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PROJECTIVE C*-ALGEBRAS

TERRY A. LORING*

Abstract.

We show that if A and B are σ-unital, projective C^*-algebras (star-homomorphisms into quotients lift to star-homomorphisms) then $A \oplus B$ and $M_n(A)$ are projective. Corresponding results are proven for semiprojectivity. A corollary is that elements $h_1, \ldots, h_n$ in a quotient C/I satisfying

$$\begin{aligned} 0 \leqq h_i &\leqq 1, \\ h_i h_j &= h_j, \ \text{if } i \prec j, \\ \text{and } h_i h_j &= 0, \ \text{if } i \not\preceq j \text{ and } j \not\preceq i \end{aligned}$$

can be lifted to elements of C satisfying the same relations. Here $\preceq$ is any partial order on $\{1, \ldots, n\}$ such that $a \preceq c$ and $b \preceq c$ implies $a \preceq b$ or $b \preceq a$.

1. Projectivity.

A C^*-algebra is called *projective* if, given any star-homomorphism $\varphi: A \to B/I$, with I any ideal in any C^*-algebra B, there is a lift $\bar{\varphi}: A \to B$ so that $\pi \circ \bar{\varphi} = \varphi$. Projective C^*-algebras are contractable by [3, Proposition 3.1] and hence are rather rare.

An important example is $\oplus_1^n C_0(0, 1]$. This is universally generated by $h_1, \ldots, h_n$ satisfying

$$(1) \qquad 0 \leqq h_i \leqq 1$$

and

$$(2) \qquad h_i h_j = 0 \text{ for } i \neq j.$$

Akemann and Pedersen [1, Proposition 2.6] show that $h_i \in (B/I)_+$ satisfying (2) lift to $k_i \in B_+$ also satisfying (2). Adjusting the norms is easy (see [4, Theorem 4.5]) so $\oplus_1^n C_0(0, 1]$ is projective.

We use the following C^*-algebras frequently and so introduce the notation

$$M_n(0, 1] = C_0(0, 1] \otimes M_n.$$

Using the corona C^*-algebra version of the Kasparov technical theorem, due to

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Olsen and Pedersen [5, Theorem 3.7], we showed [4, Theorem 4.8] that $M_n(0, 1]$ is projective.

Suppose that A and B contain strictly positive elements h_1 and h_2. Then $h_1 \oplus 0$ and $0 \oplus h_2$ generate a quotient of $C_0(0, 1] \oplus C_0(0, 1]$. In $A \otimes M_n$, the elements $h_1 \otimes e_{ij}$ generate a quotient of $M_n(0, 1]$. These subalgebras act enough like matrix units that their liftings are the first steps in showing $A \oplus B$ and $A \otimes M_n$ are projective whenever A and B are.

We mostly will work in $\mathscr{C}_0$, the category at all C^*-algebras and star-homomorphisms. By $\mathscr{C}_1$, we mean the category of unital star-homomorphisms and C^*-algebras. By *projective in* $\mathscr{C}_0$, we mean the above definition, while *projective in* $\mathscr{C}_1$ is defined assuming A, B, φ and $\bar{\varphi}$ above are unital.

LEMMA 1.1. *A is projective in $\mathscr{C}_0$ if and only if $\tilde{A}$ is projective in $\mathscr{C}_1$.*

2. Direct Sums.

We shall restrict attention to C^*-algebras that are σ-unital. If A is σ-unital, it contains a strictly positive element h, so $A = \overline{hAh}$, the hereditary subalgebra generated by h.

LEMMA 2.1. *If $\pi : B \to C$ is a surjective homomorphism, k is a positive element of B and $\pi(k) = h$ then $\pi(\overline{kBk}) = \overline{hCh}$.*

THEOREM 2.2. *If A_1 and A_2 are σ-unital, projective C^*-algebras then $A_1 \oplus A_2$ is projective.*

PROOF. Suppose $\pi : B \to C$ is surjective and we are given

$$\varphi = \varphi_1 \oplus \varphi_2 : A_1 \oplus A_2 \to C.$$

Let $h_i \in A_i$ such that $0 \leqq h_i \leqq 1$ and $A_i = \overline{h_i A_i h_i}$. Since $\varphi_1(h_1)\varphi_2(h_2) = 0$ and $C_0(0, 1] \oplus C_0(0, 1]$ is projective (or use [1, Proposition 2.6]), there are positive lifts k_i of $\varphi_i(h_i)$ with $k_1 k_2 = 0$.

Let $B_i = \overline{k_i B k_i}$. Clearly $B_1 B_2 = 0$. Since

$$A_i = \overline{h_i(A_1 \oplus A_2)h_i},$$

we see that

$$\begin{aligned} \varphi_i(A_i) &\subseteq \overline{\varphi(h_i)C\varphi(h_i)} \\ &= \overline{\pi(k_i)C\pi(k_i)} \\ &= \pi(\overline{k_i B k_i}) \\ &= \pi(B_i). \end{aligned}$$

By hypothesis, φ_i lifts to a homomorphism $\bar{\varphi}_i: A_i \to B_i$. Thus $\bar{\varphi}_1 \oplus \bar{\varphi}_2$ is a lift of φ.

PROPOSITION 2.3. *If X is a finite tree then $C(X)$ is projective in $\mathscr{C}_1$.*

PROOF. The proof is by induction on the number of edges. If X has one edge then $C(X) = C[0,1]$ is well-known to be projective in $\mathscr{C}_1$.

For X with more than one edge, pick a middle vertex v. Then

$$X = X_1 \cup_v X_2$$

with each X_i a sub-tree with fewer edges. Since

$$C(X) \cong C_0(X\backslash\{v\})^{\sim},$$
$$C_0(X\backslash\{v\}) \cong C_0(X_1\backslash\{v\}) \oplus C_0(X_2\backslash\{v\})$$
$$\text{and } C(X_i) \cong C_0(X_i\backslash\{v\})^{\sim}$$

we are done by the induction hypothesis, Lemma 1.1 and Theorem 2.2.

For notation, we will use $C^*\langle - | - \rangle_0$ to indicate a universal C^*-algebra and $C^*\langle - | - \rangle_1$ to indicate a universal unital C^*-algebra.

PROPOSITION 2.4. *Suppose S is a nonempty finite set and $\preceq$ is a partial order on S satisfying the axiom*

$$(a \preceq c \text{ and } b \preceq c) \Rightarrow (a \preceq b \text{ or } b \preceq a) \tag{3}$$

Let $G(S) = \{h_e | e \in S\}$ denote a set of generators and $R(\preceq)$ denote the relations

$$0 \leqq h_e \leqq 1, \text{ for } e \in S,$$
$$h_e h_f = h_f \text{ if } e \prec f,$$
$$h_e h_f = 0 \quad \text{if } (e \npreceq f \text{ and } f \npreceq e).$$

Then $C^*\langle G(S) | R(\preceq)\rangle_1$ is isomorphic to $C(X)$, where X is finite tree.

PROOF. If S has only one element,

$$C^*\langle G(S) | R(\preceq)\rangle_1 \cong C^*\langle h | 0 \leqq h \leqq 1\rangle_1 \cong C[0,1].$$

For larger S, consider its minimal elements $a_1, \ldots, a_k$. Axiom (3) implies that the sets

$$S_i = \{s \in S | a_i \preceq s\}$$

are disjoint. Let $\preceq_i$ denote the restricted relations. If $a \in S_i$ and $b \in S_j$, $i \neq j$, then a and b are incomparable and thus $h_a h_b = 0$. Therefore,

$$C^*\langle G(S)|R(\preceq)\rangle_0 \cong \bigoplus_{i=1}^{k} C^*\langle G(S_i)|R(\preceq_i)\rangle_0.$$

If $k > 1$, we are done by induction and adding units. (Two trees attached at a point form a tree.)

If $k = 1$ then S has a minimum element a. Consider a new relation $\preceq'$ on S defined so that a is incomparable to $S \backslash \{a\}$ and, for $s, t \in S\backslash\{a\}$,

$$s \preceq' t \Leftrightarrow s \preceq t.$$

Then

$$C^*\langle G(S)|R(\preceq)\rangle_1 \cong C^*\langle G(S)|R(\preceq')\rangle_1,$$

the isomorphism sending h_s to h'_s for $s \neq a$, and h_a to $(1 - h'_a)$. With this order, S has at least two minimal elements so the induction proceeds as above.

COROLLARY 2.5. *Suppose A is a C^*-algebra containing an ideal I and $\preceq$ is a partial order on $\{1,\dots,n\}$ satisfying Axiom* (3). *Let $\pi: A \to A/I$ denote the quotient map. If $h_1,\dots,h_n \in A/I$ satisfy the relations $R(\preceq)$, then there exists $k_1,\dots,k_n \in A$ satisfying $R(\preceq)$ such that $\pi(k_i) = h_i$.*

The vacuous order on $\{1,\dots,n\}$ corresponds to the relations

$$h_i h_j = 0, \ i \neq j.$$

Akemann and Pedersen [1, Proposition 2.6] proved that positive operators satisfying these relations can be lifted. The other extreme case, the linear order, corresponds to the relations,

$$h_i h_{i+1} = h_{i+1}, \ i = 1,\dots,n-1,$$

which Olsen and Pedersen [5, Lemma 6.5] proved to be liftable. In this case, the universal C^*-algebra is $C[0,1]$, so a simple proof exists. However, most of the ideas for Theorem 2.2 and Proposition 2.3 came from examining Olsen and Pedersen's proof.

3. Matrix Algebras.

We now show that a homomorphic image of $M_n(0,1]$ is an acceptable substitute for a set of matrix units.

PROPOSITION 3.1. *If $\varphi: M_n(0,1] \to B$ is a star-homomorphism to a C^*-algebra B and if $h = \varphi(t \otimes e_{11})$, then there is an injective star-homomorphism*

$$\alpha: \overline{hBh} \otimes M_n \to B$$

with image the C^*-algebra generated by $\varphi(M_n(0,1])$ and $\overline{hBh}$. For $x \in B$,

$$\alpha(hxh \otimes e_{ij}) = \varphi(t^{1/2} \otimes e_{i1})h^{1/2}xh^{1/2}\varphi(t^{1/2} \otimes e_{1j}).$$

PROOF. For $x, y \in B$,

$$\begin{aligned}\alpha(hxh \otimes e_{ij})\alpha(hyh \otimes e_{jk}) &= \varphi(t^{1/2} \otimes e_{i1})h^{1/2}xh^{1/2}\varphi(t \otimes e_{11})h^{1/2}yh^{1/2}\varphi(t^{1/2} \otimes e_{1k}) \\ &= \varphi(t^{1/2} \otimes e_{i1})h^{1/2}xh^2yh^{1/2}\varphi(t^{1/2} \otimes e_{1k}) \\ &= \alpha(hxh^2yh \otimes e_{ik}).\end{aligned}$$

For $0 < \gamma \leqq 1/2$,

$$\varphi(t^{1/2} \otimes e_{i1}) = \varphi(t^{\gamma} \otimes e_{i1})h^{((1/2)-\gamma)}.$$

Therefore,

$$\|\alpha(hxh \otimes e_{ij})\| \leqq \|h^{1-\gamma}xh^{1-\gamma}\|.$$

Taking limits, as $\gamma \to 0$, shows that the given formula does define a bounded $*$-homomorphism that extends to all of $\overline{hBh} \otimes M_n$.

Clearly $\overline{hBh}$ is in the image of α and since

$$\begin{aligned}\alpha(h^3 \otimes e_{ij}) &= \varphi(t^{1/2} \otimes e_{i1})h^{1/2}hh^{1/2}\varphi(t^{1/2} \otimes e_{1j}) \\ &= \varphi(t^{1/2} \otimes e_{i1})\varphi(t^2 \otimes e_{11})\varphi(t^{1/2} \otimes e_{1j}) \\ &= \varphi(t^3 \otimes e_{ij}),\end{aligned}$$

the image of α also contains $\varphi(M_n(0, 1])$.

To show α is injective, it suffices to show that α restricted to $\overline{hBh} \otimes e_{11}$ is isometric. For $x \in B$,

$$\alpha(hxh \otimes e_{11}) = h^{1/2}(h^{1/2}xh^{1/2})h^{1/2} = hxh$$

and thus $\|\alpha(y \otimes e_{11})\| = \|y\|$ for any $y \in \overline{hBh}$.

The isomorphism α is natural in the following sense.

PROPOSITION 3.2. *Suppose that B_1 and B_2 are C^*-algebras and*

$$\beta : B_1 \to B_2,$$

$$\varphi_i : M_n(0, 1] \to B_i$$

are star-homomorphisms. Let $h_i = \varphi_i(t \otimes e_{11})$. If $\beta \circ \varphi_1 = \varphi_2$ then

$$\beta \circ \alpha_1 = \alpha_2 \circ (\beta_0 \otimes \mathrm{id}),$$

where $\beta_0 : \overline{h_1Bh_1} \to \overline{h_2B_2h_2}$ is the restriction of β and α_i is as in Poposition 3.1.

THEOREM 3.3. *If A is a σ-unital, projective C^*-algebra then $A \otimes M_n$ is projective.*

PROOF. Suppose that $\pi : B \to C$ is surjective and we are given

$$\varphi : A \otimes M_n \to C.$$

Let $h \in A$ be such that $0 \leqq h \leqq 1$ and $A = \overline{hAh}$. Define a homomorphism

$$\varphi_2 : M_n(0, 1] \to C$$

by $\varphi_2(f \otimes e_{ij}) = \varphi(f(h) \otimes e_{ij})$. Since $M_n(0, 1]$ is projective, by [4, Theorem 4.8], there exists a homomorphism

$$\varphi_1 : M_n(0, 1] \to B$$

with $\pi \circ \varphi_1 = \varphi_2$.

Let $h_i = \varphi_i(t \otimes e_{11})$. Notice that $h_2 = \varphi(h \otimes e_{11})$. By Proposition 3.1 we have monomorphisms

$$\alpha_1 : \overline{h_1 B h_1} \otimes M_n \to B,$$
$$\alpha_2 : \overline{h_2 C h_2} \otimes M_n \to C.$$

For any $a \in A$,

$$\begin{aligned} \varphi(hah \otimes e_{ij}) &= \varphi((h^{1/2} \otimes e_{i1})(h^{1/2} \otimes e_{11})(a \otimes e_{11})(h^{1/2} \otimes e_{11})(h^{1/2} \otimes e_{1j})) \\ &= \varphi_2(t^{1/2} \otimes e_{i1}) h_2^{1/2} \varphi(a \otimes e_{11}) h_2^{1/2} \varphi_2(t^{1/2} \otimes e_{1j}) \\ &= \alpha_2((h_2 \varphi(a \otimes e_{11}) h_2) \otimes e_{ij}). \end{aligned}$$

Therefore, if we define

$$\psi : A \to \overline{h_2 C h_2}$$

by $\psi(hah) = h_2 \varphi(a \otimes e_{11}) h_2$, we obtain a homomorphism such that $\varphi = \alpha_2 \circ (\psi \otimes \mathrm{id})$. (In fact, $\psi(a) = \varphi(a \otimes e_{11})$.) By Proposition 3.2, we have the following commutative diagram:

$$\begin{array}{ccccc} & & \overline{h_1 B h_1} \otimes M_n & \xrightarrow{\alpha_1} & B \\ & & \downarrow \pi_0 \otimes \mathrm{id} & & \downarrow \pi \\ A \otimes M_n & \xrightarrow{\psi \otimes \mathrm{id}} & \overline{h_2 C h_2} \otimes M_n & \xrightarrow{\alpha_2} & C \end{array}$$

Here π_0 is the restriction of π which, by Lemma 2.1, is a surjection of $\overline{h_1 B h_1}$ onto $\overline{h_2 C h_2}$. By assumption, ψ lifts and hence so do $\psi \otimes \mathrm{id}$ and φ.

COROLLARY 3.4. *If X is a finite tree, minus a single point, then $C_0(X) \otimes M_n$ is projective.*

4. Semiprojectivity.

The above techniques work well with Blackadar's version of semiprojectivity. We just need a replacement for Lemma 2.1 to deal with hereditary subalgebras of inductive limits.

LEMMA 4.1. *Suppose that $B_n \xrightarrow{\gamma_n} B_{n+1}$ is an inductive system with surjective connecting maps γ_n. Let $C = \varinjlim B_n$ and let $\gamma_{n,m}: B_n \to B_m$ and $\gamma_{n,\infty}: B_n \to C$ denote the canonical homomorphisms. Given $h_1 \in (B_1)_+$, if $h_n = \gamma_{1,n}(h_1)$ and $h = \gamma_{1,\infty}(h_1)$ then*

$$\varinjlim \overline{h_n B_n h_n} = \overline{hCh}.$$

PROOF. Consider the commutative diagram

$$\begin{array}{ccc} \overline{h_n B_n h_n} & \to & B_n \\ \downarrow & & \downarrow \\ \overline{h_{n+1} B_{n+1} h_{n+1}} & \to & B_{n+1} \end{array}$$

Notice that both vertical maps are surjective. The horizontal maps are injective, so it follows that the limit map is also injective. Surjectivity follows from

$$\gamma_n(\overline{h_n B_n h_n}) = hCh.$$

The proofs of Theorems 2.2 and 3.3 now modify easily to prove the following theorems. In these theorems, we are referring to semiprojectivity as defined in [2]. In particular, we answer the question raised in [2, Remark 2.20].

THEOREM 4.2. *If A_1 and A_2 are σ-unital, semiprojective C^*-algebras then $A_1 \oplus A_2$ is semiprojective.*

THEOREM 4.3. *If A is a σ-unital, semiprojective C^*-algebra then $A \otimes M_n$ is semiprojective.*

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DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF NEW MEXICO
ALBUQUERQUE, NM 87131
U.S.A.