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MULTIPLICATION BY BLASCHKE PRODUCTS AND STABILITY OF IDEALS IN LIPSCHITZ ALGEBRAS

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1. Introduction.

Let D denote the open unit disk $\{z \in \mathsf{C}: |z| < 1\}$ and T its boundary. For $\alpha \in (0, +\infty)$ denote by Λ^α the classical *Lipschitz-Zygmund space* of smooth functions on the circle:

$$\Lambda^\alpha \overset{\text{def}}{=} \{f \in C(\mathsf{T}): \|\Delta_h^m f\|_\infty = O(|h|^\alpha), h \in \mathsf{R}\},$$

where m is any integer with $m > \alpha$, $\|\cdot\|_\infty$ is the usual L^∞ norm, and Δ_h^m stands for the mth order difference operator. (Recall that the Δ_h^m's are defined by induction: $\Delta_h^k = \Delta_h^1 \Delta_h^{k-1}$,

$$(\Delta_h^1 f)(\zeta) \overset{\text{def}}{=} f(e^{ih}\zeta) - f(\zeta), \quad \zeta \in \mathsf{T}.)$$

Further, let Λ_A^α be the *analytic subspace* of Λ^α: $\Lambda_A^\alpha \overset{\text{def}}{=} \Lambda^\alpha \cap H^\infty$, where as usual H^∞ stands for the algebra of bounded analytic functions on D. An equivalent definition ([Z], vol. 1) is as follows:

$$\Lambda_A^\alpha = \{f \in H^\infty: f^{(m)}(z) = O((1 - |z|)^{\alpha - m}), \quad z \in \mathsf{D}\};$$

here m is again an integer such that $m > \alpha$, and $f^{(m)}$ is the mth order derivative of f.

This paper is devoted to a certain subtle point concerning the multiplicative structure of functions in Λ_A^α.

Suppose $f \in \Lambda_A^\alpha$ and θ is an *inner function* (i.e. $\theta \in H^\infty$ and $\lim_{r \to 1^-} |\theta(r\zeta)| = 1$ for almost all $\zeta \in \mathsf{T}$). Assume that

$$f\theta \in \Lambda_A^\alpha. \tag{1}$$

It is known (see [Shi 1], [Shi 2] or Theorem B below) that in the case $0 < \alpha < 1$ (1) implies

$$f\theta^k \in \Lambda_A^\alpha \quad \text{for all } k \in \mathsf{N}, \tag{2}$$

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where N is the set of positive integers. That seems to be rather natural. However, the situation changes radically if α, the order of smoothness, becomes > 1. The implication (1) $\Rightarrow$ (2) is still valid for *singular* (cf. [G], chapter II) inner functions θ, yet it turns out that for any $\alpha > 1$ one can find a function f, $f \in \Lambda_A^\alpha$, and a *Blaschke product* B such that $fB \in \Lambda_A^\alpha$ but $fB^2 \notin \Lambda_A^\alpha$. This surprising phenomenon was discovered by N. A. Shirokov [Shi 1,2]. In fact, his ingenious construction provides a function f in $A^\infty \stackrel{\text{def}}{=} \bigcap_{\alpha>0} \Lambda_A^\alpha$ and a Blaschke product B such that $fB \in A^\infty$, but the modulus of continuity of $(fB^2)'$ need not satisfy any prescribed estimate.

Thus, the passage from fB to fB^2 is sometimes accompanied with a great loss of smoothness. This striking result displays a subtle distinction between the cases $0 < \alpha < 1$ and $\alpha > 1$, as far as factorization of Λ_A^α functions is concerned. (The Zygmund classes Λ^k, $k \in \mathrm{N}$, will not be considered in this paper, so we do not mention the case $\alpha = 1$.)

On the other hand, it has been proved that if $n < \alpha < n + 1$ ($n \in \mathrm{N}$), then the inclusion

$$fB^{n+1} \in \Lambda_A^\alpha \tag{3}$$

does imply (and is, therefore, equivalent to)

$$fB^k \in \Lambda_A^\alpha \quad \text{for all } k \in \mathrm{N}, \tag{4}$$

f being a function in Λ_A^α and B a Blaschke product. This fact is again a consequence of Shirokov's results [Shi 1,2]; it is also contained in Theorem B, due to the author, which we cite in Section 2 below. If one replaces (3) by a weaker condition

$$fB^n \in \Lambda_A^\alpha, \tag{5}$$

it turns out that, generally speaking, (5) is no longer sufficient for (4) to hold. It should be noted though that (5) implies $fB \in \Lambda_A^\alpha, \dots, fB^{n-1} \in \Lambda_A^\alpha$. This is because Λ_A^α possesses the following "division property" (cf. [H], [S]): whenever $f \in \Lambda_A^\alpha$, θ is inner and $f/\theta \in H^\infty$, it follows that $f/\theta \in \Lambda_A^\alpha$.

A natural problem in this context is: given $n \in \mathrm{N}$ and $\alpha \in (n, n + 1)$, describe the Blaschke products B in terms of their zeros, for which the implication (5) $\Rightarrow$ (4) does hold with an arbitrary $f \in \Lambda_A^\alpha$.

Before proceeding with the solution, we introduce some notation. For θ an inner function, set $I^\alpha(\theta) \stackrel{\text{def}}{=} \Lambda^\alpha \cap \theta H^\infty$, so that $I^\alpha(\theta)$ is a closed ideal in the algebra Λ_A^α. For $g \in H^\infty$, let T_g denote the multiplication map defined by $T_g f = fg$.

Obviously, the above problem is equivalent to characterizing the B's for which

$$T_B I^\alpha(B^n) \subset \Lambda_A^\alpha, \tag{6}$$

where $\alpha > 1$, $\alpha \notin \mathbb{N}$, $n = [\alpha]$. (Here $[\alpha]$ denotes the integral part of α.)

Thus we are actually concerned with a certain *stability property* of the ideal $I^\alpha(B^n)$ with respect to multiplication by "its own" Blaschke product B and/or by its powers B^k. Note that, in view of the above discussion, (6) is equivalent to

$$(7) \qquad T_{B^k} I^\alpha(B^n) \subset \Lambda_A^\alpha \quad \text{for all } k \in \mathbb{N}.$$

If $I^\alpha(B^n)$ is nontrivial and satisfies (6) or (7), B will be called *stable*. (Perhaps the term "α-stable" would sound more natural, but we shall soon see that this property does not depend on α.)

Assuming in addition that the zeros $\{z_j\}$ of B form an *interpolating sequence for* Λ_A^α (for a precise definition see Section 2 below), we now provide a complete characterization of all such B's that are stable.

THEOREM 1. *Let $n \in \mathbb{N}$, $n < \alpha < n+1$, and let B be a Λ_A^α-interpolating Blaschke product with zeros $\{z_j\}_{j=1}^\infty$. The following are equivalent.*

(i) *B is stable.*

(ii) $\inf_{k \in \mathbb{N}\setminus\{j\}} |z_j - z_k| = O(1 - |z_j|), \quad j \in \mathbb{N}.$

In fact, we prove a more general assertion. Given two exponents α and β such that $n < \beta \leqq \alpha < n+1$, a Blaschke product B will be termed (α, β)-*stable* iff $I^\alpha(B^n) \neq \{0\}$ and

$$T_B I^\alpha(B^n) \subset \Lambda_A^\beta.$$

In other words, B is (α, β)-stable iff for any $f \in \Lambda_A^\alpha$ (5) implies $fB^{n+1} \in \Lambda_A^\beta$ (and hence $fB^k \in \Lambda_A^\beta$ for all $k \in \mathbb{N}$). For the sake of completeness, we note that for $n < \alpha < n+1$ the set $T_B I^\alpha(B^n)$ is always contained in Λ_A^γ with $0 < \gamma \leqq n$ and is never contained (unless $I^\alpha(B^n) = \{0\}$) in Λ_A^γ with $\gamma > \alpha$. That is why we assume $\beta \in (n, \alpha]$.

Of course, (α, α)-stability is just "stability" as defined above, and so Theorem 1 is a special case of the next fact.

THEOREM 2. *Let $n \in \mathbb{N}$, $n < \beta \leqq \alpha < n+1$, and let B be a Λ_A^α-interpolating Blaschke product with zeros $\{z_j\}_{j=i}^\infty$. The following are equivalent.*

(i) *B is (α, β)-stable.*

(ii) $\inf_{k \in \mathbb{N}\setminus\{j\}} |z_j - z_k| = O((1 - |z_j|)^{(\beta - n)/(\alpha - n)}), \; j \in \mathbb{N}.$

The rest of the paper is organized as follows. In Section 2 we cite a few results that will be used in the sequel; we also specify the notion of a "Λ_A^α-interpolating Blaschke product" that occurs in Theorems 1 and 2. Section 3 contains the proof of Theorem 2. In Section 4 we give an application of Theorem 2 to embedding theorems for star-invariant subspaces of the Hardy classes H^p. Finally, Section 5 contains some examples and remarks.

2. Preliminaries.

Let $\alpha \in (0, +\infty)$, $\alpha \notin \mathbb{N}$, $n = [\alpha]$. Suppose $f \in \Lambda_A^\alpha$. It is well known (and easily shown) that for $\zeta_1, \zeta_2 \in \operatorname{clos} \mathbb{D} \stackrel{\text{def}}{=} \{|z| \leqq 1\}$ and for $s = 0, 1, \ldots, n$

$$\left| f^{(s)}(\zeta_1) - \sum_{m=s}^{n} \frac{f^{(m)}(\zeta_2)}{(m-s)!}(\zeta_1 - \zeta_2)^{m-s} \right| \leqq C|\zeta_1 - \zeta_2|^{\alpha - s}, \tag{8}$$

where C is a constant independent of ζ_1 and ζ_2. (Note that in the case $0 < \alpha < 1$ we have $n = 0$, and so (8) reduces to the usual Lipschitz condition of order α.)

A closed subset E of clos $\mathbb{D}$ is said to be *Λ_A^α-interpolating* iff any interpolation problem

$$f|E = \varphi_0, f'|E = \varphi_1, \ldots, f^{(n)}|E = \varphi_n$$

has a solution $f \in \Lambda_A^\alpha$, provided that the *data* φ_s: $E \to \mathbb{C}$ satisfy the necessary conditions stated above:

$$\left| \varphi_s(\zeta_1) - \sum_{m=s}^{n} \frac{\varphi_m(\zeta_2)}{(m-s)!}(\zeta_1 - \zeta_2)^{m-s} \right| \leqq C|\zeta_1 - \zeta_2|^{\alpha - s} \quad (\zeta_1, \zeta_2 \in E) \tag{9}$$

for $s = 0, 1, \ldots, n$.

The following characterization of Λ_A^α-interpolating sets is due to E. M. Dyn'kin [Dyn].

THEOREM A. *Let $\alpha \in (0, +\infty)$, $\alpha \notin \mathbb{N}$. A closed set E, $E \subset$ clos $\mathbb{D}$, is Λ_A^α-interpolating if and only if the two conditions hold:*

(a) $$\inf\{\rho(\zeta_1, \zeta_2) : \zeta_1, \zeta_2 \in E \cap \mathbb{D}, \zeta_1 \neq \zeta_2\} > 0,$$

where $\rho(\zeta_1, \zeta_2) \stackrel{\text{def}}{=} |\zeta_1 - \zeta_2| / |1 - \bar{\zeta}_1 \zeta_2|$.

(b) *There is a constant $c > 0$ such that for any arc I, $I \subset \mathbb{T}$, we have*

$$\sup_{\zeta \in I} \operatorname{dist}(\zeta, E) \geqq c|I|,$$

where as usual $\operatorname{dist}(\zeta, E) \stackrel{\text{def}}{=} \inf_{z \in E} |\zeta - z|$ *and $|I|$ is the length of I.*

Of course, condition (a) means that the set $E \cap \mathbb{D}$ is countable and its points $\{z_j\}$ form a *separated sequence* with respect to the *pseudohyperbolic distance* $\rho(\cdot, \cdot)$. Moreover, (a) and (b) together imply [Dyn] that this sequence is, in fact, *uniformly separated* (or *H^∞-interpolating*), i.e.

(10) $$\inf_j \prod_{k:k\neq j} \left| \frac{z_j - z_k}{1 - \bar{z}_j z_k} \right| > 0.$$

It should be noted also that the class of Λ_A^α-interpolating sets does not actually depend on α.

Given a Blaschke product

$$B(z) = B_{\{z_j\}}(z) = \prod_{j=1}^{\infty} \frac{\bar{z}_j}{|z_j|} \frac{z_j - z}{1 - \bar{z}_j z}$$

with pairwise distinct zeros (set $\bar{z}_j/|z_j| \stackrel{\text{def}}{=} -1$ if $z_j = 0$), we call it *Λ_A^α-interpolating* if the closure of its zeros, clos $\{z_j\}$, is a Λ_A^α-interpolating set. As mentioned above, for such B's we have (10); on the other hand, the set $E =$ clos $\{z_j\}$ must satisfy the *Beurling-Carleson condition*

$$\int_{\mathbf{T}} \log \operatorname{dist}(\zeta, E)\, |d\zeta| > -\infty$$

(i.e. the non-uniqueness condition for Λ_A^α [C]), whence the ideals $I^\alpha(B^k)$ are nontrivial for all $k \in \mathbf{N}$.

As another auxiliary result we cite the next Theorem B, due to the author [D 1]. (The most essential part of it is also contained in a previous paper [D 2].) In order to state it, we introduce the following notation: given $\theta \in H^\infty$ and $\varepsilon > 0$, let $\Omega(\theta, \varepsilon) \stackrel{\text{def}}{=} \{z \in \mathbf{D}\colon |\theta(z)| < \varepsilon\}$.

THEOREM B. *Let $\alpha \in (0, +\infty)$, $m \in \mathbf{N}$, $m > \alpha$. Suppose $f \in \Lambda_A^\alpha$ and θ is an inner function. The following statements are equivalent.*

(i) $f/\theta^m \in \Lambda^\alpha$.

(ii) $f\theta^m \in \Lambda_A^\alpha$.

(iii) $f\theta^k \in \Lambda^\alpha \quad \forall k \in \mathbf{Z}$.

(iv) *For some* $\varepsilon \in (0, 1)$ (*or, equivalently, for any* $\varepsilon \in (0, 1)$), *we have*

(11) $$f(z) = O((1 - |z|)^\alpha) \quad \text{as} \quad |z| \to 1,\ z \in \Omega(\theta, \varepsilon).$$

It is this last quantitative condition that will be used as a multiplication criterion.

Finally, the following lemma will be needed (cf. [G], Chapter X, Lemma 1.4).

LEMMA C. *Let B be an interpolating* (i.e. *H^∞-interpolating) Blaschke product with zeros $\{z_j\}$ such that the infimum occurring in* (10) *equals δ. Then there exist $\lambda = \lambda(\delta)$, $0 < \lambda < 1$, and $\varepsilon = \varepsilon(\delta)$, $0 < \varepsilon < 1$, such that*

(12) $$\Omega(B, \varepsilon) \subset \bigcup_j \{z \in \mathbf{D}\colon \rho(z, z_j) < \lambda\}.$$

(Recall that the non-euclidean metric $\rho(\cdot,\cdot)$ is defined by $\rho(z,w) = |z-w|/|1-\bar{z}w|$).

3. Proof of Theorem 2.

For the reader's convenience, we reproduce the theorem itself and then proceed with the proof.

Given a sequence $\{z_j\}_{j=1}^{\infty} \subset \mathsf{D}$, we set $d_j \overset{\text{def}}{=} \inf_{k \in \mathsf{N} \setminus \{j\}} |z_j - z_k|$.

THEOREM 2. *Let $n \in \mathsf{N}$, $n < \beta \leqq \alpha < n+1$, and let B be a Λ_A^{α}-interpolating Blaschke product with zeros $\{z_j\}_{j=1}^{\infty}$. The following are equivalent.*

(i) *B is (α,β)-stable (see sect. 1).*

(ii) $d_j = O((1-|z_j|)^{(\beta-n)/(\alpha-n)})$, $j \in \mathsf{N}$.

PROOF. (i) $\Rightarrow$ (ii). Let $E = \operatorname{clos}\{z_j\}$. Define the interpolation data $\varphi_0, \varphi_1, \ldots, \varphi_{n-1}$ to be zero on E, and let $\varphi_n \colon E \to \mathsf{C}$ be defined by

$$\varphi_n(z_j) = d_j^{\alpha-n}\ (j \in \mathsf{N}), \quad \varphi_n | E \cap \mathsf{T} = 0.$$

This done, conditions (9) are easily verified. Indeed, the φ_s's being continuous on E, it suffices to check that

$$\left| \varphi_s(z_j) - \sum_{m=s}^{n} \frac{\varphi_m(z_k)}{(m-s)!}(z_j - z_k)^{m-s} \right| \leqq C|z_j - z_k|^{\alpha-s} \tag{13}$$

for $s = 0, 1, \ldots, n$, where z_j and z_k are two distinct zeros of B, and C is a constant.

Case 1. $s = 0, 1, \ldots, n-1$. The left-hand side in (13) equals

$$\left| \frac{\varphi_n(z_k)}{(n-s)!}(z_j - z_k)^{n-s} \right| = \frac{1}{(n-s)!} d_k^{\alpha-n} |z_j - z_k|^{n-s} \leqq |z_j - z_k|^{\alpha-s},$$

where we have used the obvious inequalities

$$1/(n-s)! \leqq 1 \quad \text{and} \quad d_k \leqq |z_j - z_k|.$$

Case 2. $s = n$. The left-hand side in (13) equals

$$|\varphi_n(z_j) - \varphi_n(z_k)| = |d_j^{\alpha-n} - d_k^{\alpha-n}| \leqq d_j^{\alpha-n} + d_k^{\alpha-n} \leqq 2|z_j - z_k|^{\alpha-n},$$

because both d_j and d_k are $\leqq |z_j - z_k|$.

Thus (13) is established (with $C = 2$), and so is (9). Recalling that E is a Λ_A^{α}-interpolating set, one can find a function $f \in \Lambda_A^{\alpha}$ such that

$$f|E = f'|E = \ldots = f^{(n-1)}|E = 0, \quad f^{(n)}|E = \varphi_n.$$

Hence for all $j \in \mathsf{N}$ we have

(14) $$f(z_j) = f'(z_j) = \dots = f^{(n-1)}(z_j) = 0, \quad f^{(n)}(z_j) = d_j^{\alpha-n}.$$

Thus, in each of z_j's f has a zero of multiplicity n, whence $f \in I^\alpha(B^n)$.

The (α, β)-stability of B now implies $fB \in \Lambda_A^\beta$, which is equivalent to

$$(fB)^{(n+1)}(z) = O((1-|z|)^{\beta-n-1}), \quad z \in \mathbb{D}.$$

(See the Introduction for the definition of Λ_A^β in terms of derivatives.) In particular,

(15) $$(fB)^{(n+1)}(z_j) = O((1-|z_j|)^{\beta-n-1}).$$

Further, the Leibniz formula says

$$(fB)^{(n+1)}(z_j) = \sum_{m=0}^{n+1} \binom{n+1}{m} f^{(m)}(z_j) B^{(n+1-m)}(z_j).$$

Clearly, the only non-zero summand here is the one arising for $m = n$ (recall (14) and the obvious fact that $B(z_j) = 0$).
Therefore,

$$(fB)^{(n+1)}(z_j) = (n+1) f^{(n)}(z_j) B'(z_j) = (n+1) d_j^{\alpha-n} B'(z_j).$$

Now (15) yields

$$d_j^{\alpha-n} |B'(z_j)| \leqq \text{const} \cdot (1-|z_j|)^{\beta-n-1}$$

Multiplying both sides by $1-|z_j|$ and noting that $\inf_j |B'(z_j)|(1-|z_j|) > 0$ (this is but a well-known restatement of (10)), we get

$$d_j^{\alpha-n} \leqq \text{const} \cdot (1-|z_j|)^{\beta-n},$$

which clearly coincides with (ii).

(ii) $\Rightarrow$ (i). Let $f \in I^\alpha(B^n)$. Since a Λ_A^α-interpolating Blaschke product is also H^∞-interpolating, we have (10). Denote the left-hand side of (10) by δ and let $\lambda = \lambda(\delta)$ and $\varepsilon = \varepsilon(\delta)$ be the same as in Lemma C. Our plan is to use (ii) in order to derive condition (11) with α replaced by β and θ replaced by B. This done, an application of Theorem B will complete the proof.

As mentioned in Section 2 above, f satisfies (8) where now we set $s = 0$:

(16) $$\left| f(\zeta_1) - \sum_{m=0}^{n} \frac{f^{(m)}(\zeta_2)}{m!} (\zeta_1 - \zeta_2)^m \right| \leqq C|\zeta_1 - \zeta_2|^\alpha;$$

here ζ_1 and ζ_2 are arbitrary points in clos $\mathbb{D}$ and C a positive constant. (A direct way of verifying (16) is to observe that the left-hand side equals

$$\left| \int_{\zeta_1}^{\zeta_2} dt_1 \int_{\zeta_1}^{t_1} dt_2 \ldots \int_{\zeta_1}^{t_{n-1}} dt_n \int_{\zeta_1}^{t_n} f^{(n+1)}(t_{n+1})\, dt_{n+1} \right|$$

and to make the obvious estimates on the integrals.)

First we let z_j and z_k be two distinct zeros of B and apply (16) with $\zeta_1 = z_k$, $\zeta_2 = z_j$. Noting that

$$f(z_j) = f'(z_j) = \ldots = f^{(n-1)}(z_j) = 0 \quad \forall j \in \mathrm{N} \tag{17}$$

(recall that f is divisible by B^n), we get

$$\frac{1}{n!}|f^{(n)}(z_j)|\,|z_j - z_k|^n \leqq C|z_j - z_k|^\alpha,$$

whence

$$|f^{(n)}(z_j)| \leqq Cn!\,|z_j - z_k|^{\alpha - n}.$$

Since k was an arbitrary number is $\mathrm{N} \setminus \{j\}$, it follows that

$$|f^{(n)}(z_j)| \leqq \text{const} \cdot d_j^{\alpha - n}. \tag{18}$$

Rewriting (ii) as

$$d_j^{\alpha - n} \leqq \text{const} \cdot (1 - |z_j|)^{\beta - n}$$

and substituting this in (18), we obtain

$$|f^{(n)}(z_j)| \leqq \text{const} \cdot (1 - |z_j|^{\beta - n}, \quad j \in \mathrm{N}. \tag{19}$$

Now suppose $z \in \Omega(B, \varepsilon)$. In view of (12) there is a $j \in \mathrm{N}$ such that $\rho(z, z_j) < \lambda$. Another application of (16) (this time we set $\zeta_1 = z, \zeta_2 = z_j$) gives

$$\left| f(z) - \frac{1}{n!} f^{(n)}(z_j)(z - z_j)^n \right| \leqq C|z - z_j|^\alpha,$$

where we have once again used (17). Hence

$$\begin{aligned} |f(z)| &\leqq \frac{1}{n!}|f^{(n)}(z_j)|\,|z - z_j|^n + C|z - z_j|^\alpha \leqq \\ &\leqq \text{const} \cdot (1 - |z_j|)^{\beta - n}|z - z_j|^n + C|z - z_j|^\alpha. \end{aligned} \tag{20}$$

(The last inequality relies on (19).)

It is not hard to see (cf. [G], Chapter I, Section 1) that if $\rho(z, z_j) < \lambda < 1$ then there are positive constants $c_1 = c_1(\lambda)$ and $c_2 = c_2(\lambda)$ such that

$$1 - |z_j| \leqq c_1(1 - |z|)$$

and

$$|z - z_j| \leqq c_2(1 - |z|).$$

Combining these inequalities with (20) we get

$$|f(z)| \leqq C_1(1 - |z|)^\beta + C_2(1 - |z|)^\alpha \leqq C_3(1 - |z|)^\beta,$$

where C_1, C_2 and C_3 are some new constants.

Thus, for an arbitrary $f \in I^\alpha(B^n)$ we have established the estimate

$$f(z) = O((1 - |z|)^\beta), \quad z \in \Omega(B, \varepsilon),$$

which coincides with (11) up to the obvious replacements.

Conditions (iii) and (iv) in Theorem B being equivalent, we conclude that

$$f B^k \in \Lambda^\beta \quad \forall k \in \mathbf{Z}.$$

In particular, $fB \in \Lambda_A^\beta$. Therefore B is (α, β)-stable, as required.

4. Embedding theorems for star-invariant subspaces.

For $p > 0$, let H^p denote the classical *Hardy space* (see [G] or [K]) in the unit disk, $\mathbf{D}$. For $p \in [1, +\infty]$ and θ an inner function, let K_θ^p stand for the corresponding *star-invariant subspace*:

$$K_\theta^p \stackrel{\text{def}}{=} H^p \cap \theta \overline{H_0^p},$$

where $H_0^p \stackrel{\text{def}}{=} \{f \in H^p : f(0) = 0\}$ and the bar denotes complex conjugation. The term "star-invariant" here means "invariant under the backward shift operator". (It is a matter of common knowledge that the totality of K_θ^p, as θ ranges over all inner functions, coincides with the family of all closed star-invariant subspaces in H^p, $p \in [1, +\infty)$.)

In the case $p \in (0, 1)$ we set

$$K_\theta^p \stackrel{\text{def}}{=} \operatorname{clos}_{H^p} K_\theta^\infty;$$

here $\operatorname{clos}_{H^p}$ denotes the closure with respect to the H^p metric.

This section deals with some embedding theorems of the form $T_f K_\theta^p \subset H^q$, where p and q are positive exponents satisfying

$$0 < \max(1, p) < q < +\infty, \tag{21}$$

f is a function holomorphic in $\mathbf{D}$ and smooth up to the boundary, and T_f is the multiplication map defined by $T_f g = fg$.

The following proposition was proved by the author in [D 1] along with Theorem B (see Section 2 above).

THEOREM B′. *Let p and q satisfy* (21). *Set* $\alpha = 1/p - 1/q$, *and let m be an integer for which* $mp > 1$. *Given a function f,* $f \in \Lambda_A^\alpha$, *and an inner function* θ, *each of the conditions* (i) – (iv) *in Theorem* B *is equivalent to*

$$\text{(v)} \qquad T_f K_{\theta^m}^p \subset H^q.$$

In fact, from the proof [D 1] one sees that the implication (ii) ⇒ (v) holds when θ^m is replaced by an arbitrary inner function θ_1, i.e. under the above assumptions on p, q, α and f

$$f\theta_1 \in \Lambda_A^\alpha \Rightarrow T_f K_{\theta_1}^p \subset H^q. \tag{22}$$

Our next result is

THEOREM 3. *Let* $0 < p < 1 < q < +\infty$, $\alpha \overset{\text{def}}{=} 1/p - 1/q$, *and suppose there is a positive integer n for which* $n < \alpha < 1/p < n + 1$. *Suppose further that B is an* Λ_A^α*-interpolating Blaschke product with zeros* $\{z_j\}$. *If*

$$\sup_j \frac{d_j}{1 - |z_j|} = +\infty \tag{23}$$

then there exists an f, $f \in \Lambda_A^\alpha$, *such that*

$$T_f K_{B^n}^p \subset H^q \tag{24}$$

but

$$T_f K_{B^{n+1}}^p \not\subset H^q. \tag{25}$$

Before proceeding with the proof, we remark that $K_{B^n}^p$ coincides with the H^p-closed linear span of the family of rational fractions

$$\left\{\frac{1}{(1 - \bar{z}_j z)^k}: \ j \in \mathbf{N}, \quad k = 1, 2, \dots, n\right\}.$$

Thus, when we enlarge this family by letting in addition $k = n + 1$, the effect may be fatal (i.e., the corresponding embedding theorem may become false). It should be noted that in the case where $1 < p < q < +\infty$ such a phenomenon does not occur.

PROOF. By Theorem 1, (23) means that B is not stable, i.e. $T_B I^\alpha(B^n) \not\subset \Lambda_A^\alpha$. This in turn implies the existence of an f, $f \in \Lambda_A^\alpha$, such that $fB^n \in \Lambda_A^\alpha$ but $fB^{n+1} \notin \Lambda_A^\alpha$. Applying (22) with $\theta_1 = B^n$, we obtain (24). Applying Theorem B′ with $\theta = B$, $m = n + 1$, we arrive at (25).

The following generalization of Theorem 3 can be derived in a similar fashion with recourse to Theorem 2.

THEOREM 4. *Let* $0 < p < 1 < r \leqq q < +\infty$, $\alpha \overset{\text{def}}{=} 1/p - 1/q$, $\beta \overset{\text{def}}{=} 1/p - 1/r$. *Suppose that for some* n, $n \in \mathbb{N}$, *we have* $n < \beta < p^{-1} < n+1$. *If* B *is a* Λ_A^α*-interpolating Blaschke product with zeros* $\{z_j\}$ *for which*

$$\sup_j d_j(1 - |z_j|)^{-(\beta - n)/(\alpha - n)} = +\infty$$

then there exists an f, $f \in \Lambda_A^\alpha$, *such that* (24) *holds but* $T_f K^p_{B^{n+1}} \not\subset H^r$.

5. Remarks and examples.

1. In connection with embedding theorems for the K^p_θ spaces we cite [Co 1,2,3], [TV] and [D 2,3] where some partial information can be found on the embeddings $K^p_\theta \subset L^p(\mu)$ or $K^p_\theta \subset L^q(\mu)$, μ being a suitable measure on $\operatorname{clos} \mathbb{D}$ and θ an inner function. A complete characterization of the pairs (θ, μ) for which the embedding holds still seems to be unknown.

2. Suppose that the sequence $\{z_j\} \subset \mathbb{D}$ is Λ_A^α-interpolating and satisfies, in addition, the following regularity conditions:

$$d_j = |z_j - z_{j+1}|, \quad \sup_j \frac{1 - |z_j|}{1 - |z_{j+1}|} < +\infty.$$

Under these assumptions we are able to prove the converse of Theorem 3: if $B = B_{\{z_j\}}$ is stable then $T_f K^p_{B^n} \subset H^q$ implies $T_f K^p_{B^{n+1}} \subset H^q$. A similar supplement to Theorem 4 can be provided.

3. We proceed by giving a few examples.

(a) Let $\{z_j\}$ be a sequence in $\mathbb{D}$ tending to 1 nontangentially (i.e. $\sup_j |1 - z_j|/(1 - |z_j|) < +\infty$) such that $\inf_{j \neq k} \rho(z_j, z_k) > 0$. The arising Blaschke product $B = B_{\{z_j\}}$ is easily shown to be stable.

(b) Suppose $n < \beta < \alpha < n+1$, $n \in \mathbb{N}$. Fix $\gamma \geqq 1$ and let $\{z_j\}$ be defined by

$$|1 - z_j| = 2^{-j}, \quad 1 - |z_j| = 2^{-\gamma j}, \quad \operatorname{Im} z_j > 0.$$

It is not hard to see that $c_1 \cdot 2^{-j} \leqq d_j \leqq c_2 \cdot 2^{-j}$ (here c_1 and c_2 are absolute constants), and so condition (ii) in Theorem 2 holds iff $\gamma \leqq (\alpha - n)/(\beta - n)$. In particular, taking $\gamma = (\alpha - n)/(\beta - n)$ one obtains a Blaschke product that is (α, β)-stable but not (α, β_1)-stable whenever $\beta < \beta_1 \leqq \alpha$.

(c) Furthermore, consider the "super-tangential" sequence $\{z_j\}$ defined by

$$|1 - z_j| = 2^{-j}, \quad 1 - |z_j| = 2^{-2^j}, \quad \operatorname{Im} z_j > 0.$$

Clearly, condition (ii) in Theorem 2 is never fulfilled and so, for any values of α and β, the Blaschke product $B_{\{z_j\}}$ fails to be (α, β)-stable.

In order to make sure that the sequences constructed in (a), (b) and (c) above are Λ_A^α-interpolating, one may use either Theorem A or, still better, the following proposition [Kot]: if

$$\{z_j\} \subset \mathrm{D}, \lim_{j\to\infty} z_j = 1, \ |z_j - 1| \geqq |z_{j+1} - 1| \text{ and } \sup_{j\neq k} \frac{|1 - z_j||1 - z_k|}{|z_j - z_k|^2} < +\infty,$$

then $\{z_j\} \cup \{1\}$ is a Λ_A^α-interpolating set.

4. Results obtained in [D1], [D2] imply that if $B = B_{\{z_j\}}$ is not stable then it must necessarily be *sparse*, i.e.

$$\sup_j \prod_{k:k\neq j} \left| \frac{z_k - z_j}{1 - \bar{z}_k z_j} \right| = 1.$$

On the other hand, there are sparse Blaschke products that are stable; e.g. let $z_j = 1 - (j!)^{-2}$, $B = B_{\{z_j\}}$.

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