$Tf(x) = \int_0^1 f(x - \gamma(t))\,dt, \quad x \in \mathrm{R}^n.$$

An interesting problem is to find the *type set* of T. That is, we wish to determine for which points $(1/p, 1/q) \in [0,1] \times [0,1]$ it is true that T is a bounded operator from L^p to L^q.

Three necessary conditions are known. A theorem of Hörmander [H] implies that $(1/p, 1/q)$ is in the type set of T only if $p \leqq q$. In [O2], Oberlin observed that estimating the norms of f and Tf when f is the characteristic function of a small ball shows that $(1/p, 1/q)$ must lie on or above the line joining $P_1 = (1,1)$ and $P_2 = (n/(2n-1), (n-1)/(2n-1))$, and hence by duality also on or above the line joining $P_0 = (0,0)$ and P_2. The third necessary condition is an observation due to Anthony Carbery and Michael Christ. They noted that comparing the norms of f and Tf when f is the characteristic function of a small box of dimension $\delta \times \delta^2 \times \ldots \times \delta^n$ shows that $(1/p, 1/q)$ must lie on or above the line joining $P_3 = ((n^2 - n + 2)/(n^2 + n), (n-1)/(n+1))$ and $P_4 = (2/(n+1), (2n-2)/(n^2+n))$. To summarize, $(1/p, 1/q)$ must lie in the triangle $P_0P_1P_2$ if $n = 2$, and the trapezoid $P_0P_1P_3P_4$ if $n \geqq 3$.

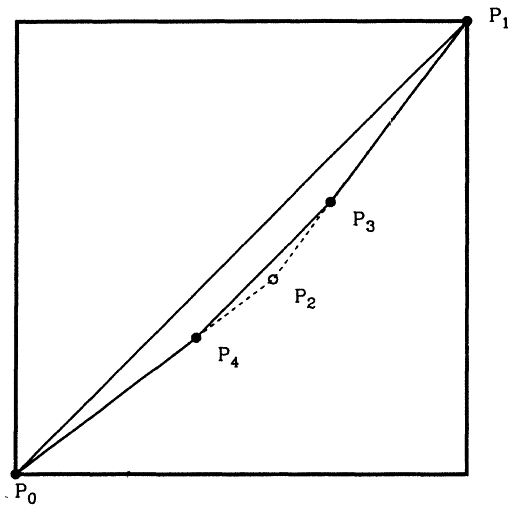

Figure 1.

Now let us consider the hypotheses known to be sufficient for $(1/p, 1/q)$ to be in the type set of T. Since $Tf = \mu * f$ where μ is a finite measure, the type set always contains the closed segment P_0P_1.

For $n = 2$, an argument involving Stein's analytic interpolation theorem [S1] and van der Corput's Lemma may be used to prove that T is a bounded operator from L^p to $L^{p'}$ when $p = 3/2$ (cf. [L,S2]). An application of the Riesz-Thorin Interpolation Theorem [Z] then shows that the necessary triangle $P_0P_1P_2$ is also sufficient.

For $n = 3$, Oberlin [O1] has proved that the necessary trapezoid is sufficient. Whether the trapezoid is sufficient for $n \geqq 4$ is unknown.

The proof of Theorem 4 is modeled on the $n = 2$ case. Theorem 3 provides a substitute for van der Corput's Lemma, allowing us to obtain the sharp $L^p - L^{p'}$ result for all n.

Proof of Theorem 4. Define the family of distributions D_z on R by

$$\langle D_z, \phi \rangle = \frac{1}{\Gamma\left(\dfrac{z+1}{2}\right)} \int \phi(u)|u|^z \, du, \quad \Re z > -1$$

and then by analytic continuation as in [GS] to an entire family of distributions. We now define an analytic family of operators by

$$T_z f(x) = \int_0^1 \left\langle D_z, \dots, \left\langle D_z, f\left(x - \gamma(t) - \sum_{j=2}^{n} u_j \gamma^{(j)}(t)\right)\right\rangle_{u_2} \cdots \right\rangle_{u_n} dt, \quad x \in \mathbf{R}^n.$$

The change of variables $y = \gamma(t) + \sum_{j=2}^{n} u_j \gamma^{(j)}(t)$ shows that

$$\|T_z f\|_{L^\infty} \leqq C_z \|f\|_{L^1}, \quad \Re z = 0,$$

where C_z has no worse than exponential growth in $|\Im z|$.

If $z = -\dfrac{n(n+1)}{(n+2)(n-1)} + is$, a computation using the formula for the Fourier transform of D_z in [GS] gives

$$\widehat{T_z f}(\xi) = C_z \hat{f}(\xi) \int_0^1 e^{-i\xi\cdot\gamma(t)} \left(\prod_{j=2}^{n} |\xi \cdot \gamma^{(j)}(t)| \right)^{2/(n^2+n-2) - is} dt.$$

Thus with $p(t) := \xi \cdot \gamma(t)$, we see that

$$|\widehat{T_z f}(\xi)| \leqq C_z |\hat{f}(\xi)| \left| \int_0^1 e^{ip(t)} \left(\prod_2^n |p^{(j)}(t)|^{2/(n^2+n-2)} \right)^{1 + i(n^2+n-2)s/2} dt \right|.$$

Now $p \in \mathscr{P}_n$ for all $\xi \in \mathbf{R}^n$, so we may apply Theorem 3 to obtain $|\widehat{T_z f}(\xi)| \leqq C_z |\hat{f}(\xi)|$. Hence by the Plancherel Theorem

$$\|T_z f\|_{L^2} \leqq C_z \|f\|_{L^2}, \quad \Re z = -\frac{n(n+1)}{(n+2)(n-1)}.$$

Since T is a constant multiple of T_{-1}, analytic interpolation implies

$$\|Tf\|_{L^{p'}} \leqq C \|f\|_{L^p}, \quad p = \frac{2n(n+1)}{n^2+n+2}.$$

An application of the Riesz-Thorin Theorem finishes the proof of the sufficiency of the hypotheses in Theorem 4. The necessity of the hypotheses follows from the requirement that $(1/p, 1/p')$ lie inside the trapezoid $P_0 P_1 P_3 P_4$.

NOTE ADDED IN PROOF. Yibiao Pan has independently proved Oberlin's conjecture.

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