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APPROXIMATION BY NEAREST INTEGER CONTINUED FRACTIONS (II)

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Abstract.

In a paper with the same title recently published in this journal, a recurrence relation of a Diophantine inequality is established: $\min(\theta_{n-1}, \theta_n, \ldots, \theta_{n+k}) < 2/(3 + \sqrt{5} - 2\alpha_k)$, where $\alpha_1 = 2/5$ and $\alpha_i = 1/(3 - \alpha_{i-1})$. In this note, we give the explicit form of this inequality: $\min(\theta_{n-1}, \theta_n, \ldots, \theta_{n+k}) < 1/\sqrt{5} + ((3 - \sqrt{5})/2)^{2k+3}/\sqrt{5}$.

Let x be an irrational number, $x = [\varepsilon_0 b_0; \varepsilon_1 b_1, \ldots, \varepsilon_n b_n, \ldots]$ be its expansion in nearest integer continued fraction. Let $A_n/B_n = [\varepsilon_0 b_0; \varepsilon_1 b_1, \ldots, \varepsilon_n b_n]$ be the nth convergent and $\theta_n = B_n^2|x - A_n/B_n|$. It was proved in [2] that $\min(\theta_{n-1}, \theta_n, \theta_{n+1}) < 5(5\sqrt{5} - 11)/2$. The present author generalized this result. It is proved in [4] that $\min(\theta_{n-1}, \theta_n, \ldots, \theta_{n+k}) < 2/(3 + \sqrt{5} - \alpha_k)$, where $\alpha_1 = 2/5$, $\alpha_i = 1/(3 - \alpha_{i-1})$. In this note, using the Fibonacci sequence, we give an explicit estimation of the value $\min(\theta_{n-1}, \theta_n, \ldots, \theta_{n+k})$ as a function of k directly.

THEOREM 1. $\min(\theta_{n-1}, \theta_n, \ldots, \theta_{n+k}) < 1/\sqrt{5} + ((3 - \sqrt{5})/2)^{2k+3}/\sqrt{5}$.

PROOF. Let $f_1 = 1$, $f_2 = 1$, $f_{n+2} = f_{n+1} + f_n$ be the Fibonacci sequence.

We first prove that the recurrence relation $\alpha_i = 1/(3 - \alpha_{i-1})$ and $\alpha_1 = 2/5$ imply that $\alpha_i = f_{2i+1}/f_{2i+3}$.

If $i = 1$, then $\alpha_1 = 2/5 = f_3/f_5$. Suppose $\alpha_k = f_{2k+1}/f_{2k+3}$. Then $\alpha_{k+1} = 1/(3 - \alpha_k) = 1/(3 - f_{2k+1}/f_{2k+3}) = f_{2k+3}/(3f_{2k+3} - f_{2k+1}) = f_{2k+3}/(2f_{2k+3} + f_{k+2}) = f_{2k+3}/(f_{2k+3} + f_{2k+4}) = f_{2k+3}/f_{2k+5}$. Therefore by induction we have $\alpha_i = f_{2i+1}/f_{2i+3}$.

Replacing α_k in the expression $\min(\theta_{n-1}, \theta_n, \ldots, \theta_{n+k}) < 2/(3 + \sqrt{5} - \alpha_k)$ by f_{2k+1}/f_{2k+3}, we have

$$\min(\theta_{n-1}, \theta_n, \ldots, \theta_{n+k}) < 2f_{2k+3}/((3 + \sqrt{5})f_{2k+3} - 2f_{2k+1}).$$

By Binet's formula for the Fibonacci sequence [1], we have

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$$f_n = (\phi^n - (-\phi^{-1})^n)/\sqrt{5},$$

where $\phi = (1 + \sqrt{5})/2$, $-\phi^{-1} = (1 - \sqrt{5})/2$. Now a direct calculation of $2((\phi/(-\phi^{-1}))^{2k+3} - 1)/((3 + \sqrt{5})((\phi/(-\phi^{-1}))^{2k+3} - 1) - 2((\phi/(-\phi^{-1}))^{2k+1} - 1)/(-\phi^{-1})^2)$ yields

$$\min(\theta_{n-1}, \theta_n, \ldots, \theta_{n+k}) < 1/\sqrt{5} + ((3 - \sqrt{5})/2)^{2k+3}/\sqrt{5}.$$

Now we can have a comparison of the two approximations by simple continued fraction and by nearest integer continued fraction. Let x be an irrational number. Borel's theorem [3] asserts that among any three consecutive convergents p_i/q_i of simple continued fraction of x, there is at least one satisfies $|x - p_i/q_i| < 1/(\sqrt{5}\,q_i^2)$. As a much weaker corollary we know that there are infinitely many convergents p_i/q_i satisfying $|x - p_i/q_i| < 1/(\sqrt{5}\,q_i^2)$. For nearest integer continued fraction we only have the following even weaker form.

COROLLARY 1. *Let x be an irrational number. Then there are infinitely many convergents A_i/B_i of nearest integer continued fraction of x satisfying*

$$|x - A_i/B_i| < 1/((\sqrt{5} - \varepsilon)\,B_i^2).$$

PROOF. Since $((3 - \sqrt{5})/2)^{2k+3} \to 0$ when $k \to \infty$, we know that $\min(\theta_{n-1}, \theta_n, \ldots, \theta_{n+k}) \downarrow 1/\sqrt{5}$ for any fixed positive integer n. Therefore for any given small $\varepsilon > 0$ and any positive integer n, pick up k such that $\min(\theta_{n-1}, \theta_n, \ldots, \theta_{n+k}) < 1/(\sqrt{5} - \varepsilon)$ then we can have an integer $m(-1 < m < k)$ such that $\theta_{n+m} < 1/(\sqrt{5} - \varepsilon)$. Since there are infinitely many positive integer n, there are infinitely many i such that $\theta_i < 1/(\sqrt{5} - \varepsilon)$. Since $\theta_i = B_i^2|x - A_i/B_i|$, we have the conclusion.

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