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ON TYPE II AND TYPE III PRINCIPAL GRAPHS OF SUBFACTORS*

MASAKI IZUMI

Abstract.

For pairs of type III_λ $(0 < \lambda \leqq 1)$ factors with finite indices, we shall show a necessary and sufficient condition that the type II principal graph does not coincide with the type III principal graph, in terms of Longo's sectors.

1. Introduction.

According to recent works of T. Hamachi and H. Kosaki [HK1, 2, 3, K3], the index theory of type III factors splits into two theories, one is a "purely type III" index theory and the other is an "essentially type II" index theory. The former is closely related to the ergodic theory, and if the factors involved are approximately finite dimensional (AFD) it is completely described by extensions of ergodic flows. On the other hand the latter is reduced to the studies of inclusions of type II von Neumann algebras and automorphisms which globally preserve subalgebras [Ka1, 2, Li1, P].

In the "essentially type II" index theory (of type III factors), there appear two principal graphs, so-called the type II principal graph and the type III principal graph, and they play essential rôles in the analysis [KL]. In this paper, for type III_λ $(0 < \lambda \leqq 1)$ factors we shall characterize the condition that the above-mentioned two graphs do not coincide in terms of sectors, which were introduced by R. Longo [L2] and turns out to be an important tool for index theory [I1, 2, CK]. Namely the change of the principal graphs of the dual pair happens if and only if a modular automorphism appears as the descendant sector of the canonical endomorphism.

Basic facts on index theory can be found in [GHJ, K1]. For sector theory we shall freely use the notations in [I2].

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2. Preliminaries.

In this section we recall the constructions of the simultaneous crossed product decompositions in [HK1, KL, L1, Li1], and the relation between the canonical endomorphisms and Jones towers [L2, K2]. Throughout this paper we assume that von Neumann algebras have separable preduals.

2-1. *The simultaneous continuous decomposition.* Let $M \supset N$ be a pair of type III factors with finite index. Then we can construct the simultaneous continuous crossed product decomposition as follows. Let ψ be a faithful normal state on N and E: $M \to N$ the minimal conditional expectation of Hiai [H, KL, L4]. (In this paper we shall treat only minimal expectations.) We define a pair of type II_∞ von Neumann algebras $\tilde{M} \supset \tilde{N}$ as follows.

$$\tilde{M} = M \underset{\sigma^\varphi}{\times} \mathbf{R} \supset \tilde{N} = N \underset{\sigma^\psi}{\times} \mathbf{R}, \qquad \varphi \equiv \psi \cdot E.$$

Let τ be the usual trace on $\tilde{M}$ and θ the dual action of σ^φ, which coincides with the dual action of σ^ψ restricted to $\tilde{N}$. Note that the trace preserving expectation $\tilde{E}$: $\tilde{M} \to \tilde{N}$ is the restriction of $E \otimes \mathrm{id}$ to $\tilde{M}$, and consequently

$$\tilde{E} - \frac{1}{[M:N]_0}\mathrm{id}: \tilde{M} \to \tilde{M}$$

is completely positive. Thanks to Takesaki duality we can identify $M \supset N$ with

$$\tilde{M} \underset{\theta}{\times} \mathbf{R} \supset \tilde{N} \underset{\theta}{\times} \mathbf{R}.$$

Let

$$M_n \supset M_{n-1} \supset \cdots \supset M \supset N$$

be the tower associated with $M \supset N$ and E_i: $M_i \to M_{i-1}$ the minimal expectation. We define type II_∞ von Neumann algebras by

$$\tilde{M}_n \equiv M_n \underset{\sigma^{\varphi_n}}{\times} \mathbf{R}$$

where $\varphi_n = \varphi \cdot E_1 \cdot E_2 \cdots E_n$. Even though $\tilde{M}_n$ is not a factor in general, $\tilde{M}_n$ can be considered to be the basic extension of $\tilde{M}_{n-1}$ by $\tilde{M}_{n-2}$ in a certain sense [HK1, Ka1]. In fact $\tilde{M}_n$ is generated by $\tilde{M}_{n-1}$ and the Jones projection in $\tilde{M}_n \cap \tilde{M}'_{n-2}$. In the same way as above we can and do identify the tower for $M \supset N$ with

$$\tilde{M}_n \underset{\theta}{\times} \mathbf{R} \supset \tilde{M}_{n-1} \underset{\theta}{\times} \mathbf{R} \supset \cdots \supset \tilde{M} \underset{\theta}{\times} \mathbf{R} \supset \tilde{N} \underset{\theta}{\times} \mathbf{R}$$

where θ extends to the tower fixing the Jones projections. It is known that the following relation holds [KL].

$$(\tilde{M}_n \cap \tilde{N}')^\theta = M_n \cap N', \tag{2.1}$$

where we regard $\tilde{M}_n$ and $\tilde{N}$ as subalgebras of M_n and N.

If M and N are type III_1 factors then $\tilde{M} \supset \tilde{N}$ is an inclusion of type II_∞ factors with the minimum index $[M: N]_0$. We define the type II and the type III principal graphs of $M \supset N$ as follows.

Definition 2.1. Under the above condition, we call the principal graph of $\tilde{M} \supset \tilde{N}$ as the *type* II *principal graph of* $M \supset N$, and that of $M \supset N$ as *the type* III *principal graph of* $M \supset N$.

From the relation (2.1), in general two principal graphs do not coincide. Actually there exist examples whose two principal graphs are different [S].

2-2. *The simultaneous discrete decomposition.* Let M be a type III_λ $(0 < \lambda < 1)$ factor and N a type III_λ subfactor of M with finite index. We assume that there exist a pair of type II_∞ factors $\tilde{M} \supset \tilde{N}$ and $\theta \in \mathrm{Aut}(\tilde{M}, \tilde{N})$ $\mathrm{Aut}(\tilde{M}, \tilde{N}) \equiv \{\alpha \in \mathrm{Aut}(\tilde{M}); \alpha(\tilde{N}) = \tilde{N}\})$ scaling the trace τ on $\tilde{M}$ such that

$$(M \supset N) \cong (\tilde{M} \underset{\theta}{\times} \mathsf{Z} \supset \tilde{N} \underset{\theta}{\times} \mathsf{Z}).$$

We further assume that the minimal expectation $\tilde{E}$: $\tilde{M} \to \tilde{N}$ preserves τ. Of course we can not expect such a decomposition in general. But if the center of $M \cap N'$ is trivial, we can construct the decomposition in a similar way as in subsection 2-1 using the generalized trace and T action instead of ψ and R action [Li1]. Let

$$M_n \supset M_{n-1} \supset \cdots \supset M \supset N$$
$$\tilde{M}_n \supset \tilde{M}_{n-1} \supset \cdots \supset \tilde{M} \supset \tilde{N}$$

be the towers for $M \supset N$ and $\tilde{M} \supset \tilde{N}$. Then θ can be uniquely extended to the latter tower as before. As in the continuous case we can and do identify the former tower with

$$\tilde{M}_n \underset{\theta}{\times} \mathsf{Z} \supset \tilde{M}_{n-1} \underset{\theta}{\times} \mathsf{Z} \supset \cdots \supset \tilde{M} \underset{\theta}{\times} \mathsf{Z} \supset \tilde{N} \underset{\theta}{\times} \mathsf{Z}.$$

Then the following holds.

$$(\tilde{M}_n \cap \tilde{N}')^\theta = M_n \cap N'. \tag{2.2}$$

As in the case of type III_1 factors we define two principal graphs of $M \supset N$ as follows.

Definition 2.2. Under the above condition, we call the principal graph of

$\tilde{M} \supset \tilde{N}$ as *the type* II *principal graph of* $M \supset N$, and that of $M \supset N$ as *the type* III *principal graph.*

REMARK 2.3. Let $\tilde{M} \times_\theta \mathbf{Z} \supset \tilde{N} \times_\theta \mathbf{Z}$ be the discrete decomposition of $M \supset N$ and $\hat{M} \times_\theta \mathbf{R} \supset \hat{N} \times_\theta \mathbf{R}$ the continuous decomposition of $M \supset N$. As in [KL, Section 4.] we can see that $\hat{M}$ and $\hat{N}$ have common central decomposition, and the type II principal graph of $M \supset N$ coincides with the principal graph of the field of type II_∞ inclusions.

Examples whose two principal graphs are different are found in [IK, KL, Li1].

2-3 *Canonical endomorphisms.* Let $M \supset N$ be a pair of properly infinite von Neumann algebras. Then both of M and N can be standardly represented on a common Hilbert space H. Let J_M, J_N be the modular conjugations of M, N acting on H. We define an endomorphism γ: $M \to N$ by $\gamma(x) \equiv J_N J_M x J_M J_N$ for $x \in M$. We say that an endomorphism is canonical if it is constructed as above, even though we do not assume that γ comes from a common cyclic and separating vector. Note that γ is uniquely determined up to inner perturbation of N [L3]. We further assume that $M \supset N$ is an inclusion of factors with finite index and define M_i $(i > 0)$ by

$$M_{2n} \equiv \Gamma^{*n} M \Gamma^n, \qquad M_{2n-1} \equiv \Gamma^{*n} N \Gamma^n$$

where $\Gamma \equiv J_N J_M$. Then the modular conjugation of M_i can be easily computed. Using this we can see that

$$M_n \supset M_{n-1} \supset \cdots \supset M \supset N$$

is the tower for $M \supset N$, and consequently

$$M \supset N \supset \gamma(M) \supset \gamma(N) \supset \gamma^2(M) \supset \ldots$$

is the tunnel for $M \supset N$. Note that $\mathrm{Ad}(\Gamma)|_{M_i}$: $M_i \to M_{i-1}$ is a canonical endomorphism.

3. Main result.

Before stating the main result we shall prove two technical lemmas.

LEMMA 3.1. *Let $\tilde{M} \supset \tilde{N}$ be a pair of properly infinite von Neumann algebras and α: $G \to \mathrm{Aut}(\tilde{M}, \tilde{N})$ a continuous action of a locally compact group G. We define another pair of properly infinite von Neumann algebras $M \supset N$ by*

$$M \equiv \tilde{M} \underset{\alpha}{\times} G \supset N \equiv \tilde{N} \underset{\alpha}{\times} G.$$

Then for any canonical endomorphism $\tilde{\gamma}$: $\tilde{M} \to \tilde{N}$ there exists a canonical endomorphism γ: $M \to N$ which satisfies the following conditions.

(i) *Let* $\pi_\alpha\colon \tilde{M} \to M$ *be the usual embedding map. Then* $\gamma(\pi_\alpha(x)) = \pi_\alpha(\tilde{\gamma}(x))$ *for* $x \in \tilde{M}$.

(ii) *There exists a* α*-cocycle* $v_g, g \in G$ *in* $\tilde{N}$ *such that* $\gamma(\lambda_g) = \pi_\alpha(v_g)\lambda_g$, *where* λ_g *is the implementing unitary for* θ_g *in* M.

(iii) *If* G *is abelian and* $\hat{\alpha}\colon \hat{G} \to \mathrm{Aut}(M, N)$ *is the dual action of* α, *then* γ *commutes with* $\hat{\alpha}_\sigma$, $\sigma \in \hat{G}$.

PROOF. (i): We assume that $\tilde{M}$ and $\tilde{N}$ act on a Hilbert space H which is the standard Hilbert space for both of $\tilde{M}$ and $\tilde{N}$. Let $J_{\tilde{M}}$, $J_{\tilde{N}}$ be the modular conjugations of $\tilde{M}, \tilde{N}$, and $u_{\tilde{M}}, u_{\tilde{N}}$ the canonical implementations of $\alpha, \alpha|_{\tilde{N}}$. Thanks to [Ha], the modular conjugations J_M, J_N of M, N acting on $L^2(G, H)$ can be obtained as follows.

$$(J_M\xi)(s) = \Delta_G(s)^{-1/2} u_{\tilde{M}}(s)^* J_{\tilde{M}}\xi(s^{-1})$$

$$(J_N\xi)(s) = \Delta_G(s)^{-1/2} u_{\tilde{N}}(s)^* J_{\tilde{N}}\xi(s^{-1})$$

where $\xi \in L^2(G, H)$, $s \in G$ and Δ_G is the modular function of G. Let $\tilde{\Gamma} \equiv J_{\tilde{N}}J_{\tilde{M}}$, $\Gamma \equiv J_N J_M$, $\tilde{\gamma} \equiv \mathrm{Ad}(\tilde{\Gamma})|_{\tilde{M}}$ and $\gamma \equiv \mathrm{Ad}(\Gamma)|_M$. Then we have the following by using the above equations.

$$(\Gamma\xi)(s) = (J_N J_M\xi)(s) = u_{\tilde{N}}(s)^* \tilde{\Gamma} u_{\tilde{M}}(s)\xi(s).$$

Hence, for $x \in \tilde{M}$ we obtain

$$\begin{aligned}(\gamma(\pi_\alpha(x))\xi)(s) &= (\Gamma\pi_\alpha(x)\Gamma^*\xi)(s)\\ &= u_{\tilde{N}}(s)^*\tilde{\Gamma}u_{\tilde{M}}(s)\alpha_{s^{-1}}(x)u_{\tilde{M}}(s)^*\tilde{\Gamma}^*u_{\tilde{N}}(s)\xi(s)\\ &= u_{\tilde{N}}(s)^*\tilde{\gamma}(x)u_{\tilde{N}}(s)\xi(s)\\ &= (\pi_\alpha(\tilde{\gamma}(x))\xi)(s).\end{aligned}$$

(ii): Using the above expression of Γ we get

$$\begin{aligned}(\gamma(\lambda_g)\xi)(s) = (\Gamma\lambda_g\Gamma^*\xi)(s) &= u_{\tilde{N}}(s)^*\tilde{\Gamma}u_{\tilde{M}}(s)(\lambda_g\Gamma^*\xi)(s)\\ &= u_{\tilde{N}}(s)^*J_{\tilde{N}}J_{\tilde{M}}u_{\tilde{M}}(s)u_{\tilde{M}}(g^{-1}s)^*J_{\tilde{M}}J_{\tilde{N}}u_{\tilde{N}}(g^{-1}s)(\lambda_g\xi)(s)\\ &= u_{\tilde{N}}(s)^*J_{\tilde{N}}u_{\tilde{M}}(g)u_{\tilde{N}}(g)^*J_{\tilde{N}}u_{\tilde{N}}(s)(\lambda_g\xi)(s).\end{aligned}$$

Let $v_g \equiv J_{\tilde{N}}u_{\tilde{M}}(g)u_{\tilde{N}}(g)^*J_{\tilde{N}}$. Since $u_{\tilde{M}}(g)u_{\tilde{N}}(g)^*$ commutes with any element in $\tilde{N}$, we have $v_g \in \tilde{N}$ and so $\gamma(\lambda_g) = \pi_\alpha(v_g)\lambda_g$. It is easy to show that v_g is a α-cocycle. (iii) is a direct consequence of (i) and (ii).

The following lemma enables us to compare the descendant $M - M(M - N)$ sectors with the descendant $\tilde{M} - \tilde{M}(\tilde{M} - \tilde{N})$ sectors.

LEMMA 3.2. *Let* $M \supset N$ *be a pair of type* III *factors with finite index, and*

$$M = \tilde{M} \underset{\theta}{\times} \mathbf{R} \supset N = \tilde{N} \underset{\theta}{\times} \mathbf{R}$$

its simultaneous continuous crossed product decomposition explained in subsection 2-1. Then for any positive integer n, there exist canonical endomorphisms

$$\gamma: M \to N, \quad \tilde{\gamma}: \tilde{M} \to \tilde{N}$$

satisfying the following.

(i) $\pi_\theta(\tilde{\gamma}(x)) = \gamma(\pi_\theta(x))$ *for* $x \in \tilde{M}$.

(ii) γ *commutes with* $\sigma_t^{\hat{\tau}}$ $t \in \mathbf{R}$ *where* $\hat{\tau}$ *is the dual weight of the trace* τ *on* $\tilde{M}$.

(iii) *For any integer i,* $(0 < i \leqq n)$, θ *globally preserves* $\tilde{\gamma}^i(\tilde{M})$ *and* $\tilde{\gamma}^i(\tilde{N})$. *Moreover* θ *fixes the Jones projections associated with the following tunnel.*

$$M \supset N \supset \gamma(M) \supset \gamma(N) \supset \cdots \supset \gamma^n(N).$$

(iv) *The following formulas hold for* $0 \leqq i \leqq n$.

$$M \cap \gamma^i(M)' = \pi_\theta((\tilde{M} \cap \tilde{\gamma}^i(\tilde{M})')^\theta).$$

$$M \cap \gamma^i(N)' = \pi_\theta((\tilde{M} \cap \tilde{\gamma}^i(\tilde{N})')^\theta).$$

PROOF. We use the notations in the proof of Lemma 3.1. In the present case $G = \mathbf{R}$ and $\alpha = \theta$. Let $M_{2i} \equiv \Gamma^{*i} M \Gamma^i$ and $M_{2i-1} \equiv \Gamma^{*i} N \Gamma^i$ $(1 \leqq i \leqq n)$. Then as we saw in subsection 2-3,

$$M_{2n} \supset M_{2n-1} \supset \cdots \supset M \supset N$$

is the tower for $M \supset N$ and $\mathrm{Ad}(\Gamma)|_{M_{2n}}: M_{2n} \to M_{2n-1}$ is a canonical endomorphism. Let $E_j: M_j \to M_{j-1}$, $E: M \to N$ be the minimal expectations and $\varphi_j \equiv \hat{\tau} \cdot E_1 \cdot E_2 \cdot \cdots \cdot E_j$. From the uniqueness (up to isomorphisms) of the basic construction, we can identify the following two towers.

$$\tilde{M}_{2n} \supset \tilde{M}_{2n-1} \supset \cdots \supset \tilde{M}_1 \supset \tilde{M} \supset \tilde{N}.$$

$$(M_{2n})_{\varphi 2n} \supset (M_{2n-1})_{\varphi 2n-1} \supset \cdots \supset (M_1)_{\varphi 1} \supset \pi_\theta(\tilde{M}) \supset \pi_\theta(\tilde{N}).$$

where $\tilde{M}_j$ is as in subsection 2-1. Let $e_j \in M_j \cap M'_{j-2}$ be the Jones projection and $\lambda_t \in M$ the implementing unitary for θ_t. Then we have $e_i \in (M_j)_{\varphi j}$ because $E_{j-1} \cdot E_j$ is minimal [KL]. So if we define $\mathbf{R}$ action on $(M_j)_{\varphi j}$ by $\mathrm{Ad}(\lambda_t)|_{(M_j)_{\varphi_j}}$, $\mathrm{Ad}(\lambda_t)$ fixes e_j and hence the above identification extends to $\mathbf{R}$ actions. Suppose that the following formula holds.

$$(3.1) \qquad (M_{2i})_{\varphi 2i} = \Gamma^{*i} \pi_\theta(\tilde{M}) \Gamma^i.$$

$$(M_{2i-1})_{\varphi 2i-1} = \Gamma^{*i} \pi_\theta(\tilde{N}) \Gamma^i.$$

Then since $\mathrm{Ad}(\Gamma)|_{\pi_\theta(\tilde{M})}: \pi_\theta(\tilde{M}) \to \pi_\theta(\tilde{N})$ is a canonical endomorphism due to Lemma 3.1. (i), so is

$$\mathrm{Ad}(\Gamma)|_{(M_{2n})_{\varphi_{2n}}}: (M_{2n})_{\varphi_{2n}} \to (M_{2n-1})_{\varphi_{2n-1}}.$$

Hence we obtain (i), (iii), (iv) because $M \supset N$ is isomorphic to $M_{2n} \supset M_{2n-1}$, and (ii) by direct computation using $\Gamma\lambda_t\Gamma^* = \pi_\theta(v_t)\lambda_t$. We shall prove (3.1) in what follows.

First we show $(M_1)_{\varphi_1} = \Gamma^*\pi_\theta(\tilde{N})\Gamma$. Let $\tilde{M}_1 \equiv J_{\tilde{M}}\tilde{N}'J_{\tilde{M}} = \tilde{\Gamma}^*\tilde{N}\tilde{\Gamma}$. The extension of θ to $\tilde{M}_1$ which preserves the Jones projection in $\tilde{M}_1 \cap \tilde{N}'$ is obtained by $\mathrm{Ad}(u_{\tilde{M}}(t))|_{\tilde{M}_1}$ [Ka1, Li] and we also denote this by θ as in subsection 2-1. We show

$$\tilde{M}_1 \underset{\theta}{\times} \mathsf{R} \equiv \pi_\theta(\tilde{M}_1) \vee \{\lambda_t\} = M_1.$$

For $x \in \tilde{N}$,

$$\begin{aligned}(\Gamma^*\pi_\theta(x)\Gamma\xi)(s) &= u_{\tilde{M}}(s)^*\tilde{\Gamma}^*u_{\tilde{N}}(s)\theta_{-s}(x)u_{\tilde{N}}(s)^*\tilde{\Gamma}u_{\tilde{M}}(s)\xi(s)\\ &= u_{\tilde{M}}(s)^*\Gamma^*x\tilde{\Gamma}u_{\tilde{M}}(s)\xi(s)\\ &= (\pi_\theta(\tilde{\Gamma}^*x\tilde{\Gamma})\xi)(s).\end{aligned}$$

This means $\Gamma^*\pi_\theta(\tilde{N})\Gamma = \pi_\theta(\tilde{M}_1)$. Let v_t be the θ-cocycle in $\tilde{N}$ as before. Since $\theta|_{\tilde{N}}$ is a stable action there exists a unitary $v \in \tilde{N}$ satisfying $v_t = v^*\theta_t(v)$ [CT, Theorem 5.1]. So we have

$$\Gamma^*\lambda_t\Gamma = (\Gamma^*\pi_\theta(v)\Gamma)\lambda_t(\Gamma^*\pi_\theta(v^*)\Gamma).$$

Hence

$$\begin{aligned}M_1 &= \Gamma^*N\Gamma = \Gamma^*(\pi_\theta(\tilde{N}) \vee \{\lambda_t\})\Gamma\\ &= \pi_\theta(\tilde{M}_1) \vee \{(\Gamma^*\pi_\theta(v)\Gamma)\lambda_t(\Gamma^*\pi_\theta(v^*)\Gamma)\}\\ &= \pi_\theta(\tilde{M}_1) \vee \{\lambda_t\}\\ &= \tilde{M}_1 \underset{\theta}{\times} \mathsf{R}.\end{aligned}$$

To show $(M_1)_{\varphi_1} = \Gamma^*\pi_\theta(\tilde{N})\Gamma$ what we have to do is to prove that φ_1 is the dual weight of the canonical trace on $\tilde{M}_1$. Let $\tilde{E}_1: \tilde{M}_1 \to \tilde{M}$ be the dual conditional expectation of $\tilde{E} \equiv E|_{\tilde{M}}$, and $\tau_1 = \tau \cdot \tilde{E}_1$ the canonical trace. Since $\tilde{E}_1$ commutes with θ,

$$(\tilde{E}_1 \otimes \mathrm{id})|_{M_1}: M_1 \to M$$

is a faithful normal conditional expectation. As we stated in subsection 2-1

$$\tilde{E}_1 - \frac{1}{[M: N]_0}\mathrm{id}: \tilde{M}_1 \to \tilde{M}_1$$

is completely positive. This implies $(\tilde{E}_1 \otimes \mathrm{id})|_{M_1} = E_1$ because E_1 is minimal. Therefore we have $\varphi_1 = \hat{\tau}_1$.

Next we show $(M_2)_{\varphi_2} = \Gamma^* \pi_\theta(\tilde{M})\Gamma$. For this it suffices to show

$$\hat{\tau} \cdot \mathrm{Ad}(\Gamma)|_{M_2} = [M: N]_0 \varphi_2,$$

or equivalently

$$\hat{\tau} = [M: N]_0 \varphi_2 \cdot \mathrm{Ad}(\Gamma^*)|_M.$$

Since

$$\mathrm{Ad}(\Gamma) \cdot E_2 \cdot \mathrm{Ad}(\Gamma^*): M \to N$$

is a conditional expectation with index $[M_1: M]_0 = [M: N]_0$, it must be E. So the right hand side of the above equation is

$$[M: N]_0 \varphi_2 \cdot \mathrm{Ad}(\Gamma^*)|_M = [M: N]_0 \varphi_1 \cdot E_2 \cdot \mathrm{Ad}(\Gamma^*)|_M = [M: N]_0 \varphi_1 \cdot \mathrm{Ad}(\Gamma^*) \cdot E.$$

Due to $\hat{\tau} \cdot E = \hat{\tau}$ and $\varphi_1|_M = \hat{\tau}$, what remains to show is the following.

$$\varphi_1 \cdot \mathrm{Ad}(\Gamma)|_{M_1} = [M: N]_0 \varphi_1.$$

As we saw before we have $\mathrm{Ad}(\Gamma) \cdot \pi_\theta|_{\tilde{M}_1} = \pi_\theta \cdot \mathrm{Ad}(\tilde{\Gamma})|_{\tilde{M}_1}$ and $\mathrm{Ad}(\Gamma)(\lambda_t) = \pi_\theta(v_t)\lambda_t$. So we obtain $\mathrm{Ad}(\Gamma) \cdot \sigma_t^{\varphi_1} = \sigma_t^{\varphi_1} \cdot \mathrm{Ad}(\Gamma)|_{M_1}$. By Longo's observation [L1] we have $\tau_1 \cdot \mathrm{Ad}(\tilde{\Gamma})|_{\tilde{M}_1} = [M: \mathrm{N}]_0 \tau_1$. (Note that $\mathrm{Ad}(\tilde{\Gamma})|_{\tilde{M}_1}: \tilde{M}_1 \to \tilde{M}$ is a canonical endomorphism.) Therefore we get the following for any positive element $x \in M_1$.

$$\begin{aligned}
\varphi_1 \cdot \mathrm{Ad}(\Gamma)(x) &= \hat{\tau}_1 \cdot \mathrm{Ad}(\Gamma)(x) \\
&= \tau_1 \cdot \pi_\theta^{-1}\left(\int_{-\infty}^{\infty} \sigma_t^{\varphi_1}(\mathrm{Ad}(\Gamma)(x))\,dt\right) \\
&= \tau_1 \cdot \pi_\theta^{-1}(\mathrm{Ad}(\Gamma)\left(\int_{-\infty}^{\infty} \sigma_t^{\varphi_1}(x)\,dt)\right) \\
&= \tau_1 \cdot \mathrm{Ad}(\tilde{\Gamma}) \cdot \pi_\theta^{-1}\left(\int_{-\infty}^{\infty} \sigma_t^{\varphi_1}(x)\,dt\right) \\
&= [M: N]_0 \tau_1 \cdot \pi_\theta^{-1}\left(\int_{-\infty}^{\infty} \sigma_t^{\varphi_1}(x)\,dt\right) \\
&= [M: N]_0 \varphi_1(x).
\end{aligned}$$

In general case, we obtain (3.1) by induction using $(M_j)_{\varphi_j} = (M_{j-1})_{\varphi_{j-1}} \vee \{e_j\}(j > 2)$ and $\Gamma e_k \Gamma^* = e_{k-2}(k > 3)$.

REMARK 3.3. In the situation that we consider only the sectors of γ and $\tilde{\gamma}$, we may replace $\mathrm{Ad}(\Gamma)$ and $\mathrm{Ad}(\Gamma)$ with $\mathrm{Ad}(\pi_\theta(v)\Gamma)$ and $\mathrm{Ad}(v\tilde{\Gamma})$ in the above proof. Since $\pi_\theta(v)\Gamma$ commutes with λ_t we further have $\gamma(\lambda_t) = \lambda_t$ in this case.

REMARK 3.4. The same type of statement holds for the discrete decomposition.

Our main theorem is as follows.

THEOREM 3.5. *Let $M \supset N$ be a pair of type* III_λ $(0 < \lambda \leqq 1)$ *factors with finite index, and γ: $M \to N$ the canonical endomorphism. In type* III_λ $(0 < \lambda < 1)$ *case we assume the pair has the simultaneous discrete crossed product decomposition. Let M_1 be the basic extension of M by N and $\hat{\tau}$ be the dual weight of the trace τ on $\tilde{M}$ as in subsection* 2-1, 2-2. *Then the type* II *and the type* III *principal graphs of $M_1 \supset M$ do not coincide if and only if ${}_M[\gamma^k]_M$ contains ${}_M[\sigma_t^{\hat{\tau}}]_M$, for some $t \notin T(M)$ and for some $k > 0$.*

PROOF. In the case of type III_1 factors: We take γ as in the statement of Lemma 3.2 for sufficiently large n and use the notations of Lemma 3.2. We denote by λ_t the implementing unitary for θ_t in M as before. We shall omit π_θ in what follows. Let p be a minimal projection in $M \cap \gamma^k(M)'$ (resp. $M \cap \gamma^k(N)'$) $(0 < k < n)$. Since p belongs to $\tilde{M} \cap \gamma^k(\tilde{M})'$ (resp. $\tilde{M} \cap \gamma^k(\tilde{N})'$), there exists an isometry $v \in \tilde{M}$ satisfying $vv^* = p$. (Note that p is always an infinite projection in $\tilde{M}$.) Let ${}_M[\rho]_M$ (resp. ${}_M[\rho]_N$) be the descendant $M - M$ (resp. $M - N$) sector of γ corresponding to p. Then ρ is obtained as follows.

$$\rho(x) = v^*\gamma^k(x)v \quad \text{for} \quad x \in M \ \ (\text{resp. } x \in N).$$

Note that $\rho(\tilde{M}) \subset \tilde{M}$ (resp. $\rho(\tilde{N}) \subset \tilde{N}$) and $\rho|_{\tilde{M}}$ (*resp.* $\rho|_{\tilde{N}}$) gives the descendant sectors of $\tilde{\gamma}$ corresponding to p. This means that we can compare $M - M$ (resp. $M - N$) sectors with $\tilde{M} - \tilde{M}$ ($\tilde{M} - \tilde{N}$) sectors. In what follows, to obtain representatives of sectors we always use isometries in $\tilde{M}$, and we write as

$$_{\tilde{M}}[\tilde{\rho}]_{\tilde{M}} \equiv {}_{\tilde{M}}[\rho|_{\tilde{M}}]_{\tilde{M}} \quad (\text{resp. } {}_{\tilde{M}}[\tilde{\rho}]_{\tilde{N}} \equiv {}_{\tilde{M}}[\rho|_{\tilde{N}}]_{\tilde{N}}).$$

We compute $\hat{\tau} \cdot \rho$ in the case that ρ is an $M - M$ sector. Let $\varepsilon_k : M \to \gamma^k(M)$ be the minimal expectation. From the proof of Lemma 3.2 we may assume $\hat{\tau} \cdot \varepsilon_k = \hat{\tau}$. So we have the following for $x \in M_+$.

$$\begin{aligned}
\hat{\tau}(\rho(x)) &= \hat{\tau}(v^*\gamma^k(x)v) = \hat{\tau}(\gamma^k(x)vv^*) \\
&= \hat{\tau} \cdot \varepsilon_k(\gamma^k(x)p) = \hat{\tau}(\gamma^k(x)\varepsilon_k(p)) \\
&= \varepsilon_k(p)\hat{\tau}(\gamma^k(x)) = \varepsilon_k(p)[M:N]_0^k\hat{\tau}(x).
\end{aligned}$$

As in [I1, subsection 2.3] we obtain

$$\hat{\tau} \cdot \rho = d(\rho)\hat{\tau}$$

where $d(\rho) = [M:\rho(N)]_0^{1/2}$. Note that if ρ is an automorphism $\hat{\tau} \cdot \rho = \hat{\tau}$ holds.

Assume that ${}_M[\gamma^k]_M$ contains ${}_M[\sigma_t^{\hat\tau}]_M$ $(t \neq 0)$. Let p be the projection corresponding to ${}_M[\sigma_t^{\hat\tau}]_M$ and $v \in \tilde{M}$ an isometry satisfying $vv^* = p$ as before. We define $\alpha \in \mathrm{Aut}(M)$ by $\alpha = v^*\gamma^k(x)v$. By assumption there exists a unitary $w \in M$ satisfying $\alpha = \mathrm{Ad}(w)\cdot\sigma_t^{\hat\tau}$. Due to $\hat\tau\cdot\alpha = \hat\tau$, w belongs to $\tilde{M}$. This means ${}_{\tilde M}[\tilde\alpha]_{\tilde M} = {}_{\tilde M}[\mathrm{id}]_{\tilde M}$ and hence the two graphs are different.

Assume that the two graphs are different. Note that $\mathbf{R}$ action θ can not move the central projections in $\tilde{M} \cap \gamma^k(\tilde{M})'$ and $\tilde{M} \cap \gamma^k(\tilde{N})'$ [KL]. So there exists a minimal central projection $z \in \tilde{M} \cap \gamma^k(\tilde{M})'$ (or $\tilde{M} \cap \gamma^k(\tilde{N})'$) such that θ acts non-trivially on $z(\tilde{M} \cap \gamma^k(\tilde{M})')$ (resp. $z(\tilde{M} \cap \gamma^k(\tilde{N})')$). Note that $z(M \cap \gamma^k(M)')$ (resp. $z(M \cap \gamma^k(N)')$) is not a factor because any $\mathbf{R}$ action on a type I factor is inner. This means that there exist two non-zero central projections z_1, z_2 in $M \cap \gamma^k(M)'$ (resp. $M \cap \gamma^k(N)'$) such that $z_1 + z_2 = z$. Hence there exist two irreducible $M - M$ (resp. $M - N$) sectors ${}_M[\rho_1]_M$, ${}_M[\rho_2]_M$ (resp. ${}_M[\rho_1]_N$, ${}_M[\rho_2]_N$) contained in ${}_M[\gamma^k]_M$ (resp. ${}_M[\gamma^k]_N$) which satisfy

$$
{}_M[\rho_1]_M \neq {}_M[\rho_2]_M \quad (\text{resp. } {}_M[\rho_1]_N \neq {}_M[\rho_2]_N),
$$

$$
{}_{\tilde M}[\tilde\rho_1]_{\tilde M} = {}_{\tilde M}[\tilde\rho_2]_{\tilde M} \quad (\text{resp. } {}_{\tilde M}[\tilde\rho_1]_{\tilde N} = {}_{\tilde M}[\tilde\rho_2]_{\tilde N}),
$$

Note that ${}_{\tilde M}[\tilde\rho_1]_{\tilde M}$, ${}_{\tilde M}[\tilde\rho_2]_{\tilde M}$ (resp. ${}_{\tilde M}[\tilde\rho_1]_{\tilde N}$, ${}_{\tilde M}[\tilde\rho_2]_{\tilde N}$) are irreducible too. We may assume $\tilde\rho_1 \equiv \tilde\rho_2$. From $\gamma\cdot\sigma_t^{\hat\tau} = \sigma_t^{\hat\tau}\cdot\gamma$ the following hold.

$$
\rho_1\cdot\sigma_t^{\hat\tau} = \sigma_t^{\hat\tau}\cdot\rho_1, \quad \rho_2\cdot\sigma_t^{\hat\tau} = \sigma_t^{\hat\tau}\cdot\rho_2.
$$

So there exist θ-cocycles $w_t^1, w_t^2 \in \tilde{M}$ satisfying

$$
\rho_1(\lambda_t) = w_t^1\lambda_t, \quad \rho_2(\lambda_t) = w_t^2\lambda_t.
$$

Therefore for $x \in \tilde{M}$ (resp. $x \in \tilde{N}$) the following holds.

$$
\begin{aligned}
w_t^1\lambda_t\rho_1(x)\lambda_t^* w_t^{1*} &= \rho_1(\lambda_t x\lambda_t^*) = \rho_1(\theta_t(x)) = \rho_2(\theta_t(x)) \\
&= w_t^2\lambda_t\rho_2(x)\lambda_t^* w_t^{2*} = w_t^2\lambda_t\rho_1(x)\lambda_t^* w_t^{2*}.
\end{aligned}
$$

Due to $\tilde{M} \cap \rho_1(\tilde{M})' = \mathbf{C}$ (resp. $\tilde{M} \cap \rho_1(\tilde{N})' = \mathbf{C}$), there exists $s \neq 0$ satisfying

$$
w_t^2 = e^{-ist}w_t^1.
$$

So we have $\rho_2 = \sigma_s^{\hat\tau}\cdot\rho_1$. Therefore ${}_M[\rho_2\overline{\rho_1}]_M = {}_M[\sigma_s^{\hat\tau}\cdot\rho_1\overline{\rho_1}]_M$ contains ${}_M[\sigma_s^{\hat\tau}]_M$.

In the case of type III_λ $(0 < \lambda < 1)$ factors: From Remark 3.4 we may do the same assumption on γ as before using θ instead of θ_t. We denote by u the implementing unitary for θ in M. In contrast with the continuous case, discrete action θ can move central projections in $\tilde{M} \cap \tilde\gamma^k(\tilde{M})'$ and $\tilde{M} \cap \tilde\gamma^k(\tilde{N})'$. So what remains is to show that if θ moves central projections in $\tilde{M} \cap \tilde\gamma^k(\tilde{M})'$ or $\tilde{M} \cap \tilde\gamma^k(\tilde{N})'$ a modular automorphism appears in the irreducible components of ${}_M[\gamma^k]_M$. In this case there exist mutually orthogonal minimal central projections

$\{p_1, p_2, \ldots, p_l\} \subset \tilde{M} \cap \tilde{\gamma}^k(\tilde{M})'$ (resp. $\tilde{M} \cap \tilde{\gamma}^k(\tilde{N})'$) satisfying $\theta(p_j) = p_{j+1}(p_{l+1} = p_1)$. Let

$$z \equiv \sum_{j=1}^{l} p_j.$$

Then $z \in (\tilde{M} \cap \gamma^k(\tilde{M})')^\theta = M \cap \gamma^k(M)'$ (resp. $z \in (\tilde{M} \cap \gamma^k(\tilde{N})')^\theta = M \cap \gamma^k(N)'$). So there is an isometry $v \in (\tilde{M})^\theta$ satisfying $vv^* = z$. We define $\rho: M \to M$ (resp. $\rho: N \to M$) by

$$\rho(x) \equiv v^* \gamma^k(x) v \quad \text{for } x \in M \quad (\text{resp. } x \in N).$$

Let $q_j \equiv v^* p_j v$. Then q_j is a central projection in $\tilde{M} \cap \tilde{\rho}(\tilde{M})'$ (resp. $\tilde{M} \cap \tilde{\rho}(\tilde{N})'$) and the following hold.

$$\theta(q_j) = q_{j+1}, \quad \sum_{j=1}^{l} q_j = 1.$$

We define a unitary $w \in \tilde{M} \cap \tilde{\rho}(\tilde{M})'$ (resp. $w \in \tilde{M} \cap \tilde{\rho}(\tilde{N})'$) by

$$w \equiv \sum_{j=1}^{l} e^{\frac{2\pi i}{l} j} q_j.$$

From $\theta(w) = e^{-\frac{2\pi i}{l}} w$, w satisfies

$$w^* u w = e^{-\frac{2\pi i}{l}} u.$$

Thanks to Remark 3.3 we may assume $\rho(u) = u$. So we have

$$\mathrm{Ad}(w) \cdot \rho(x) = \begin{cases} \rho(x) & x \in \tilde{M} \ (\text{resp. } x \in \tilde{N}) \\ e^{\frac{2\pi i}{l}} \rho(x) & x = u \end{cases}.$$

Therefore we obtain

$$\sigma_s^{\hat{\tau}} \cdot \mathrm{Ad}(w) \cdot \rho = \rho, \quad s = -\frac{2\pi}{l \log \lambda}.$$

This means

$${}_M[\rho]_M = {}_M[\sigma_s^{\hat{\tau}} \cdot \rho]_M \quad (\text{resp. } {}_M[\rho]_N = {}_M[\sigma_s^{\hat{\tau}} \cdot \rho]_N).$$

Hence ${}_M[\rho\bar{\rho}]_M = {}_M[\sigma_s^{\hat{\tau}} \cdot \rho\bar{\rho}]_M$ contains ${}_M[\sigma_s^{\hat{\tau}}]_M$.

As a corollary we have the following.

COROLLARY 3.6. *Let $M \supset N$ be a pair of type III_1 factors with finite index and $\gamma: M \to N$ the canonical endomorphism. If only finitely many automorphisms appear in the descendant sectors of ${}_M[\gamma]_M$ the type* II *and type* III *principal graphs of*

$M_1 \supset M$ coincide. In particular if the depth of $M \supset N$ is finite the above condition is automatically satisfied.

PROOF. This follows from the fact that the automorphisms which appear in the descendant sectors of ${}_M[\gamma]_M$ make a group.

REMARK 3.7. In the case of finite depth S. Popa [P] and Ph. Loi [Li2] also independently obtain Corollary 3.6 in different ways.

REMARK 3.8. In the case of type III_0 factors with the common flow of weight, we can define the type II principal graph as in Remark 2.3. So we can consider the same type of problem replacing modular automorphisms with extended modular automorphisms [CT].

Our main theorem is about the "modular invariant" of "actions of paragroups" [ST] [KST] [O]. Concerning the "Connes-Takesaki module" [CT] we have the following.

PROPOSITION 3.9. *Let $M \supset N$ be a pair of type III factors with finite index and γ: $M \to N$ the canonical endomorphism. If M and N have the common flow of weight, the Connes-Takesaki modules of the automorphisms appearing in ${}_M[\gamma^k]_M$ are trivial.*

PROOF. Let $\tilde{\gamma}$, $\tilde{M}$, and $\tilde{N}$ be as in Lemma 3.2. By assumption we have $Z(\tilde{M}) = Z(\tilde{N})$. So we can see that $\tilde{\gamma}$ is trivial on $Z(\tilde{M})$. Hence we obtain the result as in the proof of Theorem 3.5.

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RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES
KOYTO UNIVERSITY
KYOTO 606
JAPAN
EMAIL: izumi @ kurims. kyoto-u.ac. jp