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REGULAR GROWTH OF SUBHARMONIC FUNCTIONS OF SEVERAL VARIABLES

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0. Introduction.

In this paper we shall study the growth characteristics of subharmonic functions in R^m and the continuity of their indicator functions. We shall denote by $SH^{\rho(r)}(\mathsf{R}^m)$ the family of subharmonic functions u in R^m which are of finite order $\rho > 0$ and of normal type with respect to the proximate order $\rho(r)$. So there exist constants A_0 and A_1 depending on u such that $u(x) \leqq A_0 + A_1 |x|^{\rho(|x|)}$ for all $x \in \mathsf{R}^m$, where the non-negative function $\rho(r)$ is defined for $r \in \mathsf{R}^+$ such that $\rho(r) \to \rho$ and $\rho'(r) r \log r \to 0$ as $r \to \infty$. For such a function u, the indicator function h_u^* of u is defined as

$$h_u^*(x) = \limsup_{x' \to x} \limsup_{r \to \infty} u_r(x'), \qquad \text{where} \qquad u_r(x) = \frac{u(rx)}{r^{\rho(r)}}.$$

This indicator function is positively homogeneous of degree ρ and subharmonic. Thus its Laplacian Δh_u^*, in the sense of distributions, is a positive measure.

In view of subharmonicity there exist several equivalent convergences of the sequence u_r. In section 1 we prove that u_r converges in $L^1_{\text{loc}}(\mathsf{R}^m)$ to h_u^* if and only if Δu_r converges as a distribution to Δh_u^* in R^m. In contrast to the case $m = 2$, the indicator h_u^* may not be continuous for general m. In section 2 we prove that the uniformly bounded masses of the Laplacians Δu_r on any compact subset imply continuity of the indicator function. This result improves earlier work by Gruman and Berndtsson, see [9] and [7]. We also get a characteristic of functions of regular growth with continuous indicators. Finally in section 3 we present some related facts in terms of the limit sets introduced by Azarin in the paper [6].

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1. Subharmonic Functions of Regular Growth in R^m.

Following [14] we let $I_u^r(x, \delta)$ denote the integral averages of the functions u_r over the ball in R^m with center x and radius δ.

DEFINITION 1.1. Given a function $u \in SH^{\rho(r)}(\mathsf{R}^m)$, we say u is of *regular growth (RG) in the direction of* $x \in \mathsf{R}^m$ if $x \notin E_u = \{x; h_u^*(x) = -\infty\}$ and

$$\liminf_{\delta \to 0} \liminf_{r \to \infty} I_u^r(x, \delta) = h_u^*(x).$$

The function u is said to be of *regular growth in a subset* $K \subset \mathsf{R}^m$ if u is of regular growth in the direction of every $x \in K \setminus E_u$ and $K \not\subset E_u$.

The RG functions was first studied by Levin and Pfluger in a different way, see [15]. The Levin-Pfluger theory of entire functions of one complex variable with completely regular growth plays an important role for the study of entire and meromorphic functions of finite order. This theory establishes a relationship between the distribution of zero set of an entire function and its asymptotic growth, and in 1962 it was extended by Azarin, see [2]–[5], to subharmonic functions in R^m. Recall that a set $E \subset \mathsf{R}^m$ is called a C_0^α-set if every subset $E \cap \{|x| < R\}$ can be covered by balls with centers in $\{|x| < R\}$ whose radii $r_j(R)$ satisfy

$$\lim_{R \to \infty} \frac{1}{R^\alpha} \sum r_j^\alpha(R) = 0,$$

see [6]. Azarin introduced the following

DEFINITION 1.2. A function $u \in SH^{\rho(r)}(\mathsf{R}^m)$ is said to be of *completely regular growth* (CRG) in R^m if there exists a C_0^{m-1}-set E such that

$$\lim_{r \to \infty} \sup_{|x| = 1, rx \notin E} |u_r(x) - h_u^*(x)| = 0.$$

We know now that the two definitions above coincide. In fact, in view of subharmonicity, the functions of regular growth can be characterized in several different ways.

THEOREM 1.3. *For $u \in SH^{\rho(r)}(\mathsf{R}^m)$, the following statements are equivalent:*
(1) *u is* RG *in* R^m.
(2) u is CRG *in* R^m.
(3) *u_r converges as a distribution to h_u^* in* R^m.
(4) *u_r converges to h_u^* in* $L^1_{loc}(\mathsf{R}^m)$.
(5) *$h_{u+v}^* = h_u^* + h_v^*$ for all $v \in SH^{\rho(r)}(\mathsf{R}^m)$.*

(1) $\Leftrightarrow$ (4) was proved by Gruman, see [14, Theorem 4.7], and (2) $\Leftrightarrow$ (5) was

given by Favarov, see [8]. (2) $\Leftrightarrow$ (3) can be founded in [1], and (3) $\Leftrightarrow$ (4) is a basic fact for subharmonic functions, see [11, Theorem 4.1.9].

Theorem 1.3 implies that if u is a RG function then Δu_r converges as a distribution to Δh_u^*. On the other hand, we know that for a function u of integral order ρ, $\Delta u_r \to \Delta h_u^*$ implies the regularity of growth of u. This follows directly from results of Gruman or Azarin, see [14, Theorems 4.15 and 4.16] and [5] respectively. But the corresponding assertion for integer ρ seems to be unknown even for $m = 2$. We shall now prove that the assertion holds for any ρ. Let $B(x_0, r)$ be the open ball in R^m with center at x_0 and radius r, and let Γ_ω^ϕ be the right circular cone with vertex at the origin, ω as axis and angular opening ϕ. We shall denote by $\mu(E)$ the mass of a positive measure μ on a subset $E \subset \mathsf{R}^m$. Our result is

THEOREM 1.4. *Let $u \in \mathrm{SH}^{\rho(r)}(\mathsf{R}^m)$. Then u is* RG *in R^m if and only if for any $\omega \neq 0$ one has*

$$\lim_{r \to \infty} \frac{\Delta u(B(0,r) \cap \Gamma_\omega^\phi)}{r^{\rho(r)+m-2}} = \Delta h_u^*(B(0,1) \cap \Gamma_\omega^\phi),$$

except perhaps for a countable set of ϕ depending on ω.

As an application of Theorem 1.4, we get that u_r converges to h_u^* in $L^1_{\mathrm{loc}}(\mathsf{R}^m)$ if and only if Δu_r converges as a distribution to Δh_u^* in R^m.

To prove Theorem 1.4 we need two simple lemmas.

LEMMA 1.5. *Let $u \in \mathrm{SH}^{\rho(r)}(\mathsf{R}^m)$ and let K be a compact subset in R^m. Then*

$$\lim_{r \to \infty} \int_K (u_r(x) - h_u^*(x))\,dx = 0 \quad \text{if and only if} \quad \lim_{r \to \infty} \int_K |u_r(x) - h_u^*(x)|\,dx = 0.$$

PROOF. The "if" part is trivial. For "only if", we introduce the notation

$$I_{h_u^*}(x, \delta) = \frac{1}{\tau_m \delta^m} \int_{|y-x|<\delta} h_u^*(y)\,dy,$$

where τ_m is the volume of the unit ball in R^m. Let us write

$$\int_K h_u^*(x)\,dx - \int_K u_r(x)\,dx = \int_K [h_u^*(x) - I_{h_u^*}(x, \delta)]\,dx$$

$$+ \int_K [I_{h_u^*}(x, \delta) - u_r(x)] \overset{\mathrm{def}}{=} C_\delta + D_\delta.$$

For any $\varepsilon > 0$ we choose $\delta_0 > 0$ such that $|C_{\delta_0}| < \varepsilon$. But

$\limsup_{r\to\infty} u_r(x) \leqq h_u^*(x) \leqq I_{h_u^*}(x, \delta_0)$ in K. So it follows from the Hartogs lemma, see [14, Corollary 1.32] that there exists a constant $R_{\varepsilon\delta_0} > 0$ such that $u_r(x) \leqq I_{h_u^*}(x, \delta_0) + \varepsilon$ for all $r \geq R_{\varepsilon\delta_0}$ and all $x \in K$. Hence, for such r we have

$$D_{\delta_0} = \int\limits_K |I_{h_u^*}(x, \delta_0) + \varepsilon - u_r(x)|\, dx - \varepsilon \int\limits_K dx$$

$$\geqq \int\limits_K |h_u^*(x) - u_r(x)|\, dx - \int\limits_K |I_{h_u^*}(x, \delta_0) - h_u^*(x)|\, dx - 2\varepsilon \int\limits_K dx$$

$$\geqq \int\limits_K |h_u^*(x) - u_r(x)|\, dx - \varepsilon - 2\varepsilon \int\limits_K dx.$$

Therefore, for r large enough

$$\int\limits_K |h_u^*(x) - u_r(x)|\, dx \leqq \int\limits_K [h_u^*(x) - u_r(x)]\, dx + 2\varepsilon + 2\varepsilon \int\limits_K dx.$$

This implies that u_r converges to h_u^* in $L^1(K)$, if the first integral on the right-hand side converges to zero, and hence Lemma 1.5 follows.

A useful consequence of Lemma 1.5, which follows by the positive homogeneity, is the following

LEMMA 1.6. *A function u is* RG *in* R^m *if and only if there exists a shell* $D = \{x \in \mathsf{R}^m;\ \delta_2 < |x| < \delta_1\}$ *such that*

$$\int\limits_D u_r(x)\, dx \to \int\limits_D h_u^*(x)\, dx, \quad \text{as} \quad r \to \infty.$$

PROOF OF THEOREM 1.4. The "only if" part is a consequence of Theorem 1.3 because the Laplacian operator Δ is continuous in the distribution space and takes all subharmonic functions to positive measures. For "if" part, the generalized Jensen formula says that for every subharmonic function f and any constants $\delta_1 > \delta_2 > 0$ the following equality holds:

$$\frac{1}{\delta_1^{m-1}} \int\limits_{|x|=\delta_1} f(x)\, d\sigma(x) - \frac{1}{\delta_2^{m-1}} \int\limits_{|x|=\delta_2} f(x)\, d\sigma(x) = \int\limits_{\delta_2}^{\delta_1} \frac{\Delta f(B(0, \delta))}{\delta^{m-1}}\, d\delta,$$

where $d\sigma$ is the Lebesgue measure of the unit sphere in R^m.

Hence for $r > 2$ we have

$$\int\limits_{\frac{1}{2}<|x|<1} u_r(x)\,dx - \frac{1}{m}[1-(\tfrac{1}{2})^m]r^{m-1} \int\limits_{|x|=\frac{1}{r}} u_r(x)\,d\sigma(x)$$

$$= \int\limits_{\frac{1}{2}}^{1} \delta_1^{m-1}\,d\delta_1 \int\limits_{\frac{1}{r}}^{\delta_1} \frac{\Delta u_r(B(0,\delta))}{\delta^{m-1}}\,d\delta$$

and

$$\int\limits_{\frac{1}{2}<|x|<1} h_u^*(x)\,dx = \int\limits_{\frac{1}{2}}^{1} \delta_1^{m-1}\,d\delta_1 \int\limits_{0}^{\delta_1} \frac{\Delta h_u^*(B(0,\delta))}{\delta^{m-1}}\,d\delta.$$

But

$$r^{m-1} \int\limits_{|x|=\frac{1}{r}} u_r(x)\,d\sigma(x) = r^{-\rho(r)} \int\limits_{|x|=1} u(x)\,d\sigma(x) \to 0, \quad \text{as} \quad r\to\infty,$$

so by Lemma 1.6 we only need to show

$$\int\limits_{\frac{1}{2}}^{1} \delta_1^{m-1}\,d\delta_1 \int\limits_{\frac{1}{r}}^{\delta_1} \frac{\Delta u_r(B(0,\delta))}{\delta^{m-1}}\,d\delta \to \int\limits_{\frac{1}{2}}^{1} \delta_1^{m-1}\,d\delta_1 \int\limits_{0}^{\delta_1} \frac{\Delta h_u^*(B(0,\delta))}{\delta^{m-1}}\,d\delta.$$

In view of the Lebesgue dominated convergence theorem, it suffices to verify that

(i) for any $\delta\in(0,\delta_1)$ we have

$$\frac{\chi_{r,\delta_1}(\delta)}{\delta^{m-1}}\Delta u_r(B(0,\delta)) \to \frac{1}{\delta^{m-1}}\Delta h_u^*(B(0,\delta)), \quad \text{as} \quad r\to\infty,$$

where χ_{r,δ_1} denotes the characteristic function of the interval $[\frac{1}{r},\delta_1]$, and

(ii) there exists a constant $C>0$ such that for all δ in $(0,\delta_1)$ and r large enough,

$$\left|\frac{\chi_{r,\delta_1}(\delta)}{\delta^{m-1}}\Delta u_r(B(0,\delta))\right| \leqq C\delta^{\frac{\rho}{2}-1}.$$

To prove assertion (i) we can assume, without loss of generality, that $\delta=1$ and $\chi_{r,\delta_1}(\delta)=1$. Let ω_N and ω_S be the north and south pole, respectively in the unit ball $B(0,1)$. By the assumption in the theorem we can define an increasing function $F_N(\phi)$ in $(0,\pi)$, such that $F_N(\phi)=\lim_{r\to\infty}\Delta u_r(\Gamma_{\omega_N}^{\phi}\cap B(0,1))$ holds for almost every ϕ in $(0,\pi)$. So for almost every ϕ in $(0,\pi)$ and small $\psi>0$,

$$\limsup_{r\to\infty}\Delta u_r(\partial\Gamma_{\omega_N}^{\phi}\cap B(0,1)\backslash\{0\})$$

$$\leqq \limsup_{r\to\infty}(\Delta u_r(\Gamma_{\omega_N}^{\phi+\psi}\cap B(0,1)) - \Delta u_r(\Gamma_{\omega_N}^{\phi}\cap B(0,1))) = F_N(\phi+\psi)-F_N(\phi).$$

The last difference converges to zero almost everywhere as $\psi \searrow 0$ because an increasing function is continuous almost everywhere. Hence we get that for almost every ϕ in $(0, \pi)$

$$\lim_{r\to\infty} \Delta u_r(\partial \Gamma^{\phi}_{\omega_N} \cap B(0,1)\backslash\{0\}) = 0.$$

Similarly, for almost every ϕ in $(0, \pi)$ we have

$$\Delta h_u^*(\partial \Gamma^{\phi}_{\omega_N} \cap B(0,1)\backslash\{0\}) = 0.$$

Therefore, there exists $\pi/2 < \phi_0 < \pi$ satisfying the following conditions:

(a) $$\lim_{r\to\infty} \Delta u_r(\partial \Gamma^{\phi_0}_{\omega_N} \cap B(0,1)\backslash\{0\}) = \Delta h_u^*(\partial \Gamma^{\phi_0}_{\omega_N} \cap B(0,1)\backslash\{0\}) = 0;$$

(b) $$\lim_{r\to\infty} \Delta u_r(\Gamma^{\phi_0}_{\omega_N} \cap B(0,1)) = \Delta h_u^*(\Gamma^{\phi_0}_{\omega_N} \cap B(0,1));$$

(c) $$\lim_{r\to\infty} \Delta u_r(\Gamma^{\pi-\phi_0}_{\omega_S} \cap B(0,1)) = \Delta h_u^*(\Gamma^{\pi-\phi_0}_{\omega_S} \cap B(0,1)).$$

But $\Delta h_u^*(\{0\}) = 0$ and

$$\Delta u_r(\{0\}) \leqq \Delta u_r\left(B\left(0, \frac{1}{r}\right)\right) = \frac{1}{r^{\rho(r)+m-2}} \Delta u(B(0,1)) \to 0, \quad \text{as} \quad r \to \infty.$$

These facts, together with (a), (b) and (c), imply that

$$\lim_{r\to\infty} \Delta u_r(B(0,1)) = \lim_{r\to\infty} \Delta u_r(B(0,1)\backslash\{0\}) = \Delta h_u^*(B(0,1)\backslash\{0\}) = \Delta h_u^*(B(0,1)),$$

and assertion (i) follows.

Assertion (ii) is the consequence of assertion (i). In fact, for any $\sigma \in (0, \delta_1)$ we have

$$\frac{\chi_{r,\delta_1}(\delta)}{\delta^{m-1}} \Delta u_r(B(0,\delta)) = \frac{\chi_{r,\delta_1}(\delta)}{\delta^{m-1} r^{\rho(r)+m-2}} \Delta u(B(0,\delta r))$$

$$= \delta^{\frac{\rho}{2}-1} \chi_{r,\delta_1}(\delta) \frac{(\delta r)^{\rho(\delta r)-\frac{\rho}{2}}}{r^{\rho(r)-\frac{\rho}{2}}} \Delta u_{\delta r}(B(0,1)) \to \delta^{\rho-1} \Delta h_u^*(B(0,1)),$$

as $r \to \infty$. On the other hand, by the definition of the proximate order $\rho(r)$, we have

$$\frac{d}{dr}(r^{\rho(r)-\frac{\rho}{2}}) = r^{\rho(r)-\frac{\rho}{2}-1}\left(rp'(r)\log r + \rho(r) - \frac{\rho}{2}\right) > 0$$

for r large enough. So $r^{\rho(r)-\frac{\rho}{2}}$ is an increasing function for such r. Hence there exists a constant $R > 0$ such that for $\delta r > R$

$$\frac{\chi_{r,\delta_1}(\delta)}{\delta^{m-1}} \Delta u_r(B(0,\delta)) \leqq \delta^{\frac{\varrho}{2}-1}(\Delta h_u^*(B(0,1)) + 1).$$

But for $0 < \delta r \leqq R$

$$\frac{\chi_{r,\delta_1}(\delta)}{\delta^{m-1}} \Delta u_r(B(0,\delta)) \leqq \delta^{\rho(r)-1} \Delta u(B(0,R)).$$

So there exists $C > 0$ such that for all $\delta \in (0, \delta_1)$ and sufficiently large r,

$$\left| \frac{\chi_{r,\delta_1}(\delta)}{\delta^{m-1}} \Delta u_r(B(0,\delta)) \right| \leqq C\delta^{\frac{\varrho}{2}-1}.$$

This is assertion (ii). Hence the proof of Theorem 1.4 is complete.

2. Continuity of Indicator Functions in $\mathbb{R}^m$.

In this section we shall discuss regions in which the mass of the Laplacian of a function is small. For an open connected subset D on the unit sphere, we define a cone K_D by $K_D = \{x \in \mathbb{R}^m; x/|x| \in D\}$. In the case $m = 2$, an indicator function h_u^* is continuous, see [15, p. 54]. However, the corresponding assertion for general m is not true. The first counterexample was given by Lelong in [13]. Nevertheless, Gruman and Berndtsson have proved the following sufficient condition for continuity of the indicator functions, see [7], [9] and also [10].

THEOREM 2.1. *Let $u \in SH^{\rho(r)}(\mathbb{R}^m)$ and let K_D be a cone in $\mathbb{R}^m$ such that for any $D' \subset\subset D$*

$$\lim_{r\to\infty} \frac{\Delta u(B(0,r) \cap K_{D'})}{r^{\rho(r)+m-2}} = 0.$$

Then h_u^* is continuous in K_D, and there exist constants $C_{D'}$ such that

$$|h_u^*(x_1) - h_u^*(x_2)| \leqq C_{D'} |x_1 - x_2|$$

for all $x_1, x_2 \in D'$.

It is clear that the density assumption in Theorem 2.1 is equivalent to the following: For any $D' \subset\subset D$ and any $x \in K_{D'}$ one has

$$\limsup_{r\to\infty} \frac{\Delta u(B(rx,r) \cap K_{D'})}{r^{\rho(r)+m-2}} = 0.$$

In the following theorem we relax this condition.

THEOREM 2.2. *Let $u \in \mathrm{SH}^{\rho(r)}(\mathbb{R}^m)$ and let K_D be a cone in $\mathbb{R}^m$. If for any $D' \subset\subset D$ there exists a constant $C_{D'}$ such that*

$$\limsup_{r\to\infty}\frac{\Delta u(B(rx,r)\cap K_{D'})}{r^{\rho(r)+m-2}}\leqq C_{D'}\quad\text{for all}\quad x\in K_{D'},$$

then h_u^* is continuous in K_D. Furthermore, there exist constants $B_{D'}$ such that

$$|h_u^*(x_1)-h_u^*(x_2)|\leqq B_{D'}\begin{cases}|x_1-x_2|^\rho, & \textit{if }\ \rho<1;\\ |x_1-x_2|\log\dfrac{1}{|x_1-x_2|}, & \textit{if }\ \rho=1;\\ |x_1-x_2|, & \textit{if }\ \rho>1\end{cases}$$

for all $x_1, x_2\in D'$.

Proof of Theorem 2.2. Given $D'\subset\subset D$ in the unit sphere, we take subdomains G' and G in R^m such that

$$D'\subset G'\subset\subset G\subset\subset K_D\cap B(0,2).$$

Since $h_u^*=\limsup_{r\to\infty}u_r$ holds for almost all $x\in G'$, it follows by the method of integral average used in [7] that it is enough to show the estimates

$$|h_u^*(x_1)-h_u^*(x_2)|\leqq C\begin{cases}|x_1-x_2|^\rho, & \text{if }\ \rho<1;\\ |x_1-x_2|\log\dfrac{1}{|x_1-x_2|}, & \text{if }\ \rho=1;\\ |x_1-x_2|, & \text{if }\ \rho>1\end{cases}$$

for such values of x_1 and x_2. (In this proof we let C denote a generic constant which does not depend on the points x_1 and x_2.) Without loss of generality, we can suppose that $h_u^*(x_1)>h_u^*(x_2)>-\infty$, since the indicator function is not identically equal to $-\infty$. We can therefore, by the Fatou lemma, find a positive constant δ depending on x_2 and $|x_1-x_2|$ such that $4\delta<|x_1-x_2|$ and

$$\limsup_{r\to\infty}\frac{1}{\tau_m\delta^m}\int\limits_{|x-x_2|<\delta}u_r(x)\,dx\leqq h_u^*(x_2)+|x_1-x_2|.$$

On the other hand, for such δ we have

$$\limsup_{r\to\infty}\frac{1}{\tau_m\delta^m}\int\limits_{|x-x_1|<\delta}u_r(x)\,dx\geqq\limsup_{r\to\infty}u_r(x_1)=h_u^*(x_1).$$

Using the general fact $\limsup A_r-\limsup B_r\leqq\limsup(A_r-B_r)$, we thus get

$$0<h_u^*(x_1)-h_u^*(x_2)\leqq|x_1-x_2|+\limsup_{r\to\infty}\frac{1}{\tau_m\delta^m}\int\limits_{|x|<\delta}|u_r(x_1+x)-u_r(x_2+x)|\,dx.$$

By the Riesz theorem there exist harmonic functions Φ_r in G such that

$$u_r(x) = -\frac{1}{\theta_m}\int\limits_G \frac{1}{|x-y|^{m-2}}\,\Delta u_r(y) + \Phi_r(x) \quad \text{for all} \quad x \in G,$$

where $\theta_m = (m-2)m\tau_m$ for $m > 2$, and $\theta_2 = 2\pi$, see [16, Theorem 1.2.1′]. Clearly, the family $\{\Phi_r\}$ has a uniform upper bound on any compact set in G. Moreover, for any $\delta > 1$ and $r > 1$

$$\frac{1}{\tau_m\delta^m}\int\limits_{G\cap\overline{B(0,\delta)}} \Phi_r(x)\,dx \geqq \frac{1}{\tau_m\delta^m}\int\limits_{G\cap\overline{B(0,\delta)}} u_r(x)\,dx \geqq \frac{1}{\tau_m\delta^m}\int\limits_{\overline{B(0,\delta)}} (u_r(x) - C_0)\,dx$$

$$\geqq \frac{1}{\tau_m r^{\rho(r)}}\int\limits_{\overline{B(0,1)}} u(x)\,dx - C_0 \to -C_0, \quad \text{as} \quad r \to \infty.$$

Here the constant $C_0 > 0$ is a common upper bound of u_r in $\overline{B(0,\delta)}$ for all $r > 1$. It follows from Theorem 4.1.9 in [11] that $\{\Phi_r\}$ forms a normal family in G. So for any $r > 1$ and small $|x_1 - x_2|$ we have

$$\frac{1}{\tau_m\delta^m}\int\limits_{|x|<\delta} |\Phi_r(x_1+x) - \Phi_r(x_2+x)|\,dx \leqq C\,|x_1 - x_2|.$$

Applying the triangle inequality and changing the order of integration, we find that $h_u^*(x_1) - h_u^*(x_2)$ can be estimated by

$$C\,|x_1 - x_2| + \limsup_{r\to\infty}\frac{C}{\delta^m}\int\limits_{y\in G}\Delta u_r(y)\int\limits_{|x|<\delta}$$

$$\times\left|\frac{1}{|x_2+x-y|^{m-2}} - \frac{1}{|x_1+x-y|^{m-2}}\right| dx.$$

To study this upper limit we first observe that it is majorized by the sum

$$A_1 + A_2 + A_1' + A_2' + A_1'' + A_2'' + A''',$$

where

$$A_j = \limsup_{r\to\infty}\frac{C}{\delta^m}\int\limits_{\substack{y\in G\\ |y-x_j|<2\delta}}\Delta u_r(y)\int\limits_{|x|<\delta}\frac{1}{|x_j+x-y|^{m-2}}\,dx,$$

$$A_j' = \limsup_{r\to\infty}\frac{C}{\delta^m}\int\limits_{\substack{y\in G\\ 2|y-x_j|<|x_1-x_2|}}\Delta u_r(y)\int\limits_{|x|<\delta}\frac{1}{|x+x_1+x_2-x_j-y|^{m-2}}\,dx,$$

$$A_j'' = \limsup_{r\to\infty} \frac{C}{\delta^m} \int\limits_{\substack{y\in G\\ 4\delta \leqq 2|y-x_j| < |x_1-x_2|}} \Delta u_r(y) \int\limits_{|x|<\delta} \frac{1}{|x_j+x-y|^{m-2}}\, dx,$$

$$A''' = \limsup_{r\to\infty} \frac{C}{\delta^m} \int\limits_{\substack{y\in G\\ 2|y-x_1|\geqq|x_1-x_2|\\ 2|y-x_2|\geqq|x_1-x_2|}} \Delta u_r(y) \int\limits_{|x|<\delta}$$

$$\times \left| \frac{1}{|x_2+x-y|^{m-2}} - \frac{1}{|x_1+x-y|^{m-2}} \right| dx.$$

We estimate these seven upper limits separately. First we have

$$A_1 \leqq \limsup_{r\to\infty} \frac{C}{\delta^m} \int\limits_{\substack{y\in G\\ |y-x_1|<2\delta}} \Delta u_r(y) \int\limits_{|\omega|<3\delta} \frac{1}{|\omega|^{m-2}}\, d\omega$$

$$\leqq C\delta^{2-m} \limsup_{r\to\infty} \int\limits_{\substack{y\in G\\ |y-x_1|<2\delta}} \Delta u_r(y) = C\delta^{2-m} \limsup_{r\to\infty} \frac{\Delta u(B(rx_1, 2\delta r)\cap G)}{r^{\rho(r)+m-2}}$$

$$\leqq C \limsup_{r\to\infty} 2^{\rho(2\delta r)+m-2} \delta^{\rho(2\delta r)} r^{\rho(2\delta r)-\rho(r)} \limsup_{r\to\infty} \frac{\Delta u\left(B\left(r\frac{x_1}{2\delta}, r\right)\cap G\right)}{r^{\rho(r)+m-2}}$$

$$\leqq C\delta^{\rho} \leqq C|x_1-x_2|^{\rho},$$

and similarly

$$A_2 \leqq C|x_1-x_2|^{\rho}.$$

Next, it follows from $4\delta < |x_1 - x_2|$ that

$$A_1' \leqq \limsup_{r\to\infty} \frac{C}{\delta^m} \int\limits_{\substack{y\in G\\ 2|y-x_1|<|x_1-x_2|}} \Delta u_r(y) \int\limits_{|x|<\delta} \left(\frac{4}{|x_1-x_2|}\right)^{m-2} dx$$

$$= C \limsup_{r\to\infty} |x_1-x_2|^{2-m} \int\limits_{\substack{y\in G\\ 2|y-x_1|<|x_1-x_2|}} \Delta u_r(y) \leqq C|x_1-x_2|^{\rho},$$

and similarly

$$A_2' \leqq C|x_1-x_2|^{\rho}.$$

To handle A_1'' we choose an integer K_δ, such that $K_\delta \log 2 \geqq \log(|x_1 - x_2|/4\delta)$. We then get

$$A_1'' \leqq C \limsup_{r\to\infty} \int\limits_{\substack{y\in G\\ 4\delta \leqq 2|y-x_1| < |x_1-x_2|}} \frac{1}{|x_1 - y|^{m-2}} \Delta u_r(y)$$

$$\leqq C \limsup_{r\to\infty} \sum_{k=1}^{K_\delta} \int\limits_{\substack{y\in G\\ |x_1-x_2| \leqq 2^{k+1}|y-x_1| < 2|x_1-x_2|}} \left(\frac{2^{k+1}}{|x_1 - x_2|}\right)^{m-2} \Delta u_r(y)$$

$$\leqq C \sum_{k=1}^{K_\delta} \left(\frac{2^{k+1}}{|x_1 - x_2|}\right)^{m-2} \limsup_{r\to\infty} \frac{\Delta u\left(B\left(rx_1, \frac{|x_1 - x_2|}{2^k} r\right) \cap G\right)}{r^{\rho(r)+m-2}}$$

$$\leqq C \sum_{k=1}^{K_\delta} \frac{|x_1 - x_2|^\rho}{2^{k\rho}} \leqq C|x_1 - x_2|^\rho,$$

and similarly

$$A_2'' \leqq C|x_1 - x_2|^\rho.$$

Now, using the inequality $|\alpha_1^{2-m} - \alpha_2^{2-m}| \leqq (m-2)|\alpha_1 - \alpha_2|(\alpha_1^{1-m} + \alpha_2^{1-m})$ for positive numbers α_1 and α_2, we can estimate A''' by

$$\limsup_{r\to\infty} \frac{C|x_1 - x_2|}{\delta^m} \int\limits_{\substack{y\in G\\ 2|y-x_1| \geqq |x_1-x_2|\\ 2|y-x_2| \geqq |x_1-x_2|}} \Delta u_r(y) \int\limits_{|x|<\delta}$$

$$\times \left(\frac{1}{|x_2 + x - y|^{m-1}} + \frac{1}{|x_1 + x - y|^{m-1}}\right) dx$$

$$\leqq C|x_1 - x_2| \limsup_{r\to\infty} \int\limits_{\substack{y\in G\\ 2|y-x_1| \geqq |x_1-x_2|}} \frac{1}{|x_1 - y|^{m-1}} \Delta u_r(y)$$

$$+ C|x_1 - x_2| \limsup_{r\to\infty} \int\limits_{\substack{y\in G\\ 2|y-x_2| \geqq |x_1-x_2|}} \frac{1}{|x_2 - y|^{m-1}} \Delta u_r(y).$$

By symmetry we only need to estimate the first of these two limits. Let K be the least integer such that $K|x_1 - x_2| \geqq 4$. Splitting the domain of integration into $S_1 \cup S_2 \ldots \cup S_{K-1}$, where $S_k = G \cap \{y;\ k|x_1 - x_2| \leqq 2|y - x_1| < (k+1)|x_1 - x_2|\}$, we can majorize it by a sum

$$C|x_1 - x_2|^{2-m} \lim_{r\to\infty} \sum_{k=1}^{K-1} \frac{1}{k^{m-1}} \int_{S_k} \Delta u_r(y).$$

Write $T_k = S_1 \cup S_2 \ldots \cup S_k$. Application of the Abel transformation, see [18, p. 3], then yields that this sum is estimated by

$$C|x_1 - x_2|^{2-m} \limsup_{r\to\infty} \sum_{k=1}^{K-2} \left(\frac{1}{k^{m-1}} - \frac{1}{(k+1)^{m-1}}\right) \int_{T_k} \Delta u_r(y)$$

$$+ C|x_1 - x_2|^{2-m} \limsup_{r\to\infty} \frac{1}{(K-1)^{m-1}} \int_{T_{K-1}} \Delta u_r(y)$$

$$\leqq C|x_1 - x_2|^{2-m} \sum_{k=1}^{K-2} \frac{1}{k^m} \limsup_{r\to\infty} \int_{T_k} \Delta u_r(y) + \frac{C|x_1 - x_2|^{2-m}}{K^{m-1}}$$

$$\leqq C|x_1 - x_2|^{2-m} \sum_{k=1}^{K-2} \frac{1}{k^m} ((k+1)|x_1 - x_2|)^{\rho+m-2} + C|x_1 - x_2|$$

$$\leqq C \begin{cases} |x_1 - x_2|^\rho, & \text{if } \rho < 1; \\ |x_1 - x_2| \log \dfrac{1}{|x_1 - x_2|}, & \text{if } \rho = 1; \\ |x_1 - x_2|, & \text{if } \rho > 1. \end{cases}$$

Hence the proof of Theorem 2.2 is complete.

Now we shall show that there exists a subharmonic function with our condition in Theorem 2.2, but not the condition in Theorem 2.1. We need the following two lemmas.

LEMMA 2.3. *Let $u \in \mathrm{SH}^{\rho(r)}(\mathrm{R}^m)$ be RG in R^m and let Δh_u^* be a continuous function in $\mathrm{R}^m \setminus \{0\}$. Then there exists a constant C such that*

$$\limsup_{r\to\infty} \frac{\Delta u(B(rx, r))}{r^{\rho(r)+m-2}} \leqq C|x|^{\rho-2} \quad \text{for} \quad |x| \geqq 1.$$

In particular, for $0 < \rho \leqq 2$ the above left-hand side must be bounded.

This lemma follows by the straight forward computation.

LEMMA 2.4. [17, Theorem 1.3.1]. *For any plurisubharmonic function g in $\mathrm{C}^n = \mathrm{R}^{2n}$ which is at most of normal type with respect to the order ρ, there exists an entire function f satisfying the condition*

$$\frac{g(rx)}{r^{\rho}} - \frac{\log|f(rx)|}{r^{\rho}} \to 0, \quad \text{as} \quad r \to \infty$$

in $L^1_{\mathrm{loc}}(\mathsf{R}^{2n})$.

EXAMPLE 2.5. By Lemma 2.4 there exists an entire function f of normal type with respect to the order $\rho(r) \equiv 1$ such that

$$u_r(x) = \frac{\log|f(rx)|}{r} \to |x|, \quad \text{as} \quad r \to \infty$$

in $L^1_{\mathrm{loc}}(\mathsf{R}^{2n})$. So $h_u^*(x) = |x|$, and by Theorem 1.3 we see that the function $u = \log|f|$ is RG in R^{2n}. It then follows from Lemma 2.3 that there exists a constant $C > 0$ such that

$$\limsup_{r \to \infty} \frac{\Delta u(B(rx, r))}{r^{2n-1}} \leqq C \quad \text{for any } x \in \mathsf{R}^{2n}.$$

That is, the condition in Theorem 2.2 holds. But, since $h_u^*(x) = |x|$ is not pluriharmonic in $\mathsf{R}^{2n}\backslash\{0\}$, it follows from Corollary 4.12 in [14] that

$$\limsup_{r \to \infty} \frac{\Delta u(B(0, r))}{r^{2n-1}} > 0.$$

Finally for RG functions we get one characteristic of continuous indicators.

THEOREM 2.6. *Let* $u \in \mathrm{SH}^{\rho(r)}(\mathsf{R}^m)$ *be* RG *in the cone* K_D. *Then* h_u^* *is continuous in* K_D *if and only if the equality*

$$\limsup_{\delta_1 \to 0} \sup_{x \in D'} \left| \int_0^{\delta_1} \delta^{\rho - 1} \lim_{r \to \infty} \frac{\Delta u\left(B\left(r\dfrac{x}{\delta}, r\right)\right)}{r^{\rho(r)+m-2}} \, d\delta \right| = 0$$

holds for any $D' \subset\subset D$.

PROOF. We first show the sufficiency. Given $D' \subset\subset D$, by the generalized Jensen formula, we obtain that

$$\frac{m}{(2^m - 1)^2 a^m b^m} \int_a^{2a} \delta_1^{m-1} d\delta_1 \int_b^{2b} \delta_2^{m-1} d\delta_2 \int_{\delta_2}^{\delta_1} \frac{\Delta u_r(B(x, \delta))}{\delta^{m-1}} \, d\delta$$

$$= \frac{1}{(2^m - 1)a^m} \int_{a \leqq |y - x| \leqq 2a} u_r(y) \, dy - \frac{1}{(2^m - 1)b^m} \int_{b \leqq |y - x| \leqq 2b} u_r(y) \, dy$$

for any $x \in D'$ and small constants a $>$ 2b $>$ 0. Since the function u is RG in K_D, u_r

converges to h_u^* in the distribution sense, and hence the limit $\lim_{r\to\infty} \Delta u_r(B(x,\delta))$ exists for almost all δ. Letting $r \to \infty$ in the above equality, and using the Lebesgue dominated convergence theorem, we have

$$\left|\frac{1}{(2^m-1)a^m}\int\limits_{a\leq|y-x|\leq 2a} h_u^*(y)\,dy - \frac{1}{(2^m-1)b^m}\int\limits_{b\leq|y-x|\leq 2b} h_u^*(y)\,dy\right|$$

$$\leq \frac{1}{m}\int_b^{2a} \delta^{1-m}\lim_{r\to\infty} \Delta u_r(B(x,\delta))\,d\delta = \frac{1}{m}\int_b^{2a}\lim_{r\to\infty}\frac{(\delta r)^{\rho(r\delta)}}{\delta r^{\rho(r)}}\frac{\Delta u(B(rx,\delta r))}{(\delta r)^{\rho(\delta r)+m-2}}\,d\delta$$

$$= \frac{1}{m}\int_b^{2a}\delta^{\rho-1}\lim_{r\to\infty}\frac{\Delta u\left(B\left(r\frac{x}{\delta},r\right)\right)}{r^{\rho(r)+m-2}}\,d\delta.$$

Since h_u^* is subharmonic, we get, letting $b \to 0$, that

$$\left|\frac{1}{\tau_m(2^m-1)a^m}\int\limits_{a\leq|y-x|\leq 2a} h_u^*(y)\,dy - h_u^*(x)\right|$$

$$\leq \frac{1}{m\tau_m}\sup_{x\in D'}\int_0^{2a}\delta^{\rho-1}\lim_{r\to\infty}\frac{\Delta u\left(B\left(r\frac{x}{\delta},r\right)\right)}{r^{\rho(r)+m-2}}\,d\delta$$

for any $x \in D'$. It then follows from the assumption that the integral in the absolute value sign converges to h_u^* uniformly in D' as $a \to 0$. But this integral is a continuous function of x, it follows that h_u^* is itself continuous in D', and hence in D. By the positive homogeneity the continuity holds in the cone K_D.

For the necessity we also start from the first equality in the proof, and similar to the above proof, we can get that for any $D' \subset\subset D$

$$\frac{1}{m\tau_m}\sup_{x\in D'}\int_0^{a}\delta^{\rho-1}\lim_{r\to\infty}\frac{\Delta u\left(B\left(r\frac{x}{\delta},r\right)\right)}{r^{\rho(r)+m-2}}\,d\delta$$

$$\leq \sup_{x\in D'}\left|\frac{1}{\tau_m(2^m-1)a^m}\int\limits_{a\leq|y-x|\leq 2a} h_u^*(y)\,dy - h_u^*(x)\right|$$

$$\leq \sup_{|y|\leq 2a}\sup_{x\in D'}|h_u^*(x+y) - h_u^*(x)|.$$

The required result then follows from the uniform continuity of h_u^*.

3. Final Remarks on Limit Sets.

In [6] Azarin generalized the concept of indicator functions, and first studied the limit sets for subharmonic functions of proximate order $\rho(r)$ in R^m. For a function $u \in \mathrm{SH}^{\rho(r)}(\mathrm{R}^m)$, the limit set $L(u)$ consists of all subharmoonic functions which are limits in $L^1_{\mathrm{loc}}(\mathrm{R}^m)$ of subsequences $u_{r_j} \to \infty$, as $r_j \to \infty$. For the corresponding positive measure Δu, the limit set $L(\Delta u)$ is defined as the set of all measures in R^m which are limits, in the distribution sense, of subsequences Δu_{r_j}, as $r_j \to \infty$. The structure of limit sets has been completely described by Hörmander and Sigurdsson in [12]. In view of Theorem 1.3 we have that a function u is RG in R^m if and only if the limit set $L(u)$ only contains the indicator function h_u^*. Therefore, if u is a RG function then $L(\Delta u)$ is a singleton. Azarin has proved the converse implication for non-integer order ρ and a weak result for integer ρ, see [6]. Our theorem 1.4 implies now that u is a RG function in R^m if and only if $L(\Delta u)$ only contains the one element Δh_u^*.

It is also clear that the relation $L(u+v) = L(u) + L(v)$ holds for all $v \in \mathrm{SH}^{\rho(r)}(\mathrm{R}^m)$, if u is a RG function. In fact, by a slight modification of the proof of Theorem 5 in [8], we have the converse assertion.

THEOREM 3.1. *Let $u \in \mathrm{SH}^{\rho(r)}(\mathrm{R}^m)$. Then u is* RG *in R^m if and only if the equality*

$$L(u+v) = L(u) + L(v)$$

holds for any $v \in \mathrm{SH}^{\rho(r)}(\mathrm{R}^m)$.

We know that an indicator function in R^2 must be locally bounded. But it is easy to find a limit set in which there exists a subharmonic function with non-empty polar set. Moreover, the following example gives a limit set with functions which are finite everywhere, but not locally bounded. Let $\rho(r) \equiv 1/2$ and $M = \{g_0(rx)/r^{\frac{1}{2}}; r > 0\} \cup \{0\}$, where

$$g_0(x) = \sum_{k=1}^{\infty} \frac{\log(\sqrt{(x_1 - k^{-\frac{1}{2}})^2 + x_2^2 + \ldots + x_{2n}^2} + e^{-k^3}) - \log(k^{-\frac{1}{2}} + e^{-k^3})}{k^2}.$$

Choose a constant C such that $g_0(x) \leqq C|x|^{\frac{1}{2}}$ for all $x \in \mathrm{R}^{2n}$. It is clear that M is a compact set of plurisubharmonic functions g in R^{2n}, such that $g(0) = 0$, $g(x) \leqq C|x|^{\frac{1}{2}}$ and $g(r_1 x)/r_1^{\frac{1}{2}} \in M$ for all $r_1 > 0$. By Theorem 1.2.1 in [17] we know that M is a subset of some limit set. Hence this limit set contains functions which are finite everywhere, but not locally bounded. Now our results are

THEOREM 3.2. *Let $u \in \mathrm{SH}^{\rho(r)}(\mathrm{R}^m)$. Then we have*

(1) *$L(u)$ consists of functions with finite values at the point $x_0 \in \mathrm{R}^m$ if and only if the inequality*

$$\int_0^1 \frac{\mu(B(x_0, \delta))}{\delta^{m-1}} d\delta < \infty$$

holds for any $\mu \in L(\Delta u)$.

(2) *$L(u)$ consists of locally bounded functions if and only if the function*

$$F_\mu(x) = \int_0^1 \frac{\mu(B(x, \delta))}{\delta^{m-1}} d\delta$$

is locally bounded for each $\mu \in L(\Delta u)$.

(3) *$L(u)$ consists of continuous functions in the domain $D \subset\subset \mathrm{R}^m$ if and only if*

$$\lim_{\delta_1 \to 0} \sup_{x \in D} \int_0^{\delta_1} \frac{\mu(B(x, \delta))}{\delta^{m-1}} d\delta = 0$$

for all $\mu \in L(\Delta u)$.

(4) *$L(u)$ consists of harmonic functions if and only if $L(\Delta u)$ is a singleton with the measure identically zero as element.*

The proof of Theorem 3.2 is essentially the same as the proofs given before and is therefore omitted.

REFERENCES

1. P. Z. Agranovič, *Functions of several variables with completely regular growth* (Russian), Teor. Funktsiĭ Funktsional Anal. i. Prilozhen 30 (1978), 3–13.
2. V. S. Arazin, *On the indicator of a function subharmonic in a multidimensional space* (Russian), .Dokl. Akad. Nauk SSSR 139 (1961), 1033–1036.
3. V. S. Azarin, *Functions of regular growth that are subharmonic in the whole space* (Russian), Har'kov. Gos. Univ. Učen. Zap 120 = Zap. Meh.-Mat. Fak. i Har'kov. Mat. Obšč. (4) 28 (1961), 128–148.
4. V. S. Azarin, *On the indicator of a function subharmonic in a multidimensional space* (Russian), Mat. Sb. 58 (100) (1962), 87–94.
5. V. S. Azarin, *Subharmonic functions of completely regular growth in multidimensional space* (Russian), Dokl. Akad. Nauk SSSR 146 (1962), 743–746.
6. V. S. Azarin, *On the asymptotic behaviour of subharmonic functions of finite order*, Math. USSR Sb. 36 (1980), 135–154.
7. B. Berndtsson, *On the asymptotic growth (pluri)subharmonic functions*, Thesis, Univ. of Göteborg, 1977.
8. S. Ju. Favorov, *The addition of the indicators of entire and subharmonic functions of several vaviable* (Russian), Mat. Sb. 105 (147) (1978), 128–140.
9. L. Gruman, *Entire functions of several variables and their asymptotic growth*, Ark. Mat. 9 (1971), 141–163.

10. L. Gruman, *The regularity of growth of entire functions whose zeros are hyperplanes*, Ark. Mat. 10 (1972), 23–31.
11. L. Hörmander, *The Analysis of Linear Partial Differential Operators* I, Grundlehren Math. Wiss. 256, 1983.
12. L. Hörmander and R. Sigurdsson, *Limit sets of plurisubharmonic functions*, Math. Scad. 65 (1989), 308–320.
13. P. Lelong, *Non-continuous indicators for entire functions of $n \geqq 2$ variables and finite order*, Proc. Sympos. Pure Math. 11 (1968), 285–297.
14. P. Lelong and L. Gruman, *Entire Functions of Several Complex Variables*, Grundlehren Math. Wiss. 282, 1986.
15. B. Ja. Levin, *Distribution of Zeros of Entire Functions*, Trans. Math. Monographs, 5, A.M.S., Providence, R. I., 1964.
16. L. I. Ronkin, *Introduction in the Theory of Entire Functions of Several Variables*, Trans. Math. Monographs, 45, A.M.S., 1974.
17. R. Sigurdsson, *Growth properties of analytic and plurisubharmonic functions of finite order*, Math. Scand. 59 (1986), 235–304.
18. A. Zygmund, *Trigonometric Series*-I, Cambridge University Press, Cambridge, 1959.

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