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A VARIATIONAL PRINCIPLE FOR THE HAUSDORFF DIMENSION OF FRACTAL SETS

C. D. CUTLER and L. OLSEN

Abstract.

Let $\mathscr{P}(E)$ denote the set of probability measures on a Borel set $E \subseteq \mathsf{R}^n$, and let $\underline{R}(\mu)$, $\bar{R}(\mu)$ denote respectively the lower and upper Rényi dimensions associated with a measure $\mu \in \mathscr{P}(E)$. We prove that the Hausdorff dimension $\dim(E)$ satisfies

$$\dim(E) \leqq \sup_{\mu \in \mathscr{P}(E)} \underline{R}(\mu)$$

while, if E is additionally bounded, the packing dimension $\operatorname{Dim}(E)$ satisfies

$$\operatorname{Dim}(E) \geqq \sup_{\mu \in \mathscr{P}(E)} \bar{R}(\mu).$$

As a consequence, for any bounded Borel set E satisfying Taylor's definition of a fractal (i.e. $\dim(E) = \operatorname{Dim}(E)$) we obtain the variational principle

$$\dim(E) = \operatorname{Dim}(E) = \sup_{\mu \in \mathscr{P}(E)} \underline{R}(\mu) = \sup_{\mu \in \mathscr{P}(E)} \bar{R}(\mu).$$

In addition we provide an example showing that the hypothesis "bounded" cannot be eliminated.

1. Introduction.

In recent papers on fractals attention has shifted from sets to measure, cf. [1, 2, 3, 4, 5, 6, 8, 9, 10, 12]. Thus it seems reasonable to make an attempt at finding a relation between the dimension of a fractal E and parameters connected with measures supported by E. Such relations have already been investigated, cf. in particular [14, Theorem 1 p. 62] and Young [18]. Our principal result states that if $E \subseteq \mathsf{R}^n$ is a bounded Borel set satisfying Taylor's definition of a fractal, i.e. the Hausdorff dimension $\dim(E)$ of E is equal to the packing dimension $\operatorname{Dim}(E)$ of E, cf. [15] and [16], then

$$\dim(E) = \operatorname{Dim}(E) = \sup_{\mu \in \mathscr{P}(E)} \underline{R}(\mu) = \sup_{\mu \in \mathscr{P}(E)} \bar{R}(\mu) \tag{1}$$

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where $\underline{R}(\mu)$ and $\bar{R}(\mu)$ denote, respectively, the lower and upper Rényi dimensions and $\mathscr{P}(E)$ is the family of all Borel probability measures on E.

Formula (1) is a variational principle – i.e. it establishes an equality between a number naturally connected with a space or a map (in this case dim E) and the supremum of certain numbers connected to a class of probability measures supported by E. It is well-known that variational principles play a major role in ergodic theory (cf. e.g. [17, Chapter 8-9]) since these principles yield a canonical way of choosing measures. Formula (1) yields in a similar way a canonical way of choosing measures – namely measures $\mu \in \mathscr{P}(E)$ such that $\underline{R}(\mu)$ and $\bar{R}(\mu)$ are close to $\dim(E)$ and $\operatorname{Dim}(E)$. It is interesting to note that our variational principle is formulated in terms of the Rényi dimension since generalised Rényi dimensions play an important part in so-called multifractal analysis, cf. e.g. Rand [13] and the references therein.

We begin in section 2 by collecting the relevant facts and setting the notation. Then in section 3 we derive some auxiliary inequalities and prove the variational principle contained in formula (1).

2. Preliminaries.

This section contains a survey of the fractal dimensions which we will consider.

Let (X, d) be a separable metric space, $E \subseteq X$ and $s \geqq 0$. Then the s-dimensional Hausdorff measure $\mathscr{H}^s(E)$ of E is defined by

$$\mathscr{H}^s(E) = \sup_{\delta>0} \inf \left\{ \sum_{i=1}^{\infty} (\operatorname{diam} E_i)^s \,\middle|\, E \subseteq \cup_{i=1}^{\infty} E_i,\ \operatorname{diam} E_i < \delta \text{ for all } i \in \mathbb{N} \right\}.$$

The Hausdorff dimension $\dim E$ of E is defined by

$$\dim E = \inf\{s \geqq 0 \mid \mathscr{H}^s(E) < \infty\} = \sup\{s \geqq 0 \mid \mathscr{H}^s(E) > 0\}.$$

The s-dimensional packing measure $\mathscr{P}^s(E)$ of E is defined in two stages. First put

$$\mathscr{P}_0^s(E) = \inf_{\delta>0} \sup \left\{ \sum_{i=1}^{\infty} (\operatorname{diam} B_i)^s \,\middle|\, B_i \cap B_j = \varnothing \text{ for } i \neq j \text{ and } B_i \text{ is a closed ball of radius at most } \delta \text{ with center in } E \text{ for all } i \in \mathbb{N} \right\}.$$

Then

$$\mathscr{P}^s(E) = \inf \left\{ \sum_{i=1}^{\infty} \mathscr{P}_0^s(E_i) \,\middle|\, E \subseteq \cup_{i=1}^{\infty} E_i \right\}.$$

The packing dimension $\operatorname{Dim} E$ of E is defined by

$$\operatorname{Dim} E = \inf\{s \geqq 0 \mid \mathscr{P}^s(E) < \infty\} = \sup\{s \geqq 0 \mid \mathscr{P}^s(E) > 0\}.$$

It is a well-known fact that $\dim E \leqq \operatorname{Dim} E$ for all $E \subseteq \mathsf{R}^n$, cf. [14].

Two other useful dimensions of a bounded set E are the upper and lower box dimensions. For each $\delta > 0$ let $N_\delta(E)$ be the least number of sets of diameter at most δ that cover E. Then the upper and lower box dimensions of E are defined by

$$\bar{C}(E) = \limsup_{\delta \to 0} \frac{\log N_\delta(E)}{-\log \delta}$$

and

$$\underline{C}(E) = \liminf_{\delta \to 0} \frac{\log N_\delta(E)}{-\log \delta}$$

respectively.

Let us introduce the Rényi dimension. Fix $\mu \in \mathscr{P}(X)$ and write

$$h_r(\mu) = \inf\left\{ -\sum_{i=1}^{\infty} \mu(E_i) \log \mu(E_i) \mid (E_i)_i \text{ is a countable Borel partition of } X \text{ and } \operatorname{diam} E_i \leqq r \right\}$$

for $r > 0$. Then the upper and lower Rényi dimensions of μ are defined by

$$\bar{R}(\mu) = \limsup_{r \to 0} -\frac{h_r(\mu)}{\log r}$$

and

$$\underline{R}(\mu) = \liminf_{r \to 0} -\frac{h_r(\mu)}{\log r}$$

respectively, (cf. [18]).

3. Inequalities and the Variational Principle.

We want to prove that

(2) $$\dim(E) \leqq \sup_{\mu \in \mathscr{P}(E)} \underline{R}(\mu)$$

for a Borel subset E of R^n, and

(3) $$\operatorname{Dim}(E) \geqq \sup_{\mu \in \mathscr{P}(E)} \bar{R}(\mu)$$

for a bounded Borel subset E of R^n. Both proofs are based on the following result:

THEOREM 1. *Let $E \subseteq \mathrm{R}^n$ be a Borel set. Then the following assertions hold:*
i)

$$\dim(E) = \sup_{\mu\in\mathscr{P}(E)} \left(\inf_{x\in E} \liminf_{r\searrow 0} \frac{\log\mu(B(x,r))}{\log r} \right).$$

ii) *If*

$$E \subseteq \left\{ x \mid \limsup_{r\searrow 0} \frac{\log\mu(B(x,r))}{\log r} \geqq \alpha \right\} \text{ and } \mu(E) > 0,$$

then

$$\operatorname{Dim}(E) \geqq \alpha.$$

PROOF. i) Follows easily from [14, Theorem 1]. ii) Follows from [14, Theorem 1], however see also Theorem 3.2 of [5].

We begin with three small technical lemmas

LEMMA 2. *Let μ be a Borel probability measure on R^n. Let E be a Borel set, $t \geqq 0$ and $\delta \in]0, 1[$. Suppose*

$$\log\mu(B(x,r)) \leqq t\log r$$

for all $x\in E$ and $r\in]0,\delta[$. Then

$$\underline{R}(\mu) \geqq \mu(E)t.$$

PROOF. Let $r\in]0,\delta[$ and $(E_i)_i$ be a partition of R^n such that $\operatorname{diam}(E_i) \leqq r$. Let $I = \{i \mid E_i \cap E \neq \varnothing\}$. If $i\in I$ then we can choose a point $x_i \in E_i \cap E$ such that $E_i \subseteq B(x_i, r)$, whence

$$\log\mu(E_i) \leqq \log\mu(B(x_i,r)) \leqq t\log r \text{ for } i\in I. \tag{4}$$

By (4) we have

$$-\sum_i \mu(E_i)\log\mu(E_i) \geqq -\sum_{i\in I}\mu(E_i)\log\mu(E_i) \geqq -\sum_{i\in I}\mu(E_i)t\log r$$

$$= -\mu\left(\bigcup_{i\in I} E_i\right)t\log r \geqq -\mu(E)\,t\log r.$$

Since the partition $(E_i)_i$ was arbitrary this inequality implies that

$$h_r(\mu) \geqq -\mu(E)t\log r \text{ for } r\in]0,\delta[$$

whence

$$\underline{R}(\mu) = \liminf_{r\to 0} -\frac{h_r(\mu)}{\log r} \geqq t\mu(E).$$

LEMMA 3. *Let $F \subseteq \mathrm{R}^n$ be a bounded Borel set and $r > 0$. Then there exists a finite collection $F_1, \ldots, F_m$ of disjoint Borel sets with* $\operatorname{diam}(F_i) \leqq r$ *such that $F \subseteq \cup_i F_i$ and such that for each i, there exists an $x_i \in F$ satisfying*

$$B(x_i, \tfrac{1}{4}r) \subseteq F_i.$$

PROOF. Construct a sequence of balls $B(x_1, \frac{1}{2}r)$, $B(x_2, \frac{1}{2}r), \ldots$ such that $x_i \in F$ and $d(x_i, x_j) > \frac{1}{2}r$ for $i \neq j$. Because F is totally bounded this process must terminate at some finite stage, giving balls $B(x_1, \frac{1}{2}r), \ldots, B(x_m, \frac{1}{2}r)$ such that any $x \in F$ must satisfy $\min_i d(x, x_i) \leqq \frac{1}{2}r$ (consequently $F \subseteq \cup_{i=1}^m B(x_i, \frac{1}{2}r)$). Note that the smaller balls $B(x_1, \frac{1}{4}r), \ldots, B(x_m, \frac{1}{4}r)$ are disjoint. Set

$$F_1 = B(x_1, \tfrac{1}{2}r) \setminus \bigcup_{j=2}^{m} B(x_j, \tfrac{1}{4}r)$$

$$F_i = B(x_i, \tfrac{1}{2}r) \setminus \left(\bigcup_{j=1}^{i-1} F_j \cup \bigcup_{j=i+1}^{m} B(x_j, \tfrac{1}{4}r) \right) \text{ for } i = 2, \ldots, m-1$$

$$F_m = B(x_m, \tfrac{1}{2}r) \setminus \bigcup_{j=1}^{m-1} F_i.$$

It is clear that the F_i's are disjoint, and since $B(x_1, \frac{1}{4}r), \ldots, B(x_m, \frac{1}{4}r)$ are disjoint we can conclude that $B(x_i, \frac{1}{4}r) \subseteq F_i$ and $F \subseteq \cup_i F_i$.

LEMMA 4. *Let $E \subseteq \mathrm{R}^n$ be a bounded Borel set and $\mu \in \mathscr{P}(E)$. Let $F \subseteq E$ be a Borel set, $t \geqq 0$ and $\delta \in]0, 1[$. Assume*

$$\log \mu(B(x, r)) \geqq t \log r$$

for all $x \in F$ and $0 < r < \delta$. Then

$$\bar{R}(\mu) \leqq t + \mu(E \setminus F) \bar{C}(E \setminus F).$$

PROOF. Let $r \in]0, \delta[$ and choose by Lemma 3 a finite pairwise disjoint covering $(F_1, \ldots, F_m)$ of F with $\operatorname{diam} F_i \leqq r$ and such that there exists points $x_i \in F$ for all i satisfying

$$B(x_i, \tfrac{1}{4}r) \subseteq F_i.$$

The set $E \setminus F$ can be covered by $N = N_r(E \setminus F)$ closed balls $B_1, \ldots, B_N$ of diameter at most r. Define $Q_1, \ldots, Q_N$ by

$$Q_1 = (B_1 \cap (E \setminus F)) \setminus \cup_j F_j$$
$$Q_i = (B_i \cap (E \setminus F)) \setminus (\cup_j F_j \cup \cup_{j=1}^{i-1} Q_j) \text{ for } i = 2, \ldots, N.$$

Then $F_1, \ldots, F_m, Q_1, \ldots, Q_N$ are disjoint sets of diameter not exceeding r, and

$$E = \cup_i (F_i \cap E) \cup \cup_i Q_i, \quad \cup_i Q_i \subseteq E \setminus F.$$

Hence

$$\begin{aligned}
h_r(\mu) &\leqq -\sum_{i=1}^{m} \mu(F_i \cap E)\log\mu(F_i \cap E) - \sum_{i=1}^{N} \mu(Q_i)\log\mu(Q_i) \\
&= -\sum_{i=1}^{m} \mu(F_i)\log\mu(F_i) - \sum_{i=1}^{N} \mu(Q_i)\log\mu(Q_i) \\
&\leqq -\sum_{i=1}^{m} \mu(F_i)\log\mu(B(x_i, \tfrac{1}{4}r)) - \sum_{i=1}^{N} \mu(Q_i)\log\mu(Q_i) \\
&\leqq -\sum_{i=1}^{m} \mu(F_i)t\log(\tfrac{1}{4}r) - \sum_{i=1}^{N} \mu(Q_i)\log\mu(Q_i) \\
&\leqq -t\log(\tfrac{1}{4}r) - \sum_{i=1}^{N} \mu(Q_i)\log\mu(Q_i).
\end{aligned}$$

We know that if $p_1, \ldots, p_k \geqq 0$ and $\sum_{i=1}^{k} p_i = s \in [0,1]$ then in fact $-\sum_{i=1}^{k} p_i \log p_i \leqq s\log k - s\log s \leqq s\log k + \frac{1}{e}$. Therefore

$$\begin{aligned}
h_r(\mu) &\leqq -t\log(\tfrac{1}{4}r) + \sum_{i=1}^{N} \mu(Q_i)\log N + \frac{1}{e} \\
&\leqq -t\log(\tfrac{1}{4}r) + \mu\left(\bigcup_{i=1}^{N} Q_i\right)\log N_r(E\setminus F) + \frac{1}{e} \\
&\leqq -t\log(\tfrac{1}{4}r) + \mu(E\setminus F)\log N_r(E\setminus F) + \frac{1}{e}
\end{aligned}$$

for $r < \delta$, whence

$$\begin{aligned}
\bar{R}(\mu) = \limsup_{r \downarrow 0} \frac{h_r(\mu)}{-\log r} &\leqq \limsup_{r \downarrow 0} \left(\frac{t\log(\frac{1}{4}r)}{\log r} + \mu(E\setminus F)\frac{\log N_r(E\setminus F)}{-\log r} - \frac{1}{e\log r} \right) \\
&\leqq t + \mu(E\setminus F)\bar{C}(E\setminus F).
\end{aligned}$$

We are now ready to prove (2) and (3).

PROPOSITION 5. *Let $E \subseteq \mathrm{R}^n$. Then the following assertions hold:*
i) *If E is a Borel set then*

$$\dim E \leqq \sup_{\mu \in \mathscr{P}(E)} \underline{R}(\mu).$$

ii) *If E is a bounded Borel set then*

$$\sup_{\mu \in \mathcal{P}(E)} \bar{R}(\mu) \leqq \operatorname{Dim} E.$$

PROOF. i) Let $t < \dim E$. Then Theorem 1 part i) implies that there exists a measure $\mu \in \mathcal{P}(E)$ such that

$$t < \liminf_{r \downarrow 0} \frac{\log \mu(\mathrm{B}(x, r))}{\log r} \text{ for all } x \in E. \tag{5}$$

Now put

$$E_m = \left\{ x \in E \,|\, \frac{\log \mu(B(x, r))}{\log r} > t \text{ for } 0 < r < \frac{1}{m} \right\}, \quad m \in \mathsf{N}.$$

Let $\varepsilon > 0$ and observe that (5) implies that $E_m \uparrow E$. We can thus choose an integer $N \in \mathsf{N}$ so $\mu(E_N) \geqq \mu(E) - \varepsilon = 1 - \varepsilon$. An application of Lemma 2 then yields

$$\sup_{\lambda \in \mathcal{P}(E)} \underline{R}(\lambda) \geqq \underline{R}(\mu) \geqq \mu(E_N) t \geqq (1 - \varepsilon) t$$

which proves the first part of the proposition since $t < \dim E$ and $\varepsilon > 0$ were arbitrary.

ii) Let $\mu \in \mathcal{P}(E)$ and $t > \operatorname{Dim}(E)$. Then Theorem 1 part ii) implies that

$$\limsup_{r \downarrow 0} \frac{\log \mu(B(x, r))}{\log r} \leqq \operatorname{Dim}(E) \quad \mu\text{-a.s.}$$

and we can thus choose a subset F of E with $\mu(F) = 1$ such that $\limsup_{r \downarrow 0} \dfrac{\log \mu(B(x, r))}{\log r} < t$ for all $x \in F$. Now put

$$F_m = \left\{ x \in F \,|\, \frac{\log \mu(B(x, r))}{\log r} < t \text{ for } 0 < r < \frac{1}{m} \right\}, \quad m \in \mathsf{N}.$$

An application of Lemma 4 then yields

$$\bar{R}(\mu) \leqq t + \mu(E \setminus F_m) \bar{C}(E) = t + \mu(F \setminus F_m) \bar{C}(E).$$

Since $F_m \uparrow F$ we conclude that $\bar{R}(\mu) \leqq t$. This completes the proof since both $\mu \in \mathcal{P}(E)$ and $t > \operatorname{Dim}(E)$ were arbitrary.

Proposition 5 immediately yields the following variational principle

PROPOSITION 6. *If $E \subseteq \mathsf{R}^n$ is a bounded Borel set satisfying* $\dim(E) = \operatorname{Dim}(E)$, *then*

$$\dim(E) = \operatorname{Dim}(E) = \sup_{\mu \in \mathcal{P}(E)} \underline{R}(\mu) = \sup_{\mu \in \mathcal{P}(E)} \bar{R}(\mu).$$

It is easily seen that the inequality in Proposition 5) ii) may not hold if the assumption "bounded" is omitted. Indeed put $E = \mathbf{N}$ and $q_n = c((n+1)(\log(n+1))^2)^{-1}$ for $n \in \mathbf{N}$ where $c = 1/\sum_{n=2}^{\infty} \frac{1}{n(\log n)^2}$, and define $\mu \in \mathscr{P}(E)$ by $\mu = \sum_n q_n \delta_n$ (here δ_x denotes the Dirac measure concentrated at x). If $0 < r < 1$ and $(E_i)_i$ is a countable partition of $E = \mathbf{N}$ then $(E_i \cap E)_i = (\{n\})_{n \in \mathbf{N}}$, whence

$$\frac{h_r(\mu)}{-\log r} = \frac{-\sum_n \mu(\{n\}) \log(\mu\{n\})}{-\log r} = \frac{-\sum_n q_n \log q_n}{-\log r} = \infty$$

which implies that $\bar{R}(\mu) = \underline{R}(\mu) = \infty > 0 = \mathrm{Dim}(E)$.

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DEPARTMENT OF STATISTICS AND ACTUARIAL SCIENCE
UNIVERSITY OF WATERLOO
WATERLOO, ONTARIO
CANADA N2L 3G1

MATEMATISK INSTITUT
UNIVERSITY OF COPENHAGEN
2100 KØBENHAVN Ø
DENMARK