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TANGENTIAL CHARACTERIZATIONS OF BMOA ON STRICTLY PSEUDOCONVEX DOMAINS

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Let Ω be a bounded strictly pseudoconvex domain in C^n with $\mathscr{C}^\infty$ boundary $\partial\Omega$. Thus, we suppose that there is a $\mathscr{C}^\infty$ defining function $r: C^n \to R$, a neighborhood $\mathscr{O}$ of $\bar{\Omega}$, the closure of Ω, and a constant $C > 0$ such that

$$\Omega = \{\zeta : r(\zeta) < 0\},$$

$$\partial\Omega = \{\zeta : r(\zeta) = 0\},$$

$$|\nabla r| \neq 0,$$

everywhere on $\partial\Omega$, and that the Levi form of r is positive definite on $\mathscr{O}$:

$$\sum_{j,k} \frac{\partial^2 r}{\partial\zeta_j \partial\bar{\zeta}_k}(\zeta) w_j \bar{w}_k \geqq C|w|^2$$

for all $\zeta \in \mathscr{O}$ and all $w \in C^n$.

For $\zeta \in \partial\Omega$ set $\bar{D}r(\zeta) = \left(\frac{\partial r}{\partial\bar{\zeta}_1}, \ldots, \frac{\partial r}{\partial\bar{\zeta}_n}\right)$. Then $\bar{D}r(\zeta)$ is normal to $\partial\Omega$. If "$\langle\ ,\ \rangle$" denotes the Hermitian inner product on C^n let ρ be the pseudo-metric defined on $\partial\Omega$ by

$$\rho(\zeta,\eta) = |\langle\zeta - \eta, \bar{D}r(\zeta)\rangle| + |\langle\zeta - \eta, \bar{D}r(\eta)\rangle| + |\zeta - \eta|^2$$

and for $\zeta \in \partial\Omega$ and $\delta > 0$ define the non-isotropic ball

$$Q(\zeta,\delta) = \{\eta : \eta \in \partial\Omega \ \text{ and } \ \rho(\zeta,\eta) < \delta\}.$$

See [St chapter II] for a discussion of the properties of ρ and the collection of balls Q.

Since $|\nabla r|$ is non-vanishing on $\partial\Omega$ it follows that there are finitely many sets $\mathscr{N}$ whose union contains a neighborhood of $\partial\Omega$ on each of which exists a $\mathscr{C}^\infty$ projection π from $\mathscr{N}$ to $\partial\Omega$ and a diffemorphism of $\mathscr{N}$ onto $(\mathscr{N} \cap \partial\Omega) \times (-\varepsilon_0, \varepsilon_0)$. Thus each point ζ in $\mathscr{N}$ may be identified with the pair $(\pi(\zeta), r(\zeta))$.

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The pseudo-metric defined on $\partial\Omega$ and the local product structure given above on a neighborhood of $\partial\Omega$ give the "homogeneous space" structure used in the study of holomorphic functions defined on Ω. Recall that for each $\eta \in \mathcal{N} \cap \partial\Omega$ we may define the approach region

$$\Gamma_t(\eta) = \{\zeta \in \mathcal{N} \cap \Omega : |\rho(\pi(\zeta), \eta)| \leqq t|r(\zeta)|\},$$

where $t > 0$. In what follows we will assume that a single t has been chosen and suppress the dependence of the approach region on the parameter t.

If G is a holomorphic function on Ω and $0 < p < \infty$, we will say that G belongs to the Hardy space $H^p = H^p(\Omega)$ if the admissible maximal function

$$MF(\eta) = \sup_{z \in \Gamma(\eta)} |G(z)|$$

defined for each $\eta \in \partial\Omega$ belongs to $L^p(d\sigma)$. Here $d\sigma$ denotes the "surface area measure" on $\partial\Omega$.

It is well known that the dual space of the Hardy space H^1 is the space BMOA of functions F in H^2 whose boundary values on $\partial\Omega$ satisfy the bounded mean oscillation condition with respect to the non-isotropic metric $\rho(\zeta, \eta)$. Precisely, an H^2 function F belongs to BMOA if there is a constant C such that

$$A_Q(|F - A_Q F|) \leqq C,$$

where A_Q denotes the average over a non-isotropic ball $Q = \{\eta : \rho(\eta, \zeta) < \delta\}$ given by the formula

$$A_Q F = \frac{1}{\sigma(Q)} \int_Q F \, d\sigma.$$

The quantity

$$\|F\|_* = \|F\|_{H^2} + \sup_Q A_Q(|F - A_Q F|)$$

defines a norm on BMOA.

The duality between H^1 and BMOA is achieved by the pairing

$$\langle G, F \rangle = \int_{\partial\Omega} G\bar{F} \, d\sigma,$$

which is well defined whenever $G \in H^2$ and $F \in$ BMOA, and satisfies the inequality

$$|\langle G, F \rangle| \leqq C\|G\|_{H^1} \|F\|_*.$$

Furthermore, an H^2 function F belongs to BMOA if and only there is a constant C such that the inequality

$$|\langle G, F\rangle| \leqq C\|G\|_{H^1}$$

holds for all $G \in H^2$.

One way to prove the duality between BMOA and H^1 involves the alternative charcterization of BMOA in terms of Carleson measure. Recall that a positive measure μ on Ω is called a Carleson measure if there is a constant C such that

$$\mu(\hat{Q}) \leqq C\sigma(Q),$$

for every non-isotropic ball Q, where if $E \subset \partial\Omega$, $\hat{E}$ is the "tent" over E given by

$$\hat{E} = \{w \in \Omega : Q(\pi(w), |r(w)|) \subset E.$$

The following result is a version of the theorem of Fefferman and Stein appropriate for the present context.

THEOREM A. *The following conditions on a function F in $H^2(\Omega)$ are equivalent:*
(i) *F belongs to* BMOA.
(ii) *The measure $|r\nabla F|^2 \frac{dm}{|r|}$ is a Carleson measure.*
(iii) *There is a constant C such that $|\langle G, F\rangle| \leqq C\|G\|_{H^1}$ for all $G \in H^2$.*

Here, dm denotes Lebesgue measure on C^n.

A proof of Theorem A might proceed along the following lines. First, If F is in BMOA then by using the integral representation for F given by formula (4) below (see [KSt]) and the argument given in [T, Chapter XV, Prop. 1.2] one can prove that $|r\nabla F|^2 \frac{dm}{|r|}$ is a Carleson measure. Next, if $|r\nabla F|^2 \frac{dm}{|r|}$ is a Carleson measure then one can use a weighted Bergman kernel to express F together with the characterization of H^1 in terms of tent spaces to show that $\langle G, F\rangle$ defines a bounded linear functional on H^1; in fact we employ this type of argument in Lemma 3 below. Finally, if $\langle G, F\rangle$ is known to define a bounded linear functional on H^1, then it can be seen that the bounded mean oscillation condition holds by integrating F against a real H^1 atom a (defined in terms of the metric ρ; see [CW]), since if P_+ denotes the Szegö projection, we have the estimate $\|P_+ a\|_{H^1} \leqq C$ for a constant C independent of a. Alternatively, it follows from the fact that H^1 is a closed subspace of $L^1(d\sigma)$ (see the corollary on page 40 of [St]) that F is the Szegö projection of a bounded function. From this the bounded mean oscillation condition follows from an argument similar to the one given in [G], Chapter VI, Theorem 1.5.

It seems to be well known that in condition (ii) the vector expression ∇F may be replaced by the (scalar) normal derivative

$$NF(z) = \sum_{j=1}^{n} \frac{\partial r}{\partial \bar{z}_j} \frac{\partial F}{\partial z_j}.$$

See [J] for further refinements on the unit ball. Our purpose here is to offer a characterization of BMOA in terms of complex tangential derivatives of an analytic function. For t real let $\Omega_t = \{z : r(z) < t\}$ and suppose $r(\zeta) = t$. Then $\bar{D}r(\zeta) = \left(\frac{\partial r}{\partial \zeta_1}, \ldots, \frac{\partial r}{\partial \zeta_n}\right)$ is normal to $\partial\Omega_t$ at ζ. Recall that if $\mathbf{T}_\zeta(\partial\Omega_t)$ denotes the (complexified) tangent space of $\partial\Omega_t$ at ζ then

$$\mathbf{T}_\zeta(\partial\Omega_t) = C^{n-1}(\zeta) \oplus \overline{C^{n-1}}(\zeta) \oplus R(i\bar{D}r(\zeta)),$$

where $C^{n-1}(\zeta)$ denotes the orthogonal complement of the complex span of $\{\bar{D}r(\zeta)\}$ in C^n and $R(i\bar{D}r(\zeta))$ denotes the real span of $i\bar{D}r(\zeta)$.

Suppose then that v is a C^∞ mapping from $\mathcal{O}$ to C^n. Write

$$v(\zeta) = (v_1(\zeta), \ldots, v_n(\zeta)).$$

Then v determines the vector fields

$$T_v = \sum_{j=1}^{n} v_j(\zeta) \frac{\partial}{\partial \zeta_j},$$

and

$$\bar{T}_v = \sum_{j=1}^{n} \bar{v}_j(\zeta) \frac{\partial}{\partial \bar{\zeta}_j}.$$

If "v is complex tangential" to $\partial\Omega_t$, i.e.

$$\langle v(\zeta), \bar{D}r(\zeta) \rangle = 0$$

for all ζ, then it follows that

$$v(\zeta) \in C^{n-1}(\zeta), \quad \bar{v}(\zeta) \in \overline{C^{n-1}}(\zeta),$$

for all ζ on $\partial\Omega_t$ and the restriction of both T_v and $\bar{T}_v$ to $\partial\Omega_t$ define vector fields on the manifold $\partial\Omega_t$. It is natural to say therefore that the vector fields T_v and $\bar{T}_v$ are *complex tangential* on Ω. Any such complex tangential vector field can be written as a linear combination (with $\mathscr{C}^\infty$ coefficients) of the vector fields

$$T_{i,j} = \frac{\partial r}{\partial \zeta_j} \frac{\partial}{\partial \zeta_i} - \frac{\partial r}{\partial \zeta_i} \frac{\partial}{\partial \zeta_j}$$

and

$$\bar{T}_{i,j} = \frac{\partial r}{\partial \bar{\zeta}_j} \frac{\partial}{\partial \bar{\zeta}_i} - \frac{\partial r}{\partial \bar{\zeta}_i} \frac{\partial}{\partial \bar{\zeta}_j},$$

where $1 \leqq i,j \leqq n$. Working on the unit ball Ahern and Bruna [AB] gave conditions on holomorphic functions F in terms of the "analytic" tangential derivatives $T_1 T_2 \dots T_k F$, that is, each T_i is chosen among the collection of $T_{j,k}$ and not the $\bar{T}_{i,j}$, which are necessary and sufficient that F belong to a Hardy-Sobolev space. For strictly pseudoconvex domains this was also done in [C1] and [Gr]; see also [C2]. Based on the results of those authors one would expect that in a "complex tangential characterization" of BMOA the expression ∇F in Theorem A would be replaced by some sort of second order tangential derivative. This turns out to be correct. We proceed now to formulate versions of the theorems of [C1] and [C2] for the space BMOA.

Suppose that there is finite covering of $\partial\Omega$ by open balls $\{\mathcal{O}_\nu\}$ and that for each ν there is a $\mathscr{C}^\infty$ mapping v^ν of $\mathcal{O}_\nu$ to C^n satisfying the following conditions:

$$|v^\nu(\zeta)| > 0$$

for all $\zeta \in \mathcal{O}_\nu$ and

$$\langle v^\nu(\zeta), \bar{D}r(\zeta)\rangle = 0$$

for all $\zeta \in \mathcal{O}_\nu$.

Let each v^ν determine "analytic" complex tangential vector field $T_\nu = T_{v^\nu}$. Our main result is the following theorem.

THEOREM 1. *The following conditions on a function F belonging to $H^2(\Omega)$ are equivalent.*

i) *F belongs to* BMOA.

ii) *For some positive integer k the measure $|r^{k/2} T_\nu^j F|^2 \dfrac{dm}{|r|}$ is a Carleson measure for each ν and all $j \leqq k$.*

iii) *For all positive integers k the measure $|r^{k/2} T_\nu^k F|^2 \dfrac{dm}{|r|}$ is a Carleson measure for each ν.*

We point out that, as in [C1] and [C2], locally, we work with only one vector field T_ν as opposed to finitely many vector fields $T_{j,k}$ which locally span the complex tangent space at a point on $\partial\Omega$. Thus this result, while similar in spirit to work of Ahern and Bruna [AB] on the unit ball and Grellier [Gr] on a strictly pseudoconvex domain, is also different than the work of those authors and probably cannot be obtained from the methods used by them.

Theorem 1 will be proved by establishing that condition (i) implies condition (iii) and that condition (ii) implies condition (i); this will suffice since it is trivial that condition (iii) implies condition (ii).

The implication that condition (ii) implies condition (i) in the theorem above actually follows from a stronger result which we now prepare to state.

For $s > -1$ let $B_s(w,z)$ be the Bergman kernel which gives the orthogonal projection of the space $L^2(dm_s)$ onto space $H \cap L^2(dm_s)$. Here $dm_s = |r|^s\, dm$, $L^2(dm_s)$ is the space of functions f defined on Ω satisfying

$$\|f\|_{2,s}^2 = \int_\Omega |f|^2\, dm_s < \infty,$$

and $H \cap L^2(dm_s)$ is the Bergman space of holomorphic functions in $L^2(dm_s)$. Thus if $F \in H \cap L^2(dm_s)$ then

$$F(z) = \int_\Omega F(w) B_s(w,z)\, dm_s(w)$$

for all $z \in \Omega$. If $f \in L^1(dm_s)$ we define

$$P_s f(z) = \int_\Omega f(w) B_s(w,z)\, dm_s(w).$$

THEOREM 2. *Let T_ν be vector fields as described above. Suppose that f is a (not necessarily holomorphic) function defined on Ω satisfying the condition that there is a positive integer k such that $|r^{k/2}\, T_\nu^j f|^2 \dfrac{dm}{|r|}$ is a Carleson measure for each ν and all $j \leqq k$. Let $s + 1 - k/2 > 0$. Then the weighted Bergman projection $P_s f$ belongs to* BMOA.

It is clear that Theorem 2 establishes the implication (ii) implies (i) in Theorem 1.

We now develop the machinery necessary for our proofs of Theorems 1 and 2.

In the study of holomorphic functions and the Bergman projection on the unit ball, the kernel $G(w,z) = 1 - \langle z, w\rangle$ plays an important role. Note that G is holomorphic in the second variable and anti-holomorphic in the first variable. There is an analogue of this kernel which is appropriate for the context of strictly pseudoconvex domains. This kernel is the modification of the kernel constructed by Kerzman and Stein in [KSt] discussed in [C1]. It can be used as the kernel of [KSt] is used to get a fairly explicit formula for either the Szegö kernel or a weighted Bergman kernel; see [KSt], [L], [R], and [C1] and [C2]. We summarize its properties in the following proposition.

PROPOSITION 1. *Let Ω be a bounded strictly pseudoconvex domain in C^n. Suppose that m is a positive integer. Then there is a kernel $G(w,z) = G_m(w,z)$ which is $\mathscr{C}^\infty$ on $C^n \times C^n$ and a finite collection of open balls $\mathcal{O}_\nu$ such that $\partial\Omega \subset \mathcal{O}_\nu$ such that the following conditions hold:*

$$G(w, w) = -r(w), \quad w \in \cup \mathcal{O}_v;$$

$$\frac{\partial G}{\partial w_j}(w, w) = 0 \quad \text{and} \quad \frac{\partial G}{\partial \bar{w}_j}(w, w) = -\frac{\partial r}{\partial \bar{w}_j}(w), \quad j = 1, \dots, n \quad w \in \cup \mathcal{O}_v;$$

$$\frac{\partial G}{\partial \bar{z}_j}(z, z) = 0, \quad \text{and} \quad \frac{\partial G}{\partial z_j}(z, z) = -\frac{\partial r}{\partial z_j}(z), \quad j = 1, \dots, n \quad z \in \cup \mathcal{O}_v;$$

$$|G(w, z)| \geqq C(|r(w)| + |r(z)| + |w - z|^2), \quad z, w \in \bar{\Omega} \cap \mathcal{O}_v;$$

and

$$\frac{\partial G}{\partial \bar{z}_j}(w, z) = 0, \quad w, z \in \mathcal{O}_v.$$

Furthermore, for any $j = 1, \dots, n$

$$\frac{\partial G_m(w, z)}{\partial w_j} = \sum_{|I| = m} (w - z)^I e_I(w, z), \tag{1}$$

where I *denotes a multi-index of length* m, $(w - z)^I$ *is a polynomial in* w *and* z *vanishing to order* m *when* $w = z$ *and each* e_I *is a* C^∞ *function of* z *and* w.

REMARK. The point of condition (1) is that the kernel G is "almost" anti-holomorphic in the first variable. Notice also that we have used the notation $\mathcal{O}_v$ to denote two different open coverings of $\partial\Omega$; by taking a refinement of the two coverings we will be able to assume that the two coverings are in fact the same.

We will need to exploit the relation between the Hardy spaces and the "tent spaces" studied by Coifman, Meyer and Stein. If f is a function on Ω, we will say that f belongs to the tent space $T_2^p = T_2^p(\Omega)$ if the admissible area function

$$Af(\eta) = \left(\int_{\Gamma(\eta)} |f(z)|^2 \frac{dm(z)}{|r(z)|^{n+1}} \right)^{1/2},$$

defined for each $\eta \in \partial\Omega$ belongs to $L^p(d\sigma)$. Tent spaces were defined and studied in the context of the upper half space R_+^{n+1} by Coifman, Meyer, and Stein in [CMSt]. Some of their results were generalized to context including the present one by Ahern and Nagel in [AN]. It can be shown that most of the results in [CMSt] have analogues in the present setting and we will comment on these when the need arises.

We will need a variant of a theorem which is proved in [C2]; see [C2], Theorem 2. The proof of that result also establishes the following proposition.

PROPOSITION 2. *Suppose* $0 < p < \infty$. *Let* $K(\zeta, z)$ *be a kernel of the form*

$$K(\zeta, z) = \frac{|r(z)|^a |r(\zeta)|^b H(\zeta, z)}{G(\zeta, z)^{n+1+a+b+l/2}}$$

or

$$K(\zeta, z) = \frac{|r(z)|^a |r(\zeta)|^b H(\zeta, z)}{G(z, \zeta)^{n+1+a+b+l/2}},$$

where l is a non-negative integer, $H(\zeta, z)$ a $\mathscr{C}^\infty$ function which satisfies the condition that

$$|H(\zeta, z)| \leqq C|z - \zeta|^l,$$

and a and b are real numbers satisfying the conditions $a > 0$, $b > -1$, and $(n + b + 1)p - n > 0$. Then the operator

$$Kf(z) = \int_\Omega f(\zeta) K(\zeta, z)\, dm(\zeta)$$

maps the space $T_2^p(\Omega)$ to itself.

We now begin the proof that condition (i) of Theorem 1 implies condition (iii). We need a way to construct H^1 functions from functions in the tent space $T_2^1(\Omega)$.

LEMMA 1. *Let $A \in T_2^1(\Omega)$ and be supported on a compact of Ω. Let k be a positive integer, and suppose that $J(w, z)$ is a kernel of the form*

$$J(w, z) = |r(w)|^{k/2-1}\, \overline{T_w^k(G(z, w)^{-n})},$$

where T_w denotes a complex tangenital derivative acting on the w variable. Then if the integer m chosen to construct the kernel G is sufficiently large and

$$JA(z) = \int_\Omega A(w) J(w, z)\, dm(w)$$

it follows that
$$\|P_+(JA)^*\|_{H^1} \leqq C\|A\|_{T_2^1},$$

where P_+ denotes the Szegö projection of $L^2(d\sigma)$ onto H^2, "" denotes the restriction to $\partial\Omega$ of a function defined on $\bar{\Omega}$, and C is a constant independent of A.*

PROOF. Using the fact that $T_w G(z, w)$ vanishes to the first order when $w = z$ one can verify that

$$T_w^k(G(z, w)^{-n}) = \sum_{j=0}^{k} \frac{h_j(z, w)}{G(z, w)^{n+j}},$$

where h_j is $\mathscr{C}^\infty$ in both variables and, if $2j - k \geqq 0$, satisfies the inequality

$$|h_j(z, w)| \leqq C|z - w|^{2j-k}.$$

Furthermore, if D_z denotes any first order derivative with respect to the variable z then it may also be verified that

$$D_z(T_w^k G(z,w)^{-n}) = \sum_{j=-1}^{k} \frac{h_j(z,w)}{G(z,w)^{n+j+1}},$$

where h_j is $\mathscr{C}^\infty$ in both variables and if $2j-k$ is non-negative, satisfies the inequality

$$|h_j(z,w)| \leqq C|z-w|^{2j-k}.$$

We may therefore apply Proposition 2 to deduce that the operator which takes A to $r(z)D_z JA(z)$ is bounded on $T_2^1(\Omega)$. Now use the fact that

$$\left|\frac{\partial}{\partial \bar{z}_j} \overline{G(z,w)}\right| \leqq C|z-w|^m$$

for $j = 1, \dots, n$ for all z in a neighborhood of $\bar{\Omega}$ to deduce that if $\mathscr{D}$ is a differential operator of order k, then if m is chosen to be sufficiently large, there is a fixed strictly pseudoconvex neighborhood $\mathcal{O}$ of $\bar{\Omega}$ (independent of A) and constants $C = C(\mathscr{D})$ (also independent of A) on which the coefficients v_j of the $\bar{\partial}$ closed (0,1) form $\bar{\partial} JA$ satisfy the estimates

$$\sup_{\bar{\mathcal{O}}} |\mathscr{D}v(z)| \leqq C\|A\|_{T_2^1}.$$

(Notice that $\bar{\partial}_z$ and T_w^k commute.) Use the formula described in [K, Theorem 9.1.2] we may solve the equation $\bar{\partial}u = \bar{\partial}JA$ (on a slightly larger domain containing the closure of $\mathcal{O}$) with a function u which satisfies the same estimates as the coefficients v_j above. We may therefore use the smoothness of u and the tent space characterization of $H^1(\Omega)$ (see [Gr]) to conclude that the holomorphic function $JA - u$ is in $H^1(\Omega)$ and there is a constant C independent of A such that

$$\|JA - u\|_{H^1} \leqq C\|A\|_{T_2^1}.$$

Since the Szegö projection of u^* is also smooth, the proof is completed by using the last estimate and the observation that $P_+(JA)^* = P_+(JA-u)^* + P_+u^* = (JA-u)^* + P_+u^*$.

We now show that condition (iii) in Theorem 1 follows from condition (i). Suppose that $f \in \mathrm{BMOA}$. Let k be a fixed positive integer and for $A \in T_2^1$ which is compactly supported in Ω let P_+JA be the H^1 function which is given by Lemma 1 above. It follows that

$$|\langle P_+JA, f\rangle| = |\langle (JA)^*, f\rangle| \leqq C\|A\|_{T_2^1}, \tag{2}$$

where C does not depend on A. Use Fubini's theorem to calculate that

$$\langle (JA)^*, f\rangle = \int_\Omega A(w)|r(w)|^{k/2-1}\overline{T_w^k(G * f(w))}\,dm(w), \tag{3}$$

where

$$G * f(w) = \int_{\partial\Omega} f(z)G(z,w)^{-n}\,d\sigma(z).$$

We now use (2) and (3) to show that

$$|r(w)^{k/2}\,T_w^k G * f(w)|^2 \frac{dm}{|r(w)|}$$

is a Carleson measure. First, in (3) let A range over all (compactly supported) tent space atoms A supported in a fixed "tent" $\hat{Q}$, that is

$$\int_{\hat{Q}} |A|^2 \frac{dm}{|r|} \leqq \sigma(Q)^{-n};$$

see [CMSt]. By (2) the right hand side of (3) remains uniformly bounded by a constant independent of A. Duality now gives that

$$|r(w)^{k/2}\,T_w^k\, G * f(w)|^2 \frac{dm}{|r(w)|}$$

is a Carleson measure. The proof will be complete if we show that $G * f$ may be replaced by f in the last statement.

It follows from the work of Kerzman and Stein [KSt] that there is a $\mathscr{C}^\infty$ function $\Psi(z,w)$ such that

$$f(w) = \int_{\partial\Omega} f(z)\frac{\Psi(z,w)}{G(z,w)^n}\,d\sigma(z) \tag{4}$$

and therefore

$$f(w) = \Psi(w,w)G * f(w) + \int_{\partial\Omega} f(z)(\Psi(z,w) - \Psi(w,w))G(z,w)^{-n})\,d\sigma(z).$$

Since $f \in \mathrm{BMOA}$, $f \in L^p(d\sigma)$ for all $0 < p < \infty$, and $\|f\|_{L^p(d\sigma)} \leqq C(p)\,\|f\|_*$ for a constant $C(p)$ depending only on p. Estimate that

$$\begin{aligned}|G * f(w)| &\leqq \left(\int_{\partial\Omega} |f|^p\,d\sigma\right)^{1/p}\left(\int_{\partial\Omega} |G(z,w)|^{-np'}\,d\sigma(z)\right)^{1/p'}\\ &\leqq C(p)\,\|f\|_*\,|r(w)|^{-(np'-n)/p'} = C(p)\,\|f\|_*\,|r(w)|^{-n/p},\end{aligned}$$

where p' is conjugate to p. If we take p sufficiently large, it is easy to see from this last inequality that $|G * f(w)|^2 \, dm(w)$ is a Carleson measure. It follows easily now that

$$|r(w)^{(k-1)/2} T_w^k(\Psi(w,w) G * f(w)|^2$$

is also a Carleson measure. To complete the proof we need only show that if

$$U(w) = r^{(k-1)/2}(w) T_w^k \int_{\partial\Omega} f(z)(\Psi(z,w) - \Psi(w,w)) G(z,w)^{-n}) \, d\sigma(z),$$

then $|U(w)|^2 \, dm(w)$ is also a Carleson measure. But this follows from the argument given above and the fact that the kernel

$$r^{(k-1)/2}(w) \, T_w^k(\Psi(z,w) - \Psi(w,w)) G(z,w)^{-n})$$

has essentially the same singularity as the kernel G^{-n}. This completes the proof.

We now begin the proof of Theorem 2 (and therefore the implication that condition (ii) implies condition (i) in Theorem 1.) If $g \in H^1$ and T denotes an "analytic" complex tangential derivative, then it follows from the work of [Gr] that $r^{k/2}(z) T^k g(z)$ belongs to the tent space T_2^1. This is obviously the idea behind the following lemma.

LEMMA 2. *Let k be a positive integer and suppose $s + 1 - k/2 > 0$. Suppose that $\Psi(w,z)$ and $h(w,z)$ are $\mathscr{C}^\infty$ functions. Suppose further that $h(w,z)$ is anti-holomorphic in the variable z and also satisfies the condition that*

$$|h(w,z)| \leqq C|w - z|^k.$$

Then the linear operator defined by either of the formulas

$$Lg(w) = |r(w)|^{s+1-k/2} \int_{\partial\Omega} g(z) \frac{\Psi(w,z) h(w,z)}{G(w,z)^{n+1+s}} \, d\sigma(z)$$

or

$$Lg(w) = |r(w)|^{s+1-k/2} \int_{\partial\Omega} g(z) \frac{\Psi(w,z) h(w,z)}{G(z,w)^{n+1+s}} \, d\sigma(z)$$

is a bounded mapping from $H^1(\Omega)$ to $T_2^1(\Omega)$.

PROOF. The argument we now outline will apply to either of the operators defined above. Suppose then that Lg is given by the second formula above. By writing $d\sigma(z) = \eta$ where η is an $(n-1, n)$ form we may use Stoke's formula to calculate that

$$|r(w)|^{-s-1+k/2} Lg(w) = \int_\Omega \partial\left(g(z) \frac{\Psi(w,z) h(w,z)}{G(w,z)^{n+1+s}} \wedge \eta \right).$$

Since differentiating the kernel $\bar{G}$ in the z variable results in a non-singular kernel, it follows that we may confine our attention to expressions of the form

$$\int_\Omega Dg(z)\frac{\Psi(w,z)h(w,z)}{G(w,z)^{n+1+s}}\,dm(z) \tag{5}$$

where Dg denotes a first order differential operator (with $\mathscr{C}^\infty$ coefficients) operating on g. Since g is holomorphic, it follows that

$$Dg(z) = a_1(z)g(z) + D_\tau g(z) + D_n g(z)$$

where a_1 is $\mathscr{C}^\infty$, D_τ is an "analytic" complex tangential derivative (with $\mathscr{C}^\infty$ coefficients) and D_n is complex normal, that is, of the form bN where b is $\mathscr{C}^\infty$ and $Ng = \sum_{j=1}^n \frac{\partial r}{\partial \bar{z}_j}\frac{\partial g}{\partial z_j}$. Since Ω is strictly pseudoconvex we may write $D_n g(z)$ as a linear combination of expression of the form $D_\tau g(z)$ and $\bar{D}_\tau D_\tau g(z)$. Integration by parts allows us to rewrite (5) as a sum of terms of the form

$$\int_\Omega (a_2(z) + D_\tau)g(z)(a_1(z) + \bar{D}_\tau)\left(\frac{\Psi(w,z)h(w,z)}{G(w,z)^{n+1+s}}\right)dm(z), \tag{6}$$

where a_2 is $\mathscr{C}^\infty$. Since both g and $r^{1/2}D_\tau g \in T_2^1$ (see [Gr]) we see that (6) is of the form

$$\int_\Omega G(z)|r(z)|^{-1/2}(a(z) + \bar{D}_\tau)\left(\frac{\Psi(w,z)h(w,z)}{G(w,z)^{n+1+s}}\right)dm(z),$$

where $G \in T_2^1$. If we multiply this last expression by $|r(w)|^{s+1-k/2}$ we obtain a formula for a linear operator which is bounded on T_2^1 by Proposition 2. This completes the proof.

LEMMA 3. *Let f satisfy the condition that there is a positive integer k such that $|r^{k/2}T_v^j f|^2 \frac{dm}{|r|}$ is a Carleson measure for each v and all $j \leqq k$. Suppose $s > -1$ and $K(w,z)$ is a kernel of the form*

$$K(w,z) = \frac{|r(w)|^s\Psi(w,z)}{G(w,z)^{n+1+s}}$$

or

$$K(w,z) = \frac{|r(w)|^s\Psi(w,z)}{G(z,w)^{n+1+s}},$$

where Ψ is a $\mathscr{C}^\infty$ function on $C^n \times C^n$. Let

$$Kf(z) = \int_{\Omega} f(w)K(w,z)\,dm(w).$$

Then if the integer m chosen to construct the kernel G is sufficiently large, there is a constant C such that

$$\left|\int_{\partial\Omega} g(z)\,\overline{Kf(z)}\,d\sigma(z)\right| \leqq C\|g\|_{H^1}$$

for all $g \in H^1$.

PROOF. Without loss of generality, we may assume that f is supported on one of the sets $\mathcal{O}_\nu$; the general case is reduced to this one by a partition of unity argument. Let us denote the function v^ν by v and the vector field T_v by T. Arguing as in [C1] integrate by parts to deduce that

$$Kf(z) = \sum_{j=0}^{k} \int_{\Omega} T^j f(w) H_j(w,z) \frac{\langle w - z, v(w)\rangle^k |r(w)|^s}{G(w,z)^{n+1+s}}\,dm(w) + E(z),$$

where H_j is a $\mathscr{C}^\infty$ function and the "error" $E(z)$ is a function arising from terms in which the kernel G has been differentiated and therefore, in view of property (1), can be made as smooth as desired provided the integer m chosen to construct the kernel G is sufficiently large.

Use Fubini's theorem to write $\langle g, Kf - E\rangle$ as a sum of terms of the form

$$\int_{\Omega} \overline{T^j f(w)} r^{k/2}(w) K^* g(w) \frac{dm(w)}{r(w)},$$

where

$$K^* g(w) = |r(w)|^{s+1-k/2} \int_{\partial\Omega} g(z) \frac{\bar{H}_j(w,z)\langle v(w), w - z\rangle^k}{\overline{G(w,z)^{n+1+s}}}\,d\sigma(z).$$

By Lemma 2, $K^* g$ belongs to the tent space $T_2^1(\Omega)$ and

$$\|K^* g\|_{T_2^1} \leqq C\|g\|_{H^1}$$

for an absolute constant C independent of g. The lemma therefore follows from the expression obtained for $\langle g, Kf - E\rangle$ by the duality between the tent space T_2^1 and Carleson measures (see [CMSt]) and the smoothness of E. This completes the proof.

LEMMA 4. *Let K be a kernel as in Lemma 3 and suppose that f is a function satisfying the condition of Lemma 3 for some positive integer k. Then Kf also satisfies this condition for all positive integers.*

PROOF. Assume first that the $\mathscr{C}^\infty$ function Ψ appearing in the formula for the kernel K is identically 1 and denote such kernels by the notation K_0. We will prove the conclusion of the lemma for functions $K_0 f$. We first argue as in the proof of Lemma 1. From the construction of the kernel G it follows that

$$\bar{\partial}_z G(w,z) = 0, \quad z, w \in \mathcal{O}_v$$

and

$$\left| \frac{\partial}{\partial \bar{z}_j} \overline{G(z,w)} \right| \leqq C|z-w|^m$$

for $j = 1, \ldots, n$ for all z in a neighborhood of $\bar{\Omega}$. It follows that there is an open neighborhood $\mathcal{O}$ (independent of f) on which we may solve the equation $\bar{\partial} u = \bar{\partial} K_0 f$ with a function u which is smooth on $\mathcal{O}$. The function $u - K_0 f$ is therefore holomorphic in Ω and it follows from the conclusion of Lemma 3 and the smoothness of u that $u - K_0 f \in \mathrm{BMOA}$. The desired conclusion for $K_0 f$ follows now from the implication that (i) implies (ii) in Theorem 1 proven above and the smoothness of u.

To handle the general case, expand the kernel $\Psi(w,z)$ in a Taylor series in the variable w about $w = z$ to write

$$\Psi(w,z) = \sum_{l=0}^{M} p_l(z) q_l(w) + E_M(w,z),$$

where p_l and q_l are polynomials and E is a $\mathscr{C}^\infty$ function vanishing to order M on the diagonal. It is not hard to see that

$$Kf(z) = \sum_{l=0}^{M} p_l(z) K_0 f_l(z) + g(z),$$

where $g(z)$ may be as smooth as desired (provided M is taken large enough) and each f_l satisfies the same hypothesis as f. The conclusion follows now from the special case just established. This completes the proof.

We may now finish the proof of Theorem 2.

Using the representation of the weighted Bergman kernel B_s given in [C2] it follows that

$$P_s f(z) = \sum_{j=0}^{M} K A^j f + g$$

where g is as smooth as we like on $\bar{\Omega}$, K is a kernel of the type described in Lemma 3, A^0 is the identity, and for $j = 1, \ldots, M$, $A^j f = A^{j-1} A f$ where A is a sum of kernels of the type described in Lemma 3. By Lemma 4, each $A^j f$ satisfies the Carleson measure condition of that Lemma. From this it follows from Lemma 3 that

$$|\langle g, P_s f \rangle| \leqq C\|g\|_{H^1}$$

for a constant C independent of g. This completes the proof.

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