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FREE ARCHIMEDEAN l-GROUPS

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Abstract.

In this paper we discuss the existence and description of the free archimedean l-group $\mathcal{F}_{\mathcal{A}r}([G, P])$ generated by a po-group [G, P], and give some properties of the free abelian l-group (the free archimedean l-group) $\mathcal{A}_\alpha$ of rank α.

We use the standard terminologies and notations of [1, 5, 9]. We assume that all groups considered will be abelian. The group operation of an l-group is written by additive notation. Let G be an l-group and $S \subseteq G$. We denote by [S] the l-subgroup of G generated by S. The convex l-subgroup generated by an element $g \in G$ is denoted by $G(g)$. A po-group is a partially ordered group $[G, P]$ where $P = \{x \in G \mid x \geqq 0\}$ is the positive semigroup of G. P is said to be semi-group if $p \in P$ whenever $p \in G$ and $np \in P$ for some positive integer n. Let G and H be two po-groups. A map φ from G into H is called a po-group homomorphism, if φ is a group homomorphism and $x \geqq y$ implies $\varphi(x) \geqq \varphi(y)$ for any x, $y \in G$. A po-group homomorphism φ is called a po-group isomorphism if φ is an injection and φ^{-1} is also a po-group homomorphism. We use N and Z for the natural numbers and the integers, respectively.

1. Sub-product Radical Class of Archimedean l-groups.

A family $\mathscr{U}$ of l-groups is called a sub-product radical class, if it is closed under taking 1) l-subgroups, 2) joins of convex l-subgroups and 3) direct products. All our sub-product radical classes are always assumed to contain along with a given l-group all its l-isomorphic copies. Let $\mathscr{U}$ be a sub-product radical class and G be an l-group. Then the join of all convex l-subgroups of G belonging to $\mathscr{U}$ is the unique largest convex l-subgroup of G belonging to $\mathscr{U}$. It is denoted by $\mathscr{U}(G)$ and is called a sub-product radical of G. $\mathscr{U}(G)$ is a characteristic l-ideal of G.

An l-group G is said to be archimedean if it satisfies one of the following three equivalent conditions:

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1. For any $0 < a, b \in G$, there exists $n \in N$ such that $nb \nleqq a$.
2. For all $a, b \in G$, if $nb \leqq a$ for all $n \in Z$, then $b = 0$.
3. For all $a, b \in G$, if $nb \leqq a$ for all $n \in N$, then $b \leqq 0$.

Let G be an l-group. An element $a \in G$ is archimedean if $a \geqq 0$ and if for all $0 < b \leqq a$, there exists $n \in N$ such that $nb \nleqq a$ [12, 18]. Let $P(G)$ be the set of all archimedean elements of G. An element $a \in G$ is said to be generally archimedean if the positive part a^+ and the negative part a^- are both archimedean. The following lemma is easy to show using [18].

LEMMA 1.1. *Let G be an l-group and $g \in G$. Then the following are equivalent:*
(1) *g is generally archimedean.*
(2) *$|g|$ is archimedean.*
(3) *$G(g)$ is archimedean.*
(4) *$G(|g|)$ is archimedean.*

Let $\mathscr{A}r$ be the family of all archimedean l-groups. $\mathscr{A}r$ is a quasi-torsion class [13], that is, $\mathscr{A}r$ is closed under taking 1) convex l-subgroups, 2) joins of convex l-subgroups and 3) complete l-homomorphisms. It is clear that $\mathscr{A}r$ is closed under taking l-subgroups and direct products. So $\mathscr{A}r$ is a sub-product radical class. Let G be an l-group. Then there exists a unique largest archimedean l-subgroup of G, the $\mathscr{A}r$ radical $\mathscr{A}r(G)$. Clearly, G is archimedean if and only if $G = \mathscr{A}r(G)$. In [18] it was proved that the l-subgroup $A(G)$ of G is the unique largest archimedean convex l-subgroup of G. In [12] J. Jakubik also proved the existence of such $A(G)$. So we have $\mathscr{A}r(G) = [P(G)]$. By Theorem 1.3 of [5] $\mathscr{A}r(G)$ consists of the elements $g = x - y$ where $x, y \in P(G)$ and $x \wedge y = 0$. In fact, $x = g^+$ and $y = g^-$. And so such g are generally archimedean. Conversely, if $g \in G$ is a generally archimedean element, then $g \in \mathscr{A}r(G)$. Thus Lemma 1.1 infers

LEMMA 1.2. $\mathscr{A}r(G) = [P(G)]$

$$\begin{aligned} &= \{g \in G \mid g \text{ is generally archimedean}\} \\ &= \{g \in G \mid |g| \in P(G)\} \\ &= \{g \in G \mid G(g) \text{ is archimedean}\} \\ &= \{g \in G \mid G(|g|) \text{ is archimedean}\}. \end{aligned}$$

COROLLARY 1.3. *The set of all generally archimedean elements of an l-group G is closed under the addition, inverse, met and join.*

So we obtain a useful result.

PROPOSITION 1.4. *Suppose that an l-group G has a set of generators which consists of generally archimedean elements. Then G is archimedean.*

In what follows we will give an application of Proposition 1.4.

2. Free Archimedean *l*-group Generated by a po-group.

A partial *l*-group G is a set with partial operations corresponding to the usual *l*-group operations $.\,,\ -1,\ 1,\ \vee$ and $\wedge$ such that whenever the operations are defined for elements of G then the *l*-group laws are satisfied. Suppose $[G, P]$ is a po-group. Then G has implicit partial operations $\vee$ and $\wedge$ as determined by the partial order. That is,

$$x \vee y = y \vee x = y \text{ if and only if } x \leqq y \text{ and}$$

$$x \wedge y = y \wedge x = x \text{ if and only if } x \leqq y.$$

Using these two partial lattice operations together with the full group operations, G can be considered as a partial *l*-group. Thus we have the following definition as a special case of the $\mathscr{U}$-free algebra generated by a partial algebra.

DEFINITION 2.1. Let $\mathscr{U}$ be a class of *l*-groups and $[G, P]$ be a po-group. The *l*-group $\mathscr{F}_{\mathscr{U}}([G, P])$ is called the *$\mathscr{U}$-free l-group generated by* $[G, P]$ (or $\mathscr{U}$-free *l*-group over $[G, P]$) if the following conditions are satisfied:

(1) $\mathscr{F}_{\mathscr{U}}([G, P]) \in \mathscr{U}$;

(2) there exists an injective po-group isomorphism $\alpha: G \rightarrow \mathscr{F}_{\mathscr{U}}([G, P])$ such that $\alpha(G)$ generates $\mathscr{F}_{\mathscr{U}}([G, P])$ as an *l*-group;

(3) if $K \in \mathscr{U}$ and $\beta: G \rightarrow K$ is a po-group homomorphism, then there exists an *l*-homomorphism γ: $\mathscr{F}_{\mathscr{U}}([G, P]) \rightarrow K$ such that $\gamma\alpha = \beta$.

$$\begin{array}{ccc} [G, P] & \xrightarrow{\quad \alpha \quad} & \mathscr{F}_{\mathscr{U}}([G, P]) \\ & \searrow^{\beta} & \downarrow \gamma \\ & & K \end{array}$$

The classes of *l*-groups which will be refered to are $\mathscr{A}r$ and the following:

$\mathscr{L}$, the class of all *l*-groups,

$\mathscr{A}$, the class of all abelian *l*-groups.

$\mathscr{L}$, $\mathscr{A}$ and $\mathscr{A}r$ are all sub-product radical classes of *l*-groups.

In 1963 and 1965, E. C. Weinberg initially considered the $\mathscr{A}$-free *l*-group generated by a po-group $[G, P]$. He has given a necessary and sufficient condition for existence and a simple description of $\mathscr{F}_{\mathscr{A}}([G, P])$ as follows:

PROPOSITION 2.2. [17, 18]. *Let* $[G, P]$ *be a torsion-free abelian* po-*group.*

(1) *There exists an $\mathscr{A}$-free l-group* $\mathscr{F}_{\mathscr{A}}([G, P])$ *generated by* $[G, P]$ *if and only if there exists a* po-*group isomorphism of* $[G, P]$ *into an abelian l-group, if and only if* P *is semi-closed.*

(2) *Let* $\mathscr{P}$ *be the set of all total orders* T *of* G *such that* $P \subseteq T$. *Then* $\mathscr{F}_{\mathscr{A}}([G, P])$ *is*

the sublattice of the direct product $\prod_{T\in\mathscr{P}} [G, T]$ *which is generated by the long constants* $\langle g\rangle$ $(g \in G)$.

The elements of $\mathscr{F}_{\mathscr{A}}([G, P])$ *have the form*

$$x = \bigvee_{i\in I} \bigwedge_{j\in J} \langle x_{ij}\rangle$$

where I *and* J *are both finite and* $x_{ij} \in G$ $(i \in I, j \in J)$.

In 1970, P. Conrad generalized Weiberg's result.

PROPOSITION 2.3 [6]. *Let* $[G, P]$ *be a torsion-free* po-*group.*

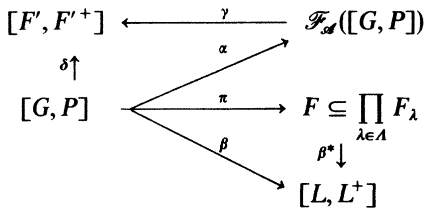
(1) *There exists an* $\mathscr{L}$*-free l-group* $\mathscr{F}_{\mathscr{L}}([G, P])$ *generated by* $[G, P]$ *if and only if there exists a* po-*group isomorphism of* $[G, P]$ *into an l-group, if and only if* P *is the intersection of right orders on* G.

(2) *Suppose that* $P = \bigcap_{\lambda\in\Lambda} P_\lambda$ *where* $\{P_\lambda \mid \lambda \in \Lambda\}$ *is the set of all right orders of* G *such that* $P_\lambda \supseteq P$. *If* G_λ *is* G *with one such right order, then denote by* $A(G_\lambda)$ *the l-group of order preserving permutations of* G_λ. *Each* $x \in G$ *corresponds to an element* ρ_x *of* $A(G_\lambda)$ *defined by* $\rho_x g = g + x$. *Then* $\mathscr{F}_{\mathscr{L}}([G, P])$ *is the sublattice of the direct product* $\prod_{\lambda\in\Lambda} A(G_\lambda)$ *which is generated by the long constants* $\langle g\rangle (g \in G)$.

In this section we will discuss the $\mathscr{A}r$-free l-group $\mathscr{F}_{\mathscr{A}r}([G, P])$ generated by a po-group $[G, P]$. Because $\mathscr{A}r$ is a sub-product radical class of l-groups, by Grätzer's existence theorem on a free algebra generated by a partial algebra (see Theorem 28.2 of [10]) we have

THEOREM 2.4. *There exists an* $\mathscr{A}r$*-free l-group* $\mathscr{F}_{\mathscr{A}r}([G, P])$ *generated by a* po-*group* $[G, P]$ *if and only if* $[G, P]$ *is* po-*group isomorphic into an archimedean l-group.*

Now suppose that a po-group $[G, P]$ is po-group isomorphic into an archimedean l-group $[F', F'^{+}]$ with the po-group isomorphism δ. Thus $[G, P]$ must be torsion-free abelian and semi-closed. By Proposition 2.2(1) there exists the $\mathscr{A}$-free l-group $\mathscr{F}_{\mathscr{A}}([G, P])$ generated by $[G, P]$ with the po-group isomorphism α of $[G, P]$ into $\mathscr{F}_{\mathscr{A}}([G, P])$. By definition 2.1 there exists an l-homomorphism γ from $\mathscr{F}_{\mathscr{A}}([G, P])$ into F' such that $\gamma\alpha = \beta$. Let $D = \{F_\lambda \mid \lambda \in \Lambda\}$ be the set of all archimedean l-homomorphism images of $\mathscr{F}_{\mathscr{A}}([G, P])$ with the l-homomorphism β_λ. Thus $\gamma\mathscr{F}_{\mathscr{A}}([G, P]) \in D$ and D is not empty. For each $\lambda \in \Lambda$, $\gamma_\lambda\alpha$ is a po-group homomorphism of $[G, P]$ into F_λ. The direct product $\prod_{\lambda\in\Lambda} F_\lambda$ is an archimedean l-group. Let π be the natural map of the po-group G



onto the subgroup G' of long constants of $\prod_{\lambda\in\Lambda} F_\lambda$. That is, $\pi(g) = (\cdots, \gamma_\lambda\alpha(g), \cdots)$ for $g \in G$. Because $\gamma\alpha = \delta$ is a po-group isomorphism, π is a po-group isomorphism of G onto G'. Let F be the sublattice of $\prod_{\lambda\in\Lambda} F_\lambda$ generated by G'. For each $g \in G$, let $g' = \pi(g)$ denote the long constant of G'. Since $\prod_{\lambda\in\Lambda} F_\lambda$ is a distributive lattice, the sublattice generated by all g' is

$$F = \left\{ \bigvee_{i\in I} \bigwedge_{j\in J} g'_{ij} \mid g_{ij} \in G, I \text{ and } J \text{ finite} \right\}.$$

Suppose that β is a po-group homomorphism of $[G, P]$ into an archimedean l-group $[L, L^+]$. Then there exists an l-homomorphism γ' of $\mathscr{F}_{\mathscr{A}}([G, P])$ into $[L, L^+]$ such that $\gamma'\alpha = \beta$. So $\gamma'\mathscr{F}_{\mathscr{A}}([G, P]) \in D$. Now we extend β to F as follows:

$$\beta^*\left(\bigvee_{i\in I} \bigwedge_{j\in J} g'_{ij}\right) = \bigvee_{i\in I} \bigwedge_{j\in J} \beta(g_{ij}).$$

To see that β^* is well defined, suppose that

$$\bigvee_{i\in I} \bigwedge_{j\in J} \beta(g_{ij}) \neq \bigvee_{m\in M} \bigwedge_{n\in N} \beta(h_{mn}).$$

Then we have

$$\bigvee_{i\in I} \bigwedge_{j\in J} \bigvee_{m\in M} \bigvee_{f\in N^M} \beta(g_{ij} - h_{mf(m)}) \neq 0$$

in $[L, L^+]$. Because $\gamma'\mathscr{F}_{\mathscr{A}}([G, P]) \in D$,

$$\bigvee_{i\in I} \bigwedge_{j\in J} \bigvee_{m\in M} \bigvee_{f\in N^M} \beta(g'_{ij} - h'_{mf(m)}) \neq 0$$

in F. That is, we have

$$\bigvee_{i\in I} \bigwedge_{j\in J} g'_{ij} \neq \bigvee_{m\in M} \bigwedge_{n\in N} h'_{mn}$$

in F. Therefore β^* is single valued.

That β^* is a lattice homomorphism is an immediate consequence of the fact that L is a distributive lattice. Now consider $g = \bigvee_{i\in I}\bigwedge_{j\in J} g'_{ij}$ and $h = \bigvee_{m\in M}\bigwedge_{n\in N} h'_{mn}$ in F.

$$\begin{aligned}\beta^*(g - h) &= \bigvee_{i\in I}\bigwedge_{j\in J}\bigwedge_{m\in M}\bigvee_{f\in N^M} (g_{ij} - h_{mf(m)})' \\ &= \bigvee_{i\in I}\bigwedge_{j\in J}\bigwedge_{m\in M}\bigvee_{f\in N^M} \beta(g_{ij} - h_{mf(m)}) \\ &= \bigvee_{i\in I}\bigwedge_{j\in J}\beta(g_{ij}) - \bigvee_{m\in M}\bigwedge_{n\in N}\beta(h_{mn}) \\ &= \beta^*(g) - \beta^*(h).\end{aligned}$$

Hence β^* is an l-homomorphism of F into L and $\beta^*\pi = \beta$.

The above discussion proves the following theorem.

THEOREM 2.5. *Suppose that a* po-*group* $[G, P]$ *is* po-*group isomorphic into an archimedean l-group. Then the* $\mathscr{A}r$*-free l-group* $\mathscr{F}_{\mathscr{A}r}([G, P])$ *generated by* $[G, P]$ *is the sublattice F of the direct product* $\prod_{\lambda\in\Lambda} F_\lambda$ *which is generated by the long constants* g' $(g \in G)$ *where* $\{F_\lambda \mid \lambda \in \Lambda\}$ *are all archimedean l-homomorphic images of the* $\mathscr{A}$*-free l-group* $\mathscr{F}_{\mathscr{A}}([G, P])$ *generated by* $[G, P]$.

NOTE. Suppose that a po-group $[G, P]$ is po-group isomorphic into an archimedean l-group. By Proposition 2.3 there exists an $\mathscr{L}$-free l-group $\mathscr{F}_{\mathscr{L}}([G, P])$ generated by $[G, P]$. If we take $\mathscr{F}_{\mathscr{L}}([G, P])$ instead of $\mathscr{F}_{\mathscr{A}}([G, P])$ in the above discussion, we obtain another description of $\mathscr{F}_{\mathscr{A}r}([G, P])$.

Let $\mathscr{U}$ be a class of algebras and X be a nonempty set. The algebra $\mathscr{F}_{\mathscr{U}}(X)$ is called the $\mathscr{U}$-free algebra on X if X generates $\mathscr{F}_{\mathscr{U}}(X)$ as an algebra, and whenever $L \in \mathscr{U}$ and λ: $X \to L$ is a map, then there exists a homomorphism σ: $\mathscr{F}_{\mathscr{U}}(X) \to L$ which extends λ. By Birkhoff's Theorem ([4]) there exists a $\mathscr{U}$-free algebra $\mathscr{F}_{\mathscr{U}}(X)$ on any nonempty set X if $\mathscr{U}$ is closed under subalgebras and direct products. Let $\mathscr{U}$ be a class of l-groups and X be a nonempty set with $|X| = \alpha$. Then the $\mathscr{U}$-free l-group $\mathscr{F}_{\mathscr{U}}(X)$ on X is said to be of rank α. We can construct the $\mathscr{U}$-free l-group $\mathscr{F}_{\mathscr{U}}(X)$ on X using the $\mathscr{U}$-free l-group generated by a trivially ordered group. Let $\mathscr{U}$ be a class of l-groups which is closed under l-subgroups and direct products. We denote by $\mathscr{G}(\mathscr{U})$ the class of all groups that can be embedded (as subgroups) into the members of $\mathscr{U}$. It is clear that $\mathscr{G}(\mathscr{U})$ is closed under subgroups and direct products.

PROPOSITION 2.6. *Let* $\mathscr{U}$ *be a class of l-groups which is closed under l-subgroups and direct products and X be a nonempty set. The* $\mathscr{U}$*-free l-group* $\mathscr{F}_{\mathscr{U}}(X)$ *on X is the* $\mathscr{U}$*-free l-group generated by the* $\mathscr{G}(\mathscr{U})$*free group on X with trivial order.*

PROOF. By Birkhoff's Theorem there exists the $\mathscr{G}(\mathscr{U})$-free group $\mathscr{F}_{\mathscr{G}(\mathscr{U})}(X)$ on X. $\mathscr{F}_{\mathscr{G}(\mathscr{U})}(X) \in \mathscr{G}(\mathscr{U})$ means $\mathscr{F}_{\mathscr{G}(\mathscr{U})}(X)$ can be embedded (as a subgroup) into a member of $\mathscr{U}$. By Theorem 28.2 of [10] there exists a $\mathscr{U}$-free l-group $\mathscr{F}_{\mathscr{U}}([\mathscr{F}_{\mathscr{G}(\mathscr{U})}(X), \{0\}])$ generated by the trivially ordered $\mathscr{G}(\mathscr{U})$-free group $\mathscr{F}_{\mathscr{G}(\mathscr{U})}(X)$. Now any map from X into an l-group $L \in \mathscr{U}$ can be extended to a group homomorphism of $\mathscr{F}_{\mathscr{G}(\mathscr{U})}(X)$ into L and hence to an l-homomorphism of $\mathscr{F}_{\mathscr{G}(\mathscr{U})}(X)$ into L and hence to an l-homomorphism of $\mathscr{F}_{\mathscr{U}}([\mathscr{F}_{\mathscr{G}(\mathscr{U})}(X), \{0\}])$ into L.

Theorem 2.7 of [14] is a special case of the above Proposition 2.6. The following theorem is a consequence of Proposition 2.6.

THEOREM 2.7. *Let X be a nonempty set. The $\mathscr{A}r$-free l-group $\mathscr{F}_{\mathscr{A}r}(X)$ on X is the $\mathscr{A}r$-free l-group generated by the $\mathscr{G}(\mathscr{A}r)$-free group $\mathscr{F}_{\mathscr{G}(\mathscr{A}r)}(X)$ with trivial order.*

$$X \to \mathscr{F}_{\mathscr{G}(\mathscr{A}r)}(X) \to \mathscr{F}_{\mathscr{A}r}(X).$$

PROPOSITION 2.8. *Suppose that $\mathscr{F}_{\mathscr{A}r}([G, P_1])$ and $\mathscr{F}_{\mathscr{A}r}([G, P_2])$ are the $\mathscr{A}r$-free l-groups generated by* po-*group $[G, P_1]$ and $[G, P_2]$, respectively. If $P_1 \subseteq P_1$. Then $\mathscr{F}_{\mathscr{A}r}([G, P_2])$ is an l-homomorphic image of $\mathscr{F}_{\mathscr{A}r}([G, P_1])$.*

PROOF. $[G, P_2]$ can be embedded into $\mathscr{F}_{\mathscr{A}r}([G, P_2])$ as a po-group and G generates $\mathscr{F}_{\mathscr{A}r}([G, P_2])$. So $[G, P_1]$ is also embedded into $\mathscr{F}_{\mathscr{A}r}([G, P_2])$ as a po-group. Hence there exists an l-homomorphism φ from $\mathscr{F}_{\mathscr{A}r}([G, P_1])$ into $\mathscr{F}_{\mathscr{A}r}([G, P_2])$. But $[G, P_1]$ can be embedded into $\mathscr{F}_{\mathscr{A}r}([G, P_1])$ as a po-group and G generates $\mathscr{F}_{\mathscr{A}r}([G, P_1])$. Therefore φ is onto $\mathscr{F}_{\mathscr{A}r}([G, P_2])$.

3. The Relation between $\mathscr{F}_{\mathscr{A}}([G, P])$ and $\mathscr{F}_{\mathscr{A}r}([G, P])$.

In [6] P. Conrad has given the relation between the $\mathscr{L}$-free l-group $\mathscr{F}_{\mathscr{L}}(X)$ and the $\mathscr{A}$-free l-group $\mathscr{F}_{\mathscr{A}}(X)$ on a nonempty set X. Let Y be the l-ideal generated by the commutator subgroup $[\mathscr{F}_{\mathscr{L}}(X), \mathscr{F}_{\mathscr{L}}(X)]$. Then $\mathscr{F}_{\mathscr{A}}(X) \cong \mathscr{F}_{\mathscr{L}}(X)/Y$.

In this section we will give the relation between $\mathscr{F}_{\mathscr{A}}([G, P])$ and $\mathscr{F}_{\mathscr{A}r}([G, P])$. Clearly, if $\mathscr{F}_{\mathscr{A}}([G, P])$ is archimedean then $\mathscr{F}_{\mathscr{A}}([G, P]) \cong \mathscr{F}_{\mathscr{A}r}([G, P])$. We will give a necessary and sufficient condition in which $\mathscr{F}_{\mathscr{A}}([G, P])$ is archimedean. First we need some concepts. Let $[G, P]$ be a torsion free abelian po-group and S be a nonempty subset of G. S is said to be positively independent if for any finite subset $\{x_1, \cdots, x_k\}$ of S and non-negative integers $\{\lambda_1, \ldots, \lambda_k\}$, $\sum_{i=1}^{k} \lambda_i x_i \in -P$ only if $\lambda_i = 0\,(i = 1, \ldots, k)$. There exists a total order P_1 of G such that $P_1 \supseteq P \cup S$ if and only if S is positively independent. Let $x = \bigvee_{i \in I} \bigwedge_{j \in J} \langle x_{ij} \rangle \in \mathscr{F}_{\mathscr{A}}([G, P])$. Then $x \nleq 0$ if and only if for some $i \in I$ the set $\{x_{ij} \mid j \in J\}$ is positively independent [3].

A po-group $[G, P]$ is said to be strong uniformly archimedean if, given $u \in G$ and a positively independent subset $\{v_1, \ldots, v_k\}$ of G, there exists $n \in N$ such that if

$\lambda_1, \cdots, \lambda_k$ are non-negative integers and $\sum_{i=1}^{k} \lambda_i \geqq mn$ with $m \in N$, then $\sum_{i=1}^{k} \lambda_i v_i \nleqq mu$.

THEOREM 3.1. *The $\mathscr{A}$-free l-group $\mathscr{F}_{\mathscr{A}}([G, P])$ generated by a* po-*group $[G, P]$ is archimedean if and only if $[G, P]$ is strong uniformly archimedean.*

PROOF. Necessity. Suppose that $u \in G$ and $\{v_1, \ldots, v_k\}$ is a positively independent subset of G. Then, $\langle v_1\rangle^+ \wedge \cdots \wedge \langle v_k\rangle^+ \neq 0$ in $\mathscr{F}_{\mathscr{A}}([G, P])$. Since $\mathscr{F}_{\mathscr{A}}([G, P])$ is archimedean, there exists $n \in N$ such that

$$n(\langle v_1\rangle \wedge \cdots \wedge \langle v_k\rangle)^+ = n(\langle v_1\rangle^+ \wedge \cdots \wedge \langle v_k\rangle^+) \nleqq \langle u\rangle^+.$$

It follows that if $\lambda \geqq n$, $\lambda(\langle v_1\rangle \wedge \cdots \wedge \langle v_k\rangle) \nleqq \langle u\rangle$. Now if $\lambda_1, \cdots, \lambda_k$ are non-negative integers and $\sum_{i=1}^{k} \lambda_i \geqq mn$ with $m \in N$, then we have

$$\sum_{i=1}^{k} \lambda_i \langle v_i\rangle \geqq \left(\sum_{i=1}^{k} \lambda_i\right)(\langle v_1\rangle \wedge \cdots \wedge \langle v_k\rangle) \nleqq m\langle u\rangle,$$

because P is semi-closed and $mn(\langle v_1\rangle \wedge \cdots \wedge \langle v_k\rangle) \leqq \left(\sum_{i=1}^{k} \lambda_i\right)(\langle v_1\rangle \wedge \cdots \wedge \langle v_k\rangle) \leqq m\langle u\rangle$ would imply $n(\langle v_1\rangle \wedge \cdots \wedge \langle v_k\rangle) \leqq \langle u\rangle$, a contradiction. Hence we have $\sum_{i=1}^{k} \lambda_i v_i \nleqq mu$ in $[G, P]$.

Sufficiency. It follows from Proposition 1.4 that it suffices to show that $\langle g\rangle$ is generally archimedean in $\mathscr{F}_{\mathscr{A}}([G, P])$ for each $g \in G$. And because G is a group and $g^- = (-g) \vee 0$, it suffices to show that g^+ is archimedean in $\mathscr{F}_{\mathscr{A}}([G, P])$ for each $g \in G$. Let $g \in G$ and $0 < x = \bigvee_{i \in I} \bigwedge_{j \in J} \langle x_{ij}\rangle \in \mathscr{F}_{\mathscr{A}}([G, P])$ where $x_{ij} \in G$. We must show there exists $n \in N$ such that $nx \nleqq g^+$. Since $x > 0$, the set $\{x_{ij} \mid j \in J\}$ is positively independent for some i. It suffices to show that there exists $n \in N$ such that $n(\bigwedge_{j \in J} \langle x_{ij}\rangle) \nleqq \langle g\rangle \vee 0$. And so it suffices to show that if $\{v_1, \cdots, v_k\}$ is a positively independent subset of G and $g \in G$, then there exists $n \in N$ and a total order T of G such that $T \supseteq P$, $v_i \in T$ and $nv_i - g \in T$ $(i = 1, \cdots, k)$. Then, lifting the identity map of $[G, P]$ onto $[G, T]$ to an l-homomorphism of $\mathscr{F}_{\mathscr{A}}([G, P])$ onto $[G, T]$ we would have $\bigwedge_{i=1}^{k} [(nv_i - g) \wedge nv_i] \nleqq 0$, and so $n\left(\bigwedge_{i=1}^{k} v_i\right) \nleqq g \vee 0$.

It therefore suffices to show that there exists $n \in N$ so that the set

$$\{v_i \mid i = 1, \ldots, k\} \cup \{nv_i - g \mid i = 1, \ldots, k\}$$

is positively independent. Because $[G, P]$ is strong uniformly archimedean, there exists $n \in N$ such that if $\lambda_1, \ldots, \lambda_k$ are non-negative integers and $\sum_{i=1}^{k} \lambda_i \geqq mn$ with $m \in N$, then $\sum_{i=1}^{k} \lambda_i v_i \nleqq mg$. Suppose that $\mu_1, \ldots, \mu_k$ and $v_1, \ldots, v_k$ are all non-negative integers and

$$\sum_{i=1}^{k} \mu_i v_i + \sum_{i=1}^{k} v_i(nv_i - g) \in -P.$$

Then $\sum_{i=1}^{k} (\mu_i + nv_i)v_i \leqq \left(\sum_{i=1}^{k} v_i\right) g$ which contradicts the choice of n unless all v_i are zero and then contradicts positive independence of the v_i unless all μ_i are zero. Thus $\{v_i \mid i = 1, \ldots, k\} \cup \{nv_i - g \mid i = 1, \ldots, k\}$ is positively independent.

COROLLARY 3.2. *Suppose that a* po-*group* $[G, P]$ *is* po-*group isomorphic into an archimedean l-group. Then* $\mathscr{F}_{\mathscr{A}}([G, P]) \cong \mathscr{F}_{\mathscr{A}r}([G, P])$ *if and only if* $[G, P]$ *is strong uniformly archimedean.*

Let G be a group. A nonempty subset S of G is said to be indepedent if for any finite subset $\{x_1, \ldots, x_k\}$ of S and non-negative integers $\{\lambda_1, \ldots, \lambda_k\}$, $\sum_{i=1}^{k} \lambda_i x_i = 0$ only if $\lambda_i = 0$ $(i = 1, \cdots, k)$. Clearly, S is independent in G if and only if S is positively independent in the po-group $[G, \{0\}]$ with the trivial order. Let G be a torsion-free and abelian group. Weinberg has proved that the $\mathscr{A}$-free l-group $\mathscr{F}_{\mathscr{A}}([G, \{0\}])$ is archimedean (Corollary 3.4 of [17]). From this we get

COROLLARY 3.3. *Suppose that G is a torsion-free and abelian group. Given* $u \in G$ *and an independent subset* $\{v_1, \ldots, v_k\}$ *of G, then there exists* $n \in N$ *such that if* $\lambda_1, \ldots, \lambda_k$ *are non-negative integers and* $\sum_{i=1}^{k} \lambda_i \geqq mn$ *with* $m \in N$, $\sum_{i=1}^{k} \lambda_i v_i \neq mu$.

4. Some properties of an archimedean *l*-group.

In order to discuss properties of $\mathscr{F}_{\mathscr{A}r}([G, P])$ we need to know some properties of an Archimedean l-group. First we introduce some concepts. Let $\{G_\alpha \mid \alpha \in A\}$ be a system of l-groups. For $g \in \prod_{\alpha \in A} G_\alpha$, we denote by g_α the α component of g. An l-group G is said to be an ideal subdirect sum of l-groups G_α, in symbol $G \subseteq^* \prod_{\alpha \in A} G_\alpha$, if G is a subdirect sum of G_α and G is an l-ideal of $\prod_{\alpha \in A} G_\alpha$. An l-group G is said to be a completely subdirect sum, if G is an l-subgroup of $\prod_{\alpha \in A} G_\alpha$ and $\sum_{\alpha \in A} G_\alpha \subseteq G$. We use the symbol $\subseteq'$ to denote subdirect sum. Let G be an l-group. We denote by vG the least cardinal α such that $|A| \leqq \alpha$ for each bounded disjoint subset A of G. G is said to be v-homogeneous if $vH = vG$ for any convex l-subgroup $H \neq 0$ of G. A v-homogeneous l-group G is said to be v-homogeneous of α type if $vG = \alpha$. By Theorem 3.7 of [11] it is easy to verify the following lemma. The proof is left to the reader.

LEMMA 4.1. *Any complete l-group is l-isomorphic to an ideal subdirect sum of complete v-homogeneous l-groups.*

By using 4.3 of [11] it is easy to verify that if an l-group G is *v-homogeneous and non-totally ordered, then* $vG \geqq \aleph_0$. It is well known that any non-zero complete totally ordered group is l-isomorphic to a real group R or an integer group Z. So from Lemma 4.1 we obtain the structure theorem of a complete l-group.

THEOREM 4.2. *Any complete l-group G is l-isomorphic to an ideal subdirect sum of real groups, integer groups and complete v-homogeneous l-groups of* $\aleph_i$ *type* $(i \geqq 0)$.

THEOREM 4.3. *Let G be an archimedean v-homogenous l-group of* $\aleph_i$ *type. Then G has the following properties:*

(1) *G has no basic element.*

(2) *G has no basis.*

(3) *The radical* $R(G) = G$.

(4) *G is not completely distributive.*

(5) *The distributive radical* $D(G) = G$.

Moreover, every non-trivial convex l-subgroup of G enjoys these same five properties.

PROOF. By Theorems 5.4 and 5.10 of [5] we need only to show (1). For any $0 < g \in G$, $vG(g) = \aleph_i > 1$. So $G(g)$ is not totally ordered, and $[0, g]$ is also not totally ordered by 4.3 of [11].

An l-group G is said to be continuous, if for any $0 < x \in G$ we have $x = x_1 + x_2$ and $x_1 \wedge x_2 = 0$ where $x_1 \neq 0, x_2 \neq 0$.

LEMMA 4.4 (Lemma 2.4 of [20]). *A complete l-group G is continuous if and only if G has no basic element.*

An l-group G is said to be projectable if each of its principal polars is a cardinal summand. The following lema is clear.

LEMMA 4.5. *Let G be a projectable (in particular, complete) and non-totally ordered l-group. Then G is directly decomposable.*

An l-group G is said to be ideal subdirectly irreducible if G cannot be expressed as an ideal subdirect sum of l-groups.

LEMMA 4.6 (Lemma 2.6 of [20]). *A complete l-group G is directly indecomposable if and only if G is ideal subdirectly irreducible.*

LEMMA 4.7 (Lemma 2.7 of [20]). *An archimedean l-group G is subdirectly irreducible if and only if the Dedekind completion $G^\wedge$ of G is ideal subdirectly irreducible.*

Now from Lemma 4.4, Lemma 4.5 and Lemma 4.6 we have

THEOREM 4.8. *Let G be a complete v-homogeneous l-groups of $\aleph_i$ type. Then*
(1) *G is continuous.*
(2) *G is directly decomposable.*
(3) *G is not ideal subdirectly irreducible.*
Moreover, every nontrivial convex l-subgroup of G enjoys these same three properties.

From Lemma 4.7 and Theorem 4.8 we obtain

COROLLARY 4.9. *An archimedean v-homogeneous l-group of $\aleph_i$ type is not subdirectly irreducible.*

A subset D in a lattice L is called a d-set if there exists $x \in L$ such that $d_1 \wedge d_2 = x$ for any pair of distinct elements of D and $d > x$ for each $d \in D$. We denote by $w[a,b]$ the least cardinal α such that $|D| \leqq \alpha$ for each d-set D of $[a,b]$.

LEMMA 4.10. *Let G be a v-homogeneous l-group of $\aleph_i$ type and be a dense l-subgroup of an l-group G′. Then G′ is also a v-homogeneous l-group of $\aleph_i$ type.*

PROOF. Suppose that H is an arbitrary convex l-subgroup of G. Let $H = H' \cap G$. Then H is dense in H' and H is a convex l-subgroup of G. We will prove that $vH' = vH$. It is clear that $vH' \geqq vH$. Let $\{x'_\alpha \in H'^+ \mid \alpha \in A\}$ be a disjoint of H' with an upper bound x'. Then there exists $x_\alpha \in H$ such that $0 < x_\alpha \leqq x'_\alpha$ for each $\alpha \in A$ and there exists $x \in H$ such that $0 < x \leqq x'$. Hence $\{x_\alpha \wedge x \mid \alpha \in A\}$ is a disjoint subset of H with an upper bound x. Hence $|A| \leqq \aleph_i$ and $vH' \leqq vH$. Therefore

$$vH' = vH = vG = \aleph_i$$

and so G' is a v-homogeneous l-group of $\aleph_i$ type.

From Theorem 2.6 and Theorem 5.2 of [8] and the above Lemma 4.10 we get

THEOREM 4.11. *Let G be an archimedean v-homogeneous l-group of $\aleph_i$ type. Then the Dedekind completion $G^\wedge$ of G and the lateral completion G^L of G are also v-homogeneous l-groups of $\aleph_i$ type.*

LEMMA 4.12. *Let G be a v-homogeneous l-group of $\aleph_i$ type and $\{x_\alpha \mid \alpha \in A\}$ be a disjoint subset in G. Then $|A| \leqq \aleph_i$.*

PROOF. Let G^L be the lateral completion of G. By Theorem 4.11 G^L is also v-homogeneous of $\aleph_i$ type. Let x be the least upper bound of a disjoint subset $\{x_\alpha \mid \alpha \in A\}$ of G in G^L. So $\{x_\alpha \mid \alpha \in A\}$ is a bounded disjoint subset in G^L. Therefore $|A| \leqq \aleph_i$.

THEOREM 4.13. *Let G be a v-homogeneous l-group of $\aleph_i$ type. Then the divisible hull G^d of G is also a v-homogeneous l-group of $\aleph_i$ type.*

PROOF. Let P be any nontrivial convex l-subgroup of G^d. For any $0 \neq x \in P$ there exists $n \in N$ such that $0 \neq nx \in P \cap G$. So $P \cap G$ is also a nontrivial convex l-subgroup of G. It is clear that $vP \geqq v(P \cap G) = \aleph_i$. On the other hand, $P = \left\{\frac{1}{n_g} \middle| g \in G \cap P, n \in N\right\}$. So if $\{c_j \mid j \in J\}$ is a bounded disjoint subset in P, let $c_j = \frac{1}{n_j} g_j$ $(j \in J,\ g_j \in G \cap P,\ n_j \in N)$. By the Bernau representation of an archimedean l-group [2] we see that $c_j \wedge c_{j'} = 0$ if and only if $g_j \wedge g_{j'} = 0\ (j \neq j')$. So $\{g_j \mid j \in J\}$ is a disjoint subset in $G \cap P$. By Lemma 4.12, $|J| \leqq \aleph_i$. Hence $vP \leqq \aleph_i$. Therefore $vP = \aleph_i$.

Now we turn to an archimedean l-group. In [19] we proved the following result.

LEMMA 4.14. *An l-group G is archimedean if and only if G is l-isomorphic to a subdirect sum of subgroups of reals and archimedean v-homogeneous l-groups of $\aleph_i$ type.*

Suppose that G is a subdirect sum of subgroups of reals and v-homogeneous l-groups of $\aleph_i$ type, $G \subseteq' \prod_{\delta \in \Delta} T_\delta$. Let $\Delta_1 = \{\delta \in \Delta \mid T_\delta$ is a subgroup of reals$\}$. If $\sum_{\delta \in \Delta_1} T_\delta \subseteq G$, G is said to be a semicomplete subdirect sum of subgroups of reals and v-homogeneous l-groups of $\aleph_i$ type, in symbols $\sum_{\delta \in \Delta_1 \subseteq \Delta} T_\delta \subseteq G \subseteq' \prod_{\delta \in \Delta} T_\delta$.

THEOREM 4.15. (Theorem 4.7 of [19]). *An l-group G is archimedean if and only if G is l-isomorphic to a semicomplete subdirect sum of subgroups of reals and archimedean v-homogeneous l-groups of $\aleph_i$ type.*

5. Properties of $\mathscr{A}_\alpha$.

We denote by $\mathscr{A}_\alpha$ the $\mathscr{A}$-free l-group $\mathscr{F}_{\mathscr{A}}(X)$ of rank α. By Proposition 2.6 $\mathscr{A}_\alpha$ is the $\mathscr{A}$-free l-group $\mathscr{F}_{\mathscr{A}}([G, \{0\}])$ generated by $\mathscr{G}(\mathscr{A})$-free group G with trivial order. It follows from Corollary 3.4 of [17] that $\mathscr{A}_\alpha$ is archimedean. Hence $\mathscr{A}_\alpha \cong \mathscr{A}r_\alpha$. We have already known some properties of $\mathscr{A}_\alpha$. For example, $\mathscr{A}_\alpha$ is a subdirect sum of integers (Theorem 2.5 of [3]); $\mathscr{A}_\alpha(\alpha > 1)$ has a countably infinite disjoint subset but no uncountable disjoint subset (Theorem 6.2 of [16]); every infinite chain in $\mathscr{A}_\alpha$ must be countable (Theorem 5.1 of [15]); the word problem for $\mathscr{A}_\alpha$ is solvable (Theorem 2.11 of [14]); $\mathscr{A}_\alpha(\alpha > 1)$ has no singular elements (Theorem 2.8 of [3]). In this section we will give further properties of $\mathscr{A}_\alpha$ using the structure theorem of an archimedean l-group.

THEOREM 5.1. *$\mathscr{A}_\alpha(\alpha > 1)$ is an archimedean v-homogeneous l-group of $\aleph_0$ type.*

PROOF. Since $\mathscr{A}_\alpha$ is archimedean, by Theorem 4.15, without loss of generality, we have

$$\sum_{\delta\in\Delta_1\subseteq\Delta} T_\delta \subseteq \mathscr{A}_\alpha \subseteq' \prod_{\delta\in\Delta} T_\delta,$$

where each T_δ $(\delta\in\Delta_1)$ is a subgroup of reals and each T_δ $(\delta\in\Delta\backslash\Delta_1)$ is an archimedean v-homogeneous l-group of $\aleph_i$ type. By Theorem 3.5 of [3] (or Theorem 1 of [18]) $\mathscr{A}_\alpha(\alpha > 1)$ has no nontrivial direct summands. Hence $\Delta_1 = \emptyset$ and $\mathscr{A}_\alpha(\alpha > 1)$ is a subdirect sum of archimedean v-homogeneous l-groups of $\aleph_i$ type. Let

$$(1) \qquad \sum_{\delta\in\Delta'} T_\delta \subseteq \mathscr{A}_\alpha^\wedge \subseteq^* \prod_{\delta\in\Delta'} T_\delta^\wedge$$

$$\mathscr{A}_\alpha \subseteq' \prod_{\delta\in\Delta'} T_\delta,$$

where $\Delta' = \Delta\backslash\Delta_1$ and each $T_\delta\,(\delta\in\Delta')$ is an archimedean v-homogeneous l-groups of $\aleph_i$ type. For any $0 < x\in\mathscr{A}_\alpha$. We denote by $\mathscr{A}_\alpha(x)$ the convex l-subgroup in $\mathscr{A}_\alpha$ generated by x and $\mathscr{A}_\alpha^\wedge(x)$ the convex l-subgroup in $\mathscr{A}_\alpha^\wedge$ generated by x. By Theorem 2 of [18] we have

$$(2) \qquad v\mathscr{A}_\alpha(x) \leqq v\mathscr{A}_\alpha \leqq \aleph_0.$$

On the other hand, $\mathscr{A}_\alpha(x)$ is dense in $\mathscr{A}_\alpha^\wedge(x)$. If $\{x_\alpha \mid \alpha\in A\}$ is a disjoint subset with an upper bound x_0 in $\mathscr{A}_\alpha^\wedge(x)$. Then there exists $x_0'\in\mathscr{A}_\alpha(x)$ such that $0 < x_0' < x_0$. Put $x_\alpha' = x_\alpha \wedge x_0'$. Then $\{x_\alpha' \mid \alpha\in A\}$ is a disjoint subset with an upper bound x_0' in $\mathscr{A}_\alpha(x)$. Hence $v\mathscr{A}_\alpha^\wedge(x) \leqq v\mathscr{A}_\alpha(x)$. And it is clear that $v\mathscr{A}_\alpha(x) \leqq v\mathscr{A}_\alpha^\wedge(x)$. Thus,

$$(3) \qquad v\mathscr{A}_\alpha(x) = v\mathscr{A}_\alpha^\wedge(x).$$

For any $\delta_0\in\Delta'$, put $\bar{x}_{\delta_0} = (\ldots 0\ldots, x_{\delta_0}, \ldots 0\ldots)$. Then $\bar{x}_{\delta_0} \leqq x$. Since $vT_{\delta_0}(x_{\delta_0}) = vT_{\delta_0} \geqq \aleph_0$ where $T_{\delta_0}(x_{\delta_0})$ is the convex l-subgroup of T_{δ_0} generated by x_{δ_0}, there exists a disjoint subset $\{x^\beta \mid \beta\in B\}$ in $T_{\delta_0}(x_{\delta_0})$ such that $x^\beta \leqq x_{\delta_0}$ and $|B| \geqq \aleph_0$. Then $\bar{x}^\beta = (\ldots 0\ldots, x^\beta, \ldots 0\ldots)\in\mathscr{A}_\alpha^\wedge$ by (1) and $\{\bar{x}^\beta \mid \beta\in B\}$ is a disjoint subset with an upper bound $\bar{x}_{\delta_0}$ in $\mathscr{A}_\alpha^\wedge(\bar{x}_{\delta_0})$. Hence

(4) $$v\mathscr{A}_\alpha^{\hat{}}(x) = v\mathscr{A}_\alpha^{\hat{}}(\bar{x}_{\delta_0}) \geqq \aleph_0.$$

Combining (2), (3) and (4) we get $v\mathscr{A}_\alpha(x) = \aleph_0$ for any $0 < x \in \mathscr{A}_\alpha$. Now for any nontrivial convex l-subgroup K in $\mathscr{A}_\alpha$. Let $0 < x \in K$. Then

$$\aleph_0 = v\mathscr{A}_\alpha(x) \leqq vK \leqq v\mathscr{A}_\alpha \leqq \aleph_0.$$

Therefore $vK = \aleph_0$ and $\mathscr{A}_\alpha$ is a v-homogeneous l-group of $\aleph_0$ type.

By Theorem 4.3 and Theorem 5.1 we obtain

THEOREM 5.2. *$\mathscr{A}_\alpha(\alpha > 1)$ has the following properties:*
(1) $\mathscr{A}_\alpha$ has no basic element.
(2) $\mathscr{A}_\alpha$ has no basis.
(3) The radical $R(\mathscr{A}_\alpha) = \mathscr{A}_\alpha$.
(4) $\mathscr{A}_\alpha$ is not completely distributive.
(5) The distributive radical $D(\mathscr{A}_\alpha) = \mathscr{A}_\alpha$.
Moreover, every nontrivial convex l-subgroup of $\mathscr{A}_\alpha$ enjoys these same five properties.

By Theorem 3.6 of [7] and the above Theorem 4.11, Theorem 4.13 and Theorem 5.1 we have

THEOREM 5.3. (1) *The Dedekind completion $\mathscr{A}_\alpha^{\hat{}}$ of $\mathscr{A}_\alpha$ is a v-homogeneous l-group of $\aleph_0$ type.*
(2) The lateral completion $\mathscr{A}_\alpha^L$ of $\mathscr{A}_\alpha$ is a v-homogeneous l-group of $\aleph_0$ type.
(3) The divisible hull $\mathscr{A}_\alpha^d$ of $\mathscr{A}_\alpha$ is a v-homogeneous l-group of $\aleph_0$ type.
(4) The essential closure $\mathscr{A}_\alpha^e$ of $\mathscr{A}_\alpha$ is a v-homogeneous l-group of $\aleph_0$ type.

From Theorem 4.8 we get

THEOREM 5.4. *The Dedekind completion $\mathscr{A}_\alpha^{\hat{}}$ of $\mathscr{A}_\alpha$ has the following properties:*
(1) $\mathscr{A}_\alpha^{\hat{}}$ is continuous.
(2) $\mathscr{A}_\alpha^{\hat{}}$ is directly decomposable.
(3) $\mathscr{A}_\alpha^{\hat{}}$ is not ideal subdirectly irreducible.
(4) $\mathscr{A}_\alpha^{\hat{}}$ has a closed l-ideal.
Moreover, each nontrivial convex l-subgroup of $\mathscr{A}_\alpha^{\hat{}}$ enjoys these same four properties.

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