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MODULAR SUBSTRUCTURES IN PSEUDOMODULAR LATTICES

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Abstract.

Pseudomodular lattices have been used in [DL 86] in order to investigate combinatorial properties of algebraic matroids and were further analyzed in [BL 87]. The purpose of our paper is to present local conditions, characterizing modular sublattices of a pseudomodular lattice. As an application, we derive a result of [HK 89], implying that Lovasz' min – max formula for matchings in projective geometries remains valid for pseudomodular lattices, and we discuss a relation woth B. Lindströms construction of subgeometries of full algebraic combinatorial geometries which are isomorphic to projective geometries over skew fields.

1. Introduction.

All lattices we consider will be geometric, i.e. of finite length, relatively complemented, graded, and the rank function defined by the grading is semimodular. Each lattice L will be endowed with a strictly increasing semimodular rank function (which may or may not be identical to the one induced by the grading), i.e. $r: L \to \mathbf{N}_0$ is strictly increasing and satisfies

$$\forall x, y \in L: r(x) + r(y) \geqq r(x \vee y) + r(x \wedge y).$$

Recall that r is called modular if this inequality is satisfied by equality for every pair $x, y \in L$.

Recently, [DL 86] and [BL 87] introduced an interesting generalization of modular lattices, the class of so called pseudomodular lattices, which was shown in [DL 86] to contain all full algebraic combinatorial geometries and which may be defined as follows:

Definition (cf. [BL 87]). A geometric lattice L endowed with a strictly increasing semimodular function $r: L \to \mathbf{N}$ is called *pseudomodular*, is for any $a, b, c \in L$ the following implications holds:
If

$$r(a \vee c) - r(a) = r(b \vee c) - r(b) = r(a \vee b \vee c) - r(a \vee b),$$

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then

$$r((a \vee c) \wedge (b \vee c)) - r(a \wedge b) = r(a \vee c) - r(a).$$

It is easy to see that pseudomodularity actually is a generalization of modularity. However, as we will see in Section 2, there is somewhat more to say about the relation between pseudomodular lattices and modular ones. Essentially, it will turn out that modular sublattices of pseudomodular ones can be chaacterized by local conditions which can be checked easily. In Section 3 we will use this fact to derive a result of [HK 89], implying that Lovasz' min – max formula for matching in projective geometries extends to pseudomodular lattices. Finally, in the last section we comment on B. Lindströms construction of large projective geometries, defined over skewfields, which are contained in full algebraic combinatorial geometries.

2. Modular Substructures.

Let L be a geometric lattice with a strictly increasing semimodular rank function $r: L \to \mathbb{N}_0$ and let M be a geometric lattice with strictly increasing modular rank function $\rho: M \to \mathbb{N}_0$. Furthermore, let $\varphi: M \to L$ be a mapping. We say that φ defines a *modular substructure* of L if

- $\varphi: M \to L$ is a homomorphism, i.e. it is $\wedge$- and $\vee$-preserving and
- $\rho(x) = r(\varphi(x))\ \forall x \in M.$

The following result states that, if L is pseudomodular, then modular substructures can be characterized by local conditions which in concrete situations are easy to check (cf. Section 3):

THEOREM 2.1. *If L is pseudomodulair, then $\varphi: M \to L$ defines a modular substructure if and only if*

(i) *φ is $\wedge$-preserving,*

(ii) *$\varphi(x \vee y) = \varphi(x) \vee \varphi(y)$ whenever $x \vee y = 1_M$,*

(iii) *if $I = [x, 1_M]$ is an interval of length 2, then $\rho(y) = r(\varphi(y))$ for all $y \in I$.*

PROOF. We show that if φ satisfies conditions (i) – (iii) above, then for any upper interval $I = [x, 1_D]$, the induced map $\varphi: I \to L$ is a homomorphism satisfying $\rho(y) = r\varphi(y)$ for all $y \in I$. The proof is by induction on $k :=$ length of I.
If $k = 2$, the claim is immediate from conditions (i) – (iii). Thus suppose that $k \geqq 3$ and assume the claim holds for smaller values of k. We have to show that for any two $y_1, y_2 \in I$ we have

$$\varphi(y_1 \vee y_2) = \varphi(y_1) \vee \varphi(y_2) \text{ and} \tag{1}$$

$$\rho(y_1 \wedge y_2) = r\varphi(y_1 \wedge y_2). \tag{2}$$

We may assume that $y_1 \wedge y_2 = x$, for otherwise the claim follows from our inductive assumption by looking at the interval $[y_1 \wedge y_2, 1_M]$. Let us first show that (1) holds. If $y_1 \vee y_2 = 1_M$, then (1) is the content of condition (ii). Hence assume that $y_1 \vee y_2 < 1_M$. Let $y_3 \in M$ be a complement of $y_1 \vee y_2$ in the interval $[x, 1_M]$.

Applying φ, we are in a situation as shown in Figure 1 below, where $u := \varphi(y_1) \vee \varphi(y_2)$, $v := \varphi(y_1) \vee \varphi(y_3)$, and $w := \varphi(y_2) \vee \varphi(y_3)$:

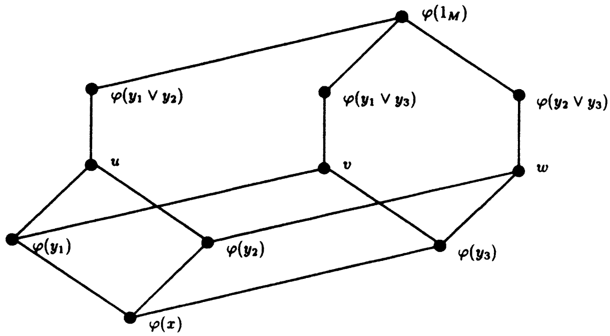

Figure 1.

Note that, due to condition (ii), we have

$$\varphi(y_1) \vee \varphi(y_2 \vee y_3) = \varphi(1_M),$$

hence

$$u \vee \varphi(y_2 \vee y_3) = \varphi(1_M),$$

as indicated in Figure 1 above. Furthermore, note that φ is order preserving since it was assumed to be $\wedge$-preserving. This implies that

$$\varphi(y_1 \vee y_2) \geqq u = \varphi(y_1) \vee \varphi(y_2)$$

etc. We are to show that actually equality holds. First note that modularity of ρ gives

$$\rho((y_1 \vee y_2) \wedge (y_2 \vee y_3)) = \rho(y_1 \vee y_2) + \rho(y_2 \vee y_3) - \rho(1_M) =$$

$$= \rho(y_1) + \rho(y_2) - \rho(x) + \rho(y_2) + \rho(y_3) - \rho(x) - \rho(1_M)$$

$$= \rho(y_1 \vee y_2) + \rho(y_3) - \rho(x) - \rho(1_M) + \rho(y_2) = \rho(y_2)$$

and therefore

$$(*) \qquad (y_1 \vee y_2) \wedge (y_2 \vee y_3) = y_2.$$

By induction on k, we know that $\varphi_{|[y_2, 1_M]}$ defines a modular substructure of L. Thus, in particular,

$$(**)\qquad r\varphi(y_1 \vee y_2) + r\varphi(y_2 \vee y_3) = r\varphi(1_M) + r\varphi(y_2).$$

Now, applying the semimodularity of L with respect to u and $\varphi(y_2 \vee y_3)$, we get – using $u \vee \varphi(y_2 \vee y_3) = \varphi(1_M)$ and $r(u \wedge \varphi(y_2 \vee y_3)) \geqq r\varphi(y_2)$ – the inequality

$$r(u) \geqq r\varphi(1_M) + r\varphi(y_2) - r\varphi(y_2 \vee y_3) \overset{(**)}{=} r\varphi(y_1 \vee y_2),$$

which together with $\varphi(y_1 \vee y_2) \geqq u$ implies (1), that is,

$$\varphi(y_1 \vee y_2) = u = \varphi(y_1) \vee \varphi(y_2)$$

in view of the strict monotonicity of r.

We are left to show that (2) holds, i.e.

$$\rho(x) = r\varphi(x).$$

Since the length k of the interval $I = [x, 1_M]$ is at least 3, we can find $y_1, y_2, y_3 \in I \setminus \{x\}$ such that each y_i is a relative complement to the join of the other two y_j's, i.e., with $y_1 \vee y_2 \vee y_3 = 1_M$ and $\rho(y_1) + \rho(y_2) + \rho(y_3) = \rho(1_M) + 2\rho(x)$. From our argument above, we conclude that, applying φ, we are in a situation as sketched in Figure 2 below:

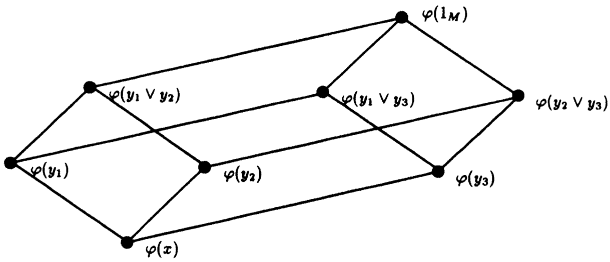

Figure 2.

Now let $a := \varphi(y_1)$, $b := \varphi(y_3)$ and $c := \varphi(y_2)$. Since, by induction, φ defines a modular substructure of L when restricted to each of the intervals $[y_i, 1_M]$, we get

$$r(a \vee c) - r(a) = r(a \vee b \vee c) - r(a \vee b) = r(b \vee c) - r(b) = \rho(y_2) - \rho(x).$$

Since L is pseudomodular, we conclude that

$$r((a \vee c) \wedge (b \vee c)) - r(a \wedge b) = r(a \vee c) - r(a), \text{ i.e. (cf. (*))},$$

$$r\varphi(y_2) - r\varphi(x) = r\varphi(y_1 \vee y_2) - r\varphi(y_1)$$

and therefore

$$r\varphi(x) = \rho(y_2) + \rho(y_1) - \rho(y_1 \vee y_2) = \rho(x),$$

as claimed.

3. An application in Matching Theory.

In this section we assume the reader to be familiar with the basic theory of matroids and geometric lattices. Suppose L is a geometric lattice with point set E and let now r denote the rank function induced by the grading. A subset $A \subseteq E$ is called a *double circuit*, if $r(A) = r(A \setminus a) = |A| - 2$ for every $a \in A$ or, equivalently, if A is the complement of a coline, i.e., a flat of codimension 2, in the dual of L. These sets play a central role in the context of matching in geometric lattices (cf. [HK 89] for more details). It is easy to see (cf. [HK 89]) that the following holds:

LEMMA 3.1. *Let $A \subseteq E$ be a double circuit. Then there exists a partition*

$$A = A_1 \dot{\cup} \ldots \dot{\cup} A_d$$

such that $C_i := A \setminus A_i$ is a circuit for $i = 1, \ldots, d$ and these are all circuits contained in A.

PROOF. If in the dual of L the hyperplanes containing the coline $E \setminus A$ are $H_1, \ldots, H_d$, then $A_1 := H_1 \cap A, \ldots, A_d := H_d \cap A$ are precisely the subsets described above.

As can be seen from [HK 89], the crucial point in proving Lovasz' min – max formula for matchings conxists in showing that in case L is pseudomodular the closures $\bar{C}_i$ of the circuits in A induce a modular sublattice of L. More precisely, one has to show (cf. Theorem 3.1 of [HK 89]) that

$$r\left(\bigcap_{i=1}^{d} \bar{C}_i\right) \geqq d - 2.$$

As we will see next, this is a simple consequence of Theorem 2.1;

Indeed, let $M = M_d$ denote the Boolean algebra of (all) subsets of $\{1, \ldots, d\}$. Obviously, M is modular relative to the map ρ: $M \to \mathbf{N}_0$, defined by

$$\rho(I) := |A| - 2 - \sum_{i \notin I} (|A_i| - 1) = \sum_{i \in I} (|A_i| - 1) + d - 2$$

for all $I \subseteq \{1, \ldots, d\}$. We claim

PROPOSITION 3.2. *If – with the above notations – we define*

$$\varphi : M \to L$$

by

$$\varphi(I) := \bigcap_{i \notin I} \bar{C}_i$$

for all proper subsets $I \subsetneqq \{1,\dots,d\}$ of $\{1,\dots,d\}$ *and*

$$\varphi(\{1,\dots,d\}) := \bar{A},$$

then φ *defines a modular substructure in* L.

PROOF. Obviously, φ is $\wedge$-preserving and for $I \cup J = \{1,\dots,d\}$ we have

$$\begin{aligned} \bar{A} \geqq \varphi(I) \vee \varphi(J) &= \cap\{\overline{A \setminus A_i} \mid i \notin I\} \vee \cap\{\overline{A \setminus A_j} \mid j \notin J\} \\ &\supseteq \cap\{A \setminus A_i \mid i \notin I\} \vee \cap\{A \setminus A_j \mid j \notin J\} \\ &= \cup\{A_i \mid i \in I\} \vee \cup\{A_j \mid j \in J\} \supseteq A \end{aligned}$$

and therefore $\varphi(I) \vee \varphi(J) = \bar{A}$. Thus we are left to show that condition (iii) of Theorem 2.1 is satisfied, too. Too check this, observe that $\rho(\{1,\dots,d\}) = |A| - 2 = r(A)$ and $\rho(\{1,\dots,d\} \setminus \{i\}) = |A| - |A_i| - 1 = r(\bar{C}_i)$, since $C_i = A \setminus A_i$ is a circuit. Finally, for $i \neq j$, we have

$$r(\bar{C}_i \wedge \bar{C}_j) = \rho(\{1,\dots,d\} \setminus \{i,j\}) = |A| - |A_i| - |A_j|$$

in view of the following sequence of inequalities

$$\begin{gathered} r(\bar{C}_i \wedge \bar{C}_j) \leqq r(\bar{C}_i) + r(\bar{C}_j) - r(\bar{C}_i \vee \bar{C}_j) = \\ (|A| - |A_i| - 1) + (|A| - |A_j| - 1) - (|A| - 2) = \\ |A| - |A_i| - |A_j| = |C_i \cap C_j| \leqq r(\bar{C}_i \wedge \bar{C}_j), \end{gathered}$$

the first one of which holds because L is semimodular, while the second one holds because $C_i \cap C_j$, being a proper subset of a circuit, must be independent.

COROLLARY 3.3. *Under the assumptions of Proposition* 3.2, *one has*

$$r\left(\bigcap_{i=1}^{d} \bar{C}_i\right) = d - 2.$$

PROOF. Since $\varphi : M \to L$ defines a modular substructure, we conclude that

$$r\left(\bigcap_{i=1}^{d} \bar{C}_i\right) = \rho(\emptyset) = |A| - 2 - \sum_{i=1}^{d} (|A_i| - 1) = d - 2,$$

as claimed.

4. Projective Geometries in Full Algebraic Matroids.

In [L 88] B. Lindström shows that for every field F of prime characteristic p the full algebraic matroid $L = L_n(F)$ of rank n, whose flats are the algebraically closed subfields of the algebraic closure $\overline{F(X_1,\dots,X_n)}$ of the purely transcendental extension $F(X_1,\dots,X_n)$ of F in n algebraically independent variables $X_1,\dots,X_n$ contains as a subgeometry a full projective space of (projective) dimension $n-1$ over a certain skew-field, defined in terms of the "p-polynomials" in the polynomial ring $F[X]$ in one variable X over F, that is, the F-linear combinations of the monomials of type X^{p^h} ($h = 0, 1, 2, \dots$). More specifically, his surprising and beautiful result asserts that

(i) the set of all p-polynomials in $F[X]$ forms an Ore-domain $R(F)$ with quotient skew-field, say, $\mathrm{Li}(F)$, when for any two such polynomials $P[X], Q[X] \in R(F)$ one defines the sum $P[X] + Q[X]$ as usual and the product $P[X] \circ Q[X]$ by composition, i.e., by

$$P[X] \circ Q[X] := P[Q[X]],$$

and that

(ii) there exists a modular substructure $\varphi: L_n(F)$, where $M = M(Li(F)^n)$ denotes the relatively complemented modular lattice of (all) subspaces $U, V, \dots,$ of the n-dimensional (right) vectorspace $\mathrm{Li}(F)^n$ over $\mathrm{Li}(F)$ (with dimension as rank function) and where for $U \leqq \mathrm{Li}(F)^n$ the field $\varphi(U)$ is the algebraic closure in $\overline{F(X_1,\dots,X_n)}$ of all polynomials

$$Q[X_1,\dots,X_n] = Q_1[X_1] + \dots + Q_n[X_n]$$

with $(Q_1[X],\dots,Q_n[X]) \in U \cap R(F)^n$.

Unfortunately, to verify that $\varphi: M \to L$ satisfies the conditions in Theorem 2.1 requires almost all the work, Lindström has done to establish his result directly. So one saves not much by invoking Theorem 2.1 in this specific context. Still, Theorem 2.1 together with the fact from [DL 86], referred to already in the Introduction, that every full algebraic matroid is pseudomodular, seems to present a conceptual framework with regard to which Lindström's amazing construction can be understood and appreciated more easily and in a more systematic way and which may help to find similar results, e.g. for full algebraic matroids defined over fields of characteristic 0.

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