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EXTENSION OF ENTIRE FUNCTIONS WITH CONTROLLED GROWTH

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Contents.

1. Introduction.

The order ρ of an entire function $F \in \mathcal{O}(\mathbf{C}^n)$ is classically defined as the infimum of all numbers $a > 0$ such that $|F(z)| \leqq C_a e^{|z|^a}$ for some constant C_a. Given the order the type σ is then defined as the infimum of all numbers $b > 0$ such that $|F(z)| \leqq C_b e^{b|z|^\rho}$ for some constant C_b. To make these notions dual in the sense of convex analysis C. O. Kiselman has introduced the concept of relative order and type, generalizing the classical order and type. This was first done in [1], but the idea is more developed in [2]. There among others he considers an extension problem for entire functions. Given two entire functions F and G in $\mathbf{C}^n$, what size can a disk $D_r = \{w \in \mathbf{C};\ |w| < r\}$ have such that there exists a holomorphic function H in $\mathbf{C}^n \times D_r$ satisfying certain growth conditions determined by F and G on the sets $C_1 = \{(z, e) \in \mathbf{C}^n \times \mathbf{C};\ |w| = 1\}$ and $C_e = \{(z, w) \in \mathbf{C}^n \times \mathbf{C};\ |w| = e\}$ respectively? It turns out that the size of the disk is determined by the relative order of G with respect to F. However using these disks we cannot let the growth of H be controlled from both above and below on both of the sets C_1 and C_e, nor can we specify that both $H(z, 1) = F(z)$ and $H(z, e) = G(z)$. This paper is a continuation of that work. Instead of disks we consider annuli. In this way we can extend F and G to H with H satisfying mainly the same growth conditions as

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above but now from both above and below on both of the sets C_1 and C_e. (The difference in the growth condition is an arbitrary $\varepsilon > 0$). It is equivalent also to assume that $H(z, 1) = F(z)$ and $H(z, e) = G(z)$. The size of the annulus turns out to be determined by both the relative order of F with respect to G and the relative order of G with respect to F. More generally we consider extensions to logarithmically convex Reinhardt domains in $\mathbf{C}^n \times \mathbf{C}^m$ and we see that all domains to which we can extend the functions can be written in an explicit way.

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2. Relative order and type.

The notion of relative order and type is introduced in Kiselman [1] and studied in detail in [2]. We give here the definitions and some simple facts.

DEFINITION 2.1. Let $f, g\colon E \to [-\infty, +\infty]$ be two functions defined on a real vector space E. We define *the order of f relative to g* as

$$\operatorname{order}(f:g) = \inf[a > 0;\ \exists c_a \in \mathbf{R},\ \forall x \in E,\ f(x) \leqq \frac{1}{a} g(ax) + c_a].$$

If g is convex and $g(0) < +\infty$ then the set above is an interval $]\rho, +\infty[$ or $[\rho, +\infty[$, where $0 \leqq \rho \leqq +\infty$.

DEFINITION 2.2. Let f, g be two functions as above. We then define *the type of f relative to g* as

$$\operatorname{type}(f:g) = \inf[b > 0;\ \exists c_b \in \mathbf{R},\ \forall x \in E,\ f(x) \leqq b g(x) + c_b].$$

The set above is an interval $]\sigma, +\infty[$ or $[\sigma, +\infty[$, where $0 \leqq \sigma \leqq +\infty$, if g is bounded from below.

Let E^* be the algebraic dual of the real vector space E and E' a fixed linear subspace of E^*. We define the spaces $\mathscr{F}(E, E')$ and $\mathscr{F}(E', E)$ in the following way: $\mathscr{F}(E, E')$ is the space of all functions from E to $[-\infty, +\infty]$ which are convex, lower semicontinuous for the weak topology $\sigma(E, E')$ and takes the value $-\infty$ only for the constant function $-\infty$. $\mathscr{F}(E', E)$ is defined similarly for functions from E' to $[-\infty, +\infty]$ but with the weak star topology $\sigma(E', E)$ instead.

Let $f\colon E \to [-\infty, +\infty]$ be a function on the real vector space E. We define the *Fenchel transform of f* by

$$\tilde{f}(\xi) = \sup_{x \in E} (\xi \cdot x - f(x)), \qquad \xi \in E' \tag{2.1}$$

We can apply the transformation twice getting

$$\tilde{\tilde{f}}(x) = \sup_{\xi \in E'} (\xi \cdot x - \tilde{f}(\xi)), \qquad x \in E. \tag{2.2}$$

A direct consequence of the definition is that we have $\tilde{f} \in \mathscr{F}(E', E)$ and $\tilde{\tilde{f}} \in \mathscr{F}(E, E')$. Obviously the transform is dependent on the subspace E' chosen. Usually in a topological vector space one takes E' as the topological dual of E. For instance in R^n, where the topological dual is isomorphic to R^n, one takes E' as R^n. Some general properties of the Fenchel transform are $\tilde{\tilde{f}} \leqq f$, $\tilde{\tilde{\tilde{f}}} = \tilde{f}$ and

$$\tilde{\tilde{f}} = \sup[v \in \mathscr{F}(E, E');\ v \leqq f]. \tag{2.3}$$

Thus $\tilde{\tilde{f}} = f$ if and only if $f \in \mathscr{F}(E, E')$. For more information see Rockafellar [3].

There is a duality theorem connecting the relative order and type via the Fenchel transform.

THEOREM 2.3 (Kiselman [2], Theorem 4.3). *Let E be a real vector space and E' a linear subspace of E^*. Assume that $f, g \in \mathscr{F}(E, E')$. Then*

$$\operatorname{order}(\tilde{g} : \tilde{f}) = \operatorname{type}(f : g) \text{ and } \operatorname{type}(\tilde{g} : \tilde{f}) = \operatorname{order}(f : g).$$

3. Growth functions.

Let F be an entire function in C^n. We then define its *growth function* as

$$f(t) = \sup[\log|F(z)|;\ z \in \mathsf{C}^n,\ |z| \leqq e^t], \qquad t \in \mathsf{R}. \tag{3.1}$$

We shall also have use for holomorphic functions $H \in \mathscr{O}(\mathsf{C}^n \times \Omega')$, where $\Omega' \subset \mathsf{C}^m$ is a logarithmically convex Reinhardt domain. We will then define *the partial growth function of H* by

$$h_w(t) = \sup_z[\log|H(z, w)|;\ z \in \mathsf{C}^n,\ |z| \leqq e^t], \qquad t \in \mathsf{R},\ w \in \Omega'. \tag{3.2}$$

and also

$$\begin{aligned} h(t, s) &= \sup_{z, w}[\log|H(z, w)|;\ (z, w) \in \mathsf{C}^n \times \Omega',\ |z| \leqq e^t,\ |w_i| = e^s,\ \forall i] \\ &= \sup_w[h_w(t);\ |w_i| = e^s,\ \forall i = 1, \dots, m], \qquad t, s \in \mathsf{R}. \end{aligned} \tag{3.3}$$

In view of Hadamard's three-circle-theorem, all functions here considered are convex in the variables t and s.

An open set in C^n is a *Reinhardt domain* if $(e^{i\theta_1}z_1, \dots, e^{i\theta_n}z_n)$ belongs to the set for all real $\theta_1, \dots, \theta_n$ if $z = (z_1, \dots, z_n)$ does. A set $S \in \mathsf{C}^n$ is *logarithmically convex* if the set $\{x \in \mathsf{R}^n;\ x = (\log|z_1|, \dots, \log|z_n|)$, for some $z \in S\}$ is convex.

If F and G are two entire functions, we define *the order of F with respect to G* as

$$\operatorname{order}(F : G) = \operatorname{order}(f : g), \tag{3.4}$$

where f and g are defined by (3.1). The order so defined is independent of the norm, since if B_j denotes the closed ball with respect to the norm j, then B_i is included in $e^{k_{ij}}B_j$, for some constant k_{ij}. Thus we have the estimate $f_i(t) \leqq f_j(t + k_{ij})$, where f_i denotes the growth function with respect to the norm i. The independence now follows from the following lemma.

LEMMA 3.1. (Kiselman [2] Lemma 3.2). *Let f_y denote the translate of $f: E \to [-\infty, +\infty]$ by the vector y: $f_y(x) = f(x - y)$. If one of f and g is convex and real valued, then*

$$\operatorname{order}(f_y : g) = \operatorname{order}(f : g_y) = \operatorname{order}(f : g).$$

We can also define what we will call the *refined growth function of F*, as

$$f_r(t) = \sup[\log|F(z)|;\ z \in \mathbf{C}^n,\ |z_i| \leqq e^{t_i}], \qquad t \in \mathbf{R}^n. \tag{3.5}$$

Also this function is convex by Hadamard.

If F and G are two entire functions in $\mathbf{C}^n$, then $\operatorname{order}(f_r : g_r)$ with f_r, g_r defined by (3.5) is in general larger than or equal to $\operatorname{order}(f : g)$, with f and g defined by (3.1), since we can always take all $t_i = t \in \mathbf{R}$. On the other hand if for example $F(z_1, z_2) = F_1(z_1)F_2(z_2)$ and $G(z_1, z_2) = F_2(z_1)F_1(z_2)$, with $\operatorname{order}(F_1 : F_2) > 1$, we get $\operatorname{order}(F : G) = 1$, but $\operatorname{order}(f_r : g_r) > 1$. But with the change of variables

$$z'_1 = \frac{z_1 + z_2}{\sqrt{2}}, \qquad z'_2 = \frac{z_1 - z_2}{\sqrt{2}},$$

we get $\operatorname{order}(f_r : g_r) = 1$ also for f_r, g_r. Thus we see that the relative order between two refined growth functions is coordinate dependent.

If we take g as the exponential function $g(t) = e^t$ and f as defined by (3.1), we get the order of f relative to g as the classical order of F. We now take g_r as the convex function defined by $g_r(t) = e^{\max_i t_i}$, $t \in \mathbf{R}^n$ and f_r as defined by (3.5). Then we have $\operatorname{order}(f : g) = \operatorname{order}(f_r : g_r)$, since $f_r(t_1, \ldots, t_n) \leqq f(\max)_i\, t_i)$ if we use the norm $|z| = \max_i |z_i|$ in (3.1). Similarly we can define $g(t) = e^{\rho t}$ and $g_r(t) = e^{\rho \max_i t_i}$ to retain the classical type.

4. Coefficient functions.

Let F be an entire function in $\mathbf{C}^n$. We can then expand F in homogeneous polynomials

$$\sum_{j=0}^{\infty} P_j(z), \tag{4.1}$$

where P_j is homogeneous of degree j. We define the norm of the polynomials P_j as

(4.2) $$\|P_j\| = \sup_{|z| \leqq 1} |P_j(z)|.$$

With this norm we define *the coefficient function of F* as

(4.3) $$p(j) = \begin{cases} -\log \|P_j\|, & j \in \mathbf{N}; \\ +\infty & j \in \mathbf{R} \setminus \mathbf{N}; \end{cases}$$

Note that in this definition we set $-\log 0 = +\infty$.

5. Properties of coefficient functions.

Let $f, g\colon E \to [-\infty, +\infty]$ be two functions on a real vector space E. We then define the *infimal convolution of f and g* by

$$f \square g(x) = \inf_y [f(y) \dot{+} g(x-y)], \qquad x \in E;$$

where $\dot{+}$ is upper addition extending the usual addition to hold from $[-\infty, +\infty]^2$ to $[-\infty, +\infty]$, so that $(+\infty) \dot{+} (-\infty) = +\infty$. In the same manner lower addition $\underset{\cdot}{+}$ is defined, so that $(+\infty) \underset{\cdot}{+} (-\infty) = -\infty$. If we apply the Fenchel transformation to an infimal convolution we get

(5.1) $$(f \square g)^{\sim} = \tilde{f} \underset{\cdot}{+} \tilde{g}.$$

See also Rockafellar [3] concerning the infimal convolution.

Define the function K as

(5.2) $$K(t) = \begin{cases} -\log(1 - e^t), & t < 0; \\ +\infty & t \geqq 0. \end{cases}$$

Then we have the following theorem connecting the growth and coefficient functions of an entire function.

THEOREM 5.1 (Kiselman [2], Theorem 6.1). *Let $F \in \mathcal{O}(\mathbf{C}^n)$ be an entire function. Let f and p be defined by* (3.1) *and* (4.3) *respectively. Then*

(5.3) $$\tilde{p} \leqq f \leqq \tilde{p} \square K \qquad \text{on } \mathbf{R}.$$

COROLLARY 5.2 (Kiselman [2], Corollary 6.5). *Let F, G be two entire functions in $\mathbf{C}^n$. Let f, g be their growth functions defined by* (3.1) *and p, q be their coefficient functions defined by* (4.3). *Then*

$$\operatorname{order}(f : g) = \operatorname{order}(\tilde{p} : \tilde{q}) = \operatorname{type}(\tilde{\tilde{q}} : \tilde{\tilde{p}}).$$

PROOF. This follows from Lemma 3.1 and Theorem 2.3. See Kiselman [2] for the details.

LEMMA 5.3. *Let $F \in \mathcal{O}(\mathbf{C}^n)$ and let p be its coefficient function defined by* (4.3). *Then $p(j)$, $j \in \mathbf{N}$, has faster growth than any linear function, or equivalently*

$$\frac{p(j)}{j} \to +\infty \qquad \text{as } j \to +\infty.$$

PROOF. Since F is entire, $\|P_j\| R^j \to 0$ as $j \to +\infty$, for all $R > 0$. Taking logarithms we get

$$j \log R - p(j) \to -\infty \qquad \text{as } j \to +\infty.$$

Since this holds for all positive R, the lemma follows.

It actually follows that

$$\frac{p(j)}{|j|} \to +\infty \qquad \text{as } |j| \to +\infty \text{ for } j \in \mathbf{R},$$

since $p(j) = +\infty$ for all $j \in \mathbf{R} \setminus \mathbf{N}$.

LEMMA 5.4. *Let $a: E \to [-\infty, +\infty]$ be a function on a finite-dimensional real vector space E which grows faster than any linear function:*

$$\frac{a(x)}{|x|} \to +\infty \qquad \text{as } |x| \to +\infty.$$

Then also

$$\frac{\tilde{\tilde{a}}(x)}{|x|} \to +\infty \qquad \text{as } |x| \to +\infty.$$

Moreover if a is lower semicontinuous, then $\tilde{a}$ and $\tilde{\tilde{a}}$ are determined by the set $M = \{(x, a(x)); a(x) = \tilde{\tilde{a}}(x)\}$. That is, if a is redefined to $+\infty$ at all other points, then $\tilde{a}$ and $\tilde{\tilde{a}}$ are unchanged.

PROOF. To prove the first part of the lemma we just observe that the affine functions are among the functions we take supremum of in (2.3). To prove the second part let a_M be the function obtained from a by setting

$$(5.4) \qquad a_M(x) = \begin{cases} a(x) & \text{if } (x, a(x)) \in M; \\ +\infty & \text{if } (x, a(x)) \notin M. \end{cases}$$

We will show that $\tilde{a}_M = \tilde{a}$. Thus also $\tilde{\tilde{a}}_M = \tilde{\tilde{a}}$. Obviously $\tilde{a}_M \leqq \tilde{a}$. Pick some $\xi \in E'$. By the growth condition and lower semicontinuity of a and the definition of the Fenchel transform there exists an $x \in E$ such that $\tilde{a}(\xi) = \xi \cdot x - a(x)$. Since $\tilde{\tilde{a}}(x) \geqq \xi \cdot x - \tilde{a}(x) = a(x)$ and $\tilde{\tilde{a}} \leqq a$, we conclude that x is such that $(x, a(x)) \in M$, which implies $a_M(x) = a(x)$. We get $\tilde{a}_M(\xi) \geqq \xi \cdot x - a_M(x) = \xi \cdot x - a(x) = \tilde{a}(\xi)$. Thus actually $\tilde{a}_M(\xi) = \tilde{a}(\xi)$. But ξ was arbitrary so if follows that $\tilde{a}_M = \tilde{a}$.

By Lemma 5.3, this holds in particular for a coefficient function, since by definition a coefficient function is obviously lower semicontinuous.

6. Extension of entire functions.

Before we proceed with the main theorem we will need the following lemma.

LEMMA 6.1 (Kiselman [2], Part of Theorem 7.2). *If the problem*

$$\begin{cases} \text{Find } F\colon E \times \mathsf{R} \to]-\infty, +\infty], & \text{such that} \\ F(x,j) = f_j(x), & x \in E,\ j = 0, 1, \text{ where} \\ f_j\colon E \to]-\infty, +\infty], & \text{are convex and lower} \\ & \text{semicontinuous for } \sigma(E, E'), \end{cases}$$

has a convex solution F which is finite at a point $(0, t)$, $1 < t < +\infty$, *then*

$$\operatorname{order}(f_1 : f_0) \leqq \frac{t}{t-1}.$$

Now let $\alpha \in \mathsf{R}^m$ satisfy $\sum_{j=1}^m \alpha_i = 1$ and $p \in \mathsf{R}\backslash\{0\}$ be a nonzero real number. We consider hyperplanes H_p^α, defined by

$$H_p^\alpha = \{x \in \mathsf{R}^m;\ x \cdot \alpha = p\}. \tag{6.1}$$

The set $\{H_p^\alpha\}_\alpha$, consists of all hyperplanes containing $\mathbf{p} = (p, \ldots, p)$ but not the origin. The set $\{H_p^\alpha\}_p$ is a set of parallel hyperplanes. In R, H_p^α is just the point p.

Let S_p^α denote the closed halfspace which is bounded by H_p^α and contains the origin. More explicitly

$$S_p^\alpha = \begin{cases} \{x \in \mathsf{R}^m;\ x \cdot \alpha \leqq p\} & \text{if } p > 0; \\ \{x \in \mathsf{R}^m;\ x \cdot \alpha \geqq p\} & \text{if } p < 0. \end{cases} \tag{6.2}$$

Now define the logarithmically convex Reinhardt domain ω_p^α by

$$\omega_p^\alpha = \{w \in \mathbf{C}^m;\ (\log|w_1|, \log|w_2|, \ldots, \log|w_m|) \in \operatorname{int} S_p^\alpha\}. \tag{6.3}$$

This is the smallest open set Ω' which satisfies

$$\operatorname{int} S_p^\alpha = \{x \in \mathsf{R}^m;\ x = (\log|w_1|, \ldots, \log|w_m|) \text{ for some } w \in \Omega'\}.$$

There is also a largest open set which satisfies this property. We will denote this set by Ω_p^α. We then have

$$\Omega_p^\alpha = \omega_p^\alpha \cup B_J \tag{6.3'}$$

where $J = \{i\colon p\alpha_i \geqq 0\}$ and B_J the union of all sets $\Pi\omega_p^\alpha$, where Π is a projection which takes some of the components with index in the set J to zero. That is, Ω_p^α is the interior of the closure of ω_p^α. The definition of Ω_p^α just says that if $m = 1$ and

$p > 0$ then we add the origin to ω_p^1 and if $m \geqq 2$ that if some component w_i of $w \in \omega_p^\alpha$ can be arbitrarily small with the other components having fixed values then we add to ω_p^α images of all points in ω_p^α with w_i projected to zero. If there are more such components we go on recursively. For instance if $p = 1$ and $\alpha = \left(\frac{1}{m}, \ldots, \frac{1}{m}\right)$ then Ω_p^α contains all points $w \in \mathbf{C}^m$ with some component $w_i = 0$, whereas ω_p^α contains none. This is why we introduce the extension (6.3′). We want to fill in unnecessary boundaries.

Define $S_{+\infty}^\alpha = S_{-\infty}^\alpha = \mathbf{R}^m$. Then (6.3) and (6.3′) can be used to define ω_p^α and Ω_p^α also for $p = -\infty, +\infty$, if we let $0 \cdot \infty = 0$ in the definition of J. We have $\omega_{-\infty}^\alpha = \omega_{+\infty}^\alpha = (\mathbf{C} \backslash \{0\})^m$, for all α. If $\alpha_i \geqq 0$ for all i then $\Omega_{+\infty}^\alpha = \mathbf{C}^m$, but there is no α such that $\Omega_{-\infty}^\alpha = \mathbf{C}^m$ since some α_i must be positive. For instance $\Omega_{+\infty}^{(1,0)} = \mathbf{C}^2$ but $\Omega_{-\infty}^{(1,0)} = (\mathbf{C} \backslash \{0\}) \times \mathbf{C}$.

THEOREM 6.2. *Let $F, G \in \mathcal{O}(\mathbf{C}^n)$ be two transcendental entire functions. Define their growth functions f, g by* (3.1) *respectively. Let λ, $\rho \in \mathbf{R}$, α, $\beta \in \mathbf{R}^m$ satisfy $0 < \lambda \leqq 1 \leqq \rho < +\infty$, and $\sum \alpha_i = \sum \beta_i = 1$. Define the domain*

$$\Omega = \{(z, w) \in \mathbf{C}^n \times \mathbf{C}^m;\ w \in \Omega_{\lambda'}^\alpha \cap \Omega_{\rho'}^\beta\}$$

where $\lambda' = \frac{\lambda}{\lambda - 1}, (-\infty \leqq \lambda' < 0), \rho' = \frac{\rho}{\rho - 1}, (1 < \rho' \leqq +\infty)$ *and* $\Omega_{\lambda'}^\alpha, \Omega_{\rho'}^\beta$ *are defined by* (6.3′). *Then the following conditions are equivalent:*

(a)
$$\operatorname{order}(G : F) \leqq \rho;$$
$$\operatorname{order}(F : G) \leqq \frac{1}{\lambda};$$

(b) *For each $\varepsilon > 0$ (or equivalently some $\varepsilon > 0$) there exists an $H \in \mathcal{O}(\Omega)$ such that*

$$\begin{cases} f \leqq h_w \square K + \varepsilon, & h_w \leqq f \square K + \varepsilon, \quad |w_i| = 1, \ \forall i = 1, \ldots, m; \\ g \leqq h_w \square K + \varepsilon, & h_w \leqq g \square K + \varepsilon, \quad |w_i| = e, \ \forall i = 1, \ldots, m, \end{cases}$$

where h_w is defined by (3.2) *and K by* (5.2),

(b′) *The condition* (b) *holds together with the extra assumption*

$$H(z, \mathbf{1}) = F(z), \quad H(z, \boldsymbol{e}) = G(z),$$

where $\mathbf{1} = (1, \ldots, 1)$ and $\boldsymbol{e} = (e, \ldots, e)$ (m times).

(c) *There exists an $H \in \mathcal{O}(\Omega)$ such that*

$$\operatorname{order}(f : h(\cdot, 0)) = \operatorname{order}(h(\cdot, 0) : f) = 1,$$
$$\operatorname{order}(g : h(\cdot, 1)) = \operatorname{order}(h(\cdot, 1) : g) = 1.$$

where $h(\cdot, \cdot)$ is defined by (3.3).

(c′) *The condition* (c) *holds together with the extra assumption*

$$H(z, \mathbf{1}) = F(z), \qquad H(z, e) = G(z).$$

(*The conditions are equivalent also for polynomials if condition* (c) *is altered to*

$$\operatorname{order}(f : h(\cdot, 0)) = \operatorname{order}(h(\cdot, 0) : f) = 0,$$

$$\operatorname{order}(g : h(\cdot, 1)) = \operatorname{order}(h(\cdot, 1) : g) = 0.$$

In this case condition (a) *is equivalent with* $\operatorname{order}(G : F) = \operatorname{order}(F : G) = 0$.)

If $H \in \mathcal{O}(\Omega)$ *satisfies condition* (b) *or* (b′) *for some* $\varepsilon > 0$ *then it also satisfies condition* (c) *or* (c′) *respectively.*

PROOF. Note first that sometimes $\Omega^{\alpha}_{\lambda'} \cap \Omega^{\beta}_{\rho'} = \omega^{\alpha}_{\lambda'} \cap \omega^{\beta}_{\rho'}$. That is, if $\operatorname{sign}(\alpha_i) = 1$ when $\operatorname{sign}(\beta_i) \geqq 0$. For instance when $m = 1$ this is always the case:

$$\Omega = \{(z, w) \in \mathbf{C}^n \times \mathbf{C};\, e^{\lambda'} < |w| < e^{\rho'}\}$$

and w is never zero even if $\lambda' = -\infty$.

(a) implies (b′). Make the expansions $F(z) = \sum_{j=0}^{\infty} P_j(z)$ and $G(z) = \sum_{j=0}^{\infty} Q_j(z)$, as in (4.1). Let p and q be the coefficient functions of F and G defined by (4.3) respectively. Put

$$(6.4)\quad H(z, w) = \frac{1}{E_N\left(\dfrac{1}{e}\right)}\left[E_N\left(\frac{w^{\gamma}}{e}\right)\sum_{j=0}^{\infty} P_j(z) w^{\mu_j} + E_N\left(\frac{1}{w^{\gamma'}}\right)\sum_{j=0}^{\infty} Q_j(z)\left(\frac{w}{e}\right)^{\nu_j}\right],$$

where $\gamma, \gamma' \in \mathbf{Z}^m$ are multi-indices satisfying $\sum_{i=1}^{m} \gamma_i = \sum_{i=1}^{m} \gamma'_i = 1$. Moreover, we choose all γ_i nonnegative. That is $\gamma_i = \delta_{ik}$ for some $k = 1, \ldots, m$, with δ_{ik} as the Kronecker delta. Some α_i must be positive and we choose γ'_i positive for this i and all the others nonpositive. In this way the functions $w \mapsto w^{\gamma}$ and $w \mapsto \dfrac{1}{w^{\gamma'}}$ are holomorphic in $\Omega^{\alpha}_{\lambda'} \cap \Omega^{\beta}_{\rho'}$. We have as usual $w^{\gamma} = w_1^{\gamma_1} \ldots w_m^{\gamma_m}$. The multi-indices $\mu_j, \nu_j \in \mathbf{Z}^m$ are chosen such that $\mu_{ji} = \nu_{ji} = 0$, if $\tilde{\tilde{p}}(j) = +\infty$, which occurs if and only if $\tilde{\tilde{q}}(j) = +\infty$, since we are demanding finite order. (Recall Corollary 5.2). Otherwise we take μ_{ji} as the integer part of $\alpha_i \min(\tilde{\tilde{p}}(j) - \tilde{\tilde{q}}(j) - N, 0)$ and ν_{ji} as the integer part of $\beta_i \max(\tilde{\tilde{p}}(j) - \tilde{\tilde{q}}((j) + N + m, 0)$. For clarity:

$$\left(\frac{w}{e}\right)^{\nu_j} = \prod_{i=1}^{m}\left(\frac{w_i}{e}\right)^{\nu_{ji}}.$$

The function $E_N(\xi) = (1 - \xi)\exp\left(\xi + \dfrac{\xi^2}{2} + \ldots + \dfrac{\xi^N}{N}\right)$, $N = 1, 2, \ldots$, is a so called Weierstrass function. We have $E_N(1) = 0$, so that $H(z, \mathbf{1}) = F(z)$, $H(z, e) = G(z)$. It can be shown (Rudin [4]), that for $|\xi| \leqq 1$

$$|1 - E_N(\xi)| \leqq |\xi|^{N+1}. \tag{6.5}$$

The integer N is chosen such that $e^{2-N} < \varepsilon$.

When $|w_i| = 1$, $i = 1, \ldots, m$, we make the estimates

$$\left\| P_j w^{\mu_j} - E_N\left(\frac{w^\gamma}{e}\right) P_j w^{\mu_j} \right\| = \left| 1 - E_N\left(\frac{w^\gamma}{e}\right) \right| \|P_j\| \leqq e^{-N-1} \|P_j\| \quad \text{and} \tag{6.6}$$

$$\begin{aligned} \left\| E_N\left(\frac{1}{w^{\gamma'}}\right) Q_j \left(\frac{w}{e}\right)^{\nu_j} \right\| &= \left| E_N\left(\frac{1}{w^{\gamma'}}\right) \right| \exp\left(-q(j) - \sum_i \nu_{ji}\right) \\ &\leqq 2\exp(-q(j) + \tilde{\tilde{q}}(j) - \tilde{\tilde{p}}(j) - N) \leqq 2\exp(-N - \tilde{\tilde{p}}(j)), \end{aligned} \tag{6.7}$$

using $\sum_i \nu_{ji} \geqq \tilde{\tilde{p}}(j) - \tilde{\tilde{q}}(j) + N$, $q(j) \geqq \tilde{\tilde{q}}(j)$ and (6.5).

When $|w_i| = e$, $i = 1, \ldots, m$, we make the estimates

$$\begin{aligned} \left\| E_N\left(\frac{w^\gamma}{e}\right) P_j w^{\mu_j} \right\| &= \left| E_N\left(\frac{w^\gamma}{e}\right) \right| \exp\left(-p(j) + \sum_i \mu_{ji}\right) \\ &\leqq 2\exp(-p(j) + \tilde{\tilde{p}}(j) - \tilde{\tilde{q}}(j) - N) \leqq 2\exp(-N - \tilde{\tilde{q}}(j)), \end{aligned} \tag{6.8}$$

using $\sum_i \mu_{ji} \leqq \tilde{\tilde{p}}(j) - \tilde{\tilde{q}}(j) - N$, $p \geqq \tilde{\tilde{p}}$ and (6.5). We also have

$$\left\| Q_j\left(\frac{w}{e}\right)^{\nu_j} - E_N\left(\frac{1}{w^{\gamma'}}\right) Q_j \left(\frac{w}{e}\right)^{\nu_j} \right\| = \left| 1 - E_N\left(\frac{1}{w^{\gamma'}}\right) \right| \|Q_j\| \leqq e^{-N-1} \|Q_j\|. \tag{6.9}$$

The partial coefficient function r_w of H is

$$r_w(j) = -\log\left\| \frac{1}{E_N(1/e)} \left[E_N\left(\frac{w^\gamma}{e}\right) P_j w^{\mu_j} + E_N\left(\frac{1}{w^{\gamma'}}\right) Q_j \left(\frac{w}{e}\right)^{\nu_j} \right] \right\|.$$

By the triangle inequality and (6.5), we have

$$\begin{aligned} -\log(1 + e^{-N-1}) - \log\Bigg(\left\| E_N\left(\frac{w^\gamma}{e}\right) P_j w^{\mu_j} \right\| \\ + \left\| E_N\left(\frac{1}{w^{\gamma'}}\right) Q_j \left(\frac{w}{e}\right)^{\nu_j} \right\| \Bigg) \leqq r_w(j) \end{aligned} \tag{6.10}$$

and

$$\begin{aligned} r_w(j) \leqq -\log(1 - e^{-N-1}) - \log\Bigg| \left\| E_N\left(\frac{w^\gamma}{e}\right) P_j w^{\mu_j} \right\| \\ - \left\| E_N\left(\frac{1}{w^{\gamma'}}\right) Q_j \left(\frac{w}{e}\right)^{\nu_j} \right\| \Bigg|. \end{aligned} \tag{6.11}$$

When $|w_i| = 1$, $i = 1, \ldots, m$, we have by our estimates (6.6), (6.7) and since $p \geqq \tilde{\tilde{p}}$

(6.12) $$-\log(1 + e^{-N-1}) - \log(1 + e^{-N-1} + 2e^{-N}) + \tilde{\tilde{p}}(j) \leqq r_w(j),$$

which implies $\tilde{\tilde{p}}(j) - e^{2-N} \leqq r_w(j)$, hence taking the Fenchel transformation $\tilde{r}_w \leqq \tilde{p} + e^{2-N}$. (Recall the general rule $\tilde{\tilde{\tilde{p}}} = \tilde{p}$.)

For j such that $p(j) = \tilde{\tilde{p}}(j)$, we also have

(6.13) $$r_w(j) \leqq -\log(1 - e^{-N-1}) - \log(1 - e^{-N-1} - 2e^{-N}) + \tilde{\tilde{p}}(j),$$

which implies $r_w(j) \leqq \tilde{\tilde{p}}(j) + e^{2-N}$, for j satisfying $p(j) = \tilde{\tilde{p}}(j)$. Define p_M from p by (5.4). Then $r_w \leqq p_M + e^{2-N}$, hence $\tilde{p}_M \leqq \tilde{r}_w + e^{2-N}$. But by Lemma 5.4 $\tilde{p}_M = \tilde{p}$, so it follows that $\tilde{p} - e^{2-N} \leqq \tilde{r}_w$. So far, we have $\tilde{p} - e^{2-N} \leqq \tilde{r}_w \leqq \tilde{p} + e^{2-N}$, thus finally by Theorem 5.1

(6.14) $$f \leqq \tilde{p} \,\square\, K \leqq (\tilde{r}_w + \varepsilon) \,\square\, K = \tilde{r}_w \,\square\, K + \varepsilon \leqq h_w \,\square\, K + \varepsilon,$$
$$h_w \leqq \tilde{r}_w \,\square\, K \leqq (\tilde{p} + \varepsilon) \,\square\, K = \tilde{p} \,\square\, K + \varepsilon \leqq f \,\square\, K + \varepsilon,$$

when $|w_i| = 1$, for all $i = 1, \ldots, m$, since we chose $e^{2-N} < \varepsilon$.

When $|w_i| = e, i = 1, \ldots, m$, we have similarily using (6.10), (6.8), (6.9) and $q \geqq \tilde{\tilde{q}}$

(6.15) $$-\log(1 + e^{-N+1}) - \log(2e^{-N} + 1 + e^{-N-1}) + \tilde{\tilde{q}}(j) \leqq r_w(j),$$

which implies $\tilde{\tilde{q}}(j) - e^{2-N} \leqq r_w(j)$, hence taking the Fenchel transformation $\tilde{r}_w \leqq \tilde{q} + e^{2-N}$. For j such that $q(j) = \tilde{\tilde{q}}(j)$, we get from (6.11), (6.8) and (6.9)

(6.16) $$r_w(j) \leqq -\log(1 - e^{-N-1}) - \log(1 - e^{-N-1} - 2e^{-N}) + \tilde{\tilde{q}}(j),$$

which implies $r_w(j) \leqq \tilde{\tilde{q}}(j) + e^{2-N}$, for j satisfying $q(j) = \tilde{\tilde{q}}(j)$. Define q_M from q by (5.4). Then $r_w \leqq q_M + e^{2-N}$, hence $\tilde{q}_M \leqq \tilde{r}_w + e^{2-N}$. But by Lemma 5.4 $\tilde{q}_M = \tilde{q}$, so it follows that $\tilde{q} - e^{2-N} \leqq \tilde{r}_w$. So far, we have $\tilde{q} - e^{2-N} \leqq \tilde{r}_w \leqq \tilde{q} + e^{2-N}$, thus finally by Theorem 5.1

(6.17) $$g \leqq h_w \,\square\, K + \varepsilon, \quad h_w \leqq g \,\square\, K + \varepsilon, \quad |w_i| = e, \quad i = 1, \ldots, m.$$

We now have to show that H is holomorphic in Ω. Directly after the definition of H in (6.4) we have chosen γ and γ' so that $w \mapsto w^\gamma$ and $w \mapsto 1/w^{\gamma'}$ becomes holomorphic in Ω. Since the Weierstrass functions are entire these do not cause any trouble. Apart from the Weierstrass functions we show that the first part of the series defining H converges locally uniformly in $\mathbf{C}^n \times \Omega^\alpha_{\lambda'}$ and that the second part converges locally uniformly in $\mathbf{C}^n \times \Omega^\beta_{\rho'}$. Then we can conclude that the whole of the series defining H actually converges locally uniformly in Ω.

It simplifies the argument to show the convergence just in $\mathbf{C}^n \times \omega^\alpha_{\lambda'}$ and $\mathbf{C}^n \times \omega^\beta_{\rho'}$ respectively. By our choice of μ and ν it is then clear that we have locally uniform convergence also in $\mathbf{C}^n \times \Omega^\alpha_{\lambda'}$ and $\mathbf{C}^n \times \Omega^\beta_{\rho'}$. This is because we have no negative exponents on components of w which may be zero. For instance if $\alpha_i < 0$ then $\sup_{j,w} |w^{\mu_j}| < +\infty$ on compact subsets of $(\mathbf{C}\setminus\{0\})^{i-1} \times \{0\} \times (\mathbf{C}\setminus\{0\})^{m-i}$

$\subset B_J$ since then w^{μ_j} becomes zero for $\mu_{ji} > 0$. If $\alpha_i = 0$ then w^{μ_j} does not depend on w_i.

The first part of the series converges locally uniformly in $\mathbf{C}^n \times \omega^\alpha_{\lambda'}$, if

$$\|P_j\| \, R^j r^{\mu_j} = \|P_j\| \, R^j \prod_{i=1}^{m} r_i^{\mu_{ji}} \to 0 \quad \text{as } j \to +\infty, \tag{6.18}$$

uniformly for all R, $0 \leqq R \leqq R_1 < +\infty$ and for all $r \in \mathsf{R}^m_+$ such that

$$(\log r_1, \ldots, \log r_m) \in S, \ S \text{ is compact and } S \subset \operatorname{int} S^\alpha_{\lambda'}, \tag{6.19}$$

since every r_i is bounded from below on compact subsets of $\omega^\alpha_{\lambda'}$. Taking logarithms, this is equivalent to

$$\frac{\sum_{i=1}^m \mu_{ji} \log r_i - p(j)}{j} \to -\infty \quad \text{as } j \to +\infty, \tag{6.20}$$

with r_i as above. From the definition of μ_j, we have $\tilde{\tilde{p}}(j) - N < \tilde{\tilde{q}}(j)$, when μ_j is nonzero. When $\mu_j = 0$, we have convergence independently of $r \in \mathsf{R}^m_+$, by Lemma 5.3. For μ_j not equal to zero

$$\frac{\sum_{i=1}^m \mu_{ji} \log r_i - p(j)}{j} \leqq \frac{\sum_{i=1}^m \alpha_i(\tilde{\tilde{p}}(j) - \tilde{\tilde{q}}(j) + O(1)) \log r_i - \tilde{\tilde{p}}(j)}{j}, \tag{6.21}$$

since $p \geqq \tilde{\tilde{p}}$. We will now use Corollary 5.2 to make estimates. We have $\tilde{\tilde{q}} \leqq b\tilde{\tilde{p}} + c_b$, for $b > 1/\lambda$, since $\operatorname{type}(\tilde{\tilde{q}} : \tilde{\tilde{p}}) = \operatorname{order}(f : g) \leqq 1/\lambda$. The expression above is linear in $\tilde{\tilde{q}}(j)$. Thus, we only need to estimate it on the endpoints of the possible values of $\tilde{\tilde{q}}(j)$. Since $\tilde{\tilde{q}}(j)$ has a bound from above and since μ_j is zero when $\tilde{\tilde{q}}(j)$ is less than $\tilde{\tilde{p}}(j) - N$, the expression cannot be larger than the value for $\tilde{\tilde{q}}(j) \leqq \tilde{\tilde{p}}(j) - N$, or $\tilde{\tilde{q}}(j) = b\tilde{\tilde{p}}(j) + c_b$. On the bound when $\tilde{\tilde{q}}(j) \leqq \tilde{\tilde{p}}(j) - N$ we have $\mu_j = 0$ and this case we have already considered. On the bound when $\tilde{\tilde{q}}(j) = b\tilde{\tilde{p}}(j) + c_b$ the expression above is equal to

$$\frac{\sum_{i=1}^m \alpha_i(\tilde{\tilde{p}}(j) - b\tilde{\tilde{p}}(j) + O(1)) \log r_i - \tilde{\tilde{p}}(j)}{j} \tag{6.22}$$

$$= \frac{[(1-b)\sum_{i=1}^m \alpha_i \log r_i - 1]\tilde{\tilde{p}}(j) + O(\max|\log r_i|)}{j} \to -\infty \quad \text{as } j \to +\infty,$$

locally uniformly for $(\log r_1, \ldots, \log r_m) \in S^\alpha_{b'}$, where $b' > \dfrac{1}{1-b} = \dfrac{1/b}{1/b - 1} > \dfrac{\lambda}{\lambda - 1} = \lambda'$, by Lemma 5.3 and 5.4. (Recall that $b > 1/\lambda \geqq 1$.) For each compact subset S of the interior of $S^\alpha_{\lambda'}$, there exist b, b', such that $S \subset S^\alpha_{b'} \subset \operatorname{int} S^\alpha_{\lambda'}$. Thus we see that the first part of the series converges locally uniformly in $\mathbf{C}^n \times \omega^\alpha_{\lambda'}$ and by our discussion above we can conclude that we have locally uniform convergence in $\mathbf{C}^n \times \Omega^{\alpha}_{\lambda'}$.

In the same way we see that the second part of (6.4) converges locally uniformly in $\mathbf{C}^n \times \omega^{\beta}_{\rho'}$. We want

$$\|Q_j\| R^j\left(\frac{r}{e}\right)^{\nu_j} = \|Q_j\| R^j \prod_{i=1}^{m}\left(\frac{r_i}{e}\right)^{\nu_{ji}} \to 0 \quad \text{as } j \to +\infty; \tag{6.23}$$

Taking logarithms, this is equivalent to

$$\frac{\sum_{i=1}^m \nu_{ji}\log\left(\frac{r_i}{e}\right) - q(j)}{j} \to -\infty \quad \text{as } j \to +\infty; \tag{6.24}$$

We have $\tilde{\tilde{p}} \leqq a\tilde{\tilde{q}} + c_a$, for $a > \rho$, since $\operatorname{type}(\tilde{\tilde{p}}:\tilde{\tilde{q}}) = \operatorname{order}(g:f) \leqq \rho$. For ν_j not to be equal to zero, we must have $\tilde{\tilde{p}}(j) > \tilde{\tilde{q}}(j) - N - m$. In this case

$$\frac{\sum_{i=1}^m \nu_{ji}\log\left(\frac{r_i}{e}\right) - q(j)}{j} \leqq \frac{\sum_{i=1}^m \beta_i(\tilde{\tilde{p}}(j) - \tilde{\tilde{q}}(j) + O(1))\log\left(\frac{r_i}{e}\right) - \tilde{\tilde{q}}(j)}{j}. \tag{6.25}$$

since $q \geqq \tilde{\tilde{q}}$. This is an expression linear in $\tilde{\tilde{p}}(j)$. We have convergence independently of $r \in \mathbf{R}^m_+$ on the bound putting $\tilde{\tilde{p}}(j) = \tilde{\tilde{q}}(j) - N - m$, when $\nu_j = 0$, by Lemma 5.3. On the upper bound for $\tilde{\tilde{p}}(j)$, that is $\tilde{\tilde{p}}(j) = a\tilde{\tilde{q}}(j) + c_a$, the expression above is equal to

$$\frac{\sum_{i=1}^m \beta_i(a\tilde{\tilde{q}}(j) - \tilde{\tilde{q}}(j) + O(1))\log r_i - a\tilde{\tilde{q}}(j)}{j} \tag{6.26}$$

$$= \frac{[(a-1)\sum_{i=1}^m \beta_i \log r_i - a]\tilde{\tilde{q}}(j) + O(\max|\log r_i|)}{j} \to -\infty \quad \text{as } j \to +\infty,$$

locally uniformly for $(\log r_1, \ldots, \log r_m) \in S^{\beta}_{a'}$, where $a' < \dfrac{a}{a-1} < \dfrac{\rho}{\rho-1} = \rho'$. But $S^{\beta}_{a'} \subset \operatorname{int} S^{\beta}_{\rho'}$. By a discussion similar to that for the first part we can conclude that the second part of the series defining H converges locally uniformly in $\mathbf{C}^n \times \Omega^{\beta}_{\rho'}$. (The Weierstrass function is an exception. It is holomorphic in $\Omega^{\alpha}_{\lambda'}$).

REMARK. It can be shown that if ν_j had not been put to zero for $\tilde{\tilde{p}}(j)$ less than $\tilde{\tilde{q}} - N - m$ then the domain of convergence of the second part of the series defining H had been only $\mathbf{C}^n \times \Omega^{\beta}_{\rho'} \cap \Omega^{\beta}_{\lambda'}$. A similar statement holds for the first part.

(b) implies (c) and (b′) implies (c′). Pick some $\varepsilon > 0$. By condition (b) there exists a holomorphic function $H \in \mathcal{O}(\Omega)$ satisfying

$$h_w(t) \leqq f \,\square\, K(t) + \varepsilon \leqq f(t+1) + K(-1) + \varepsilon, \tag{6.27}$$
$$f(t) \leqq h_w \,\square\, K(t) + \varepsilon \leqq h_w(t+1) + K(-1) + \varepsilon,$$

when $|w_i| = 1$ for all i. If also condition (b′) holds one can take a function satisfying

$$H(z, 1) = F(z), H(z, e) = G(z). \tag{6.28}$$

The estimates still hold if we take the supremum over w. Using Lemma 3.1, we get $\operatorname{order}(h(\cdot, 0) : f) \leqq 1$ and $\operatorname{order}(f : h(\cdot, 0)) \leqq 1$. In general the following property holds

$$\operatorname{order}(f : f) = \begin{cases} 1, & \text{if } F \text{ is not a polynomial;} \\ 0, & \text{if } F \text{ is a polynomial,} \end{cases} \tag{6.29}$$

as is perhaps easiest seen by Corollary 5.2 and Lemma 5.3. By submultiplicativity when F is not a polynomial

$$1 = \operatorname{order}(f : f) \leqq \operatorname{order}(f : h(\cdot, 0)) \operatorname{order}(h(\cdot, 0) : f) \leqq 1, \tag{6.30}$$

so that all orders are 1. If F is a polynomial the only possibilities for the order are 0 or $+\infty$ and the latter is excluded by the estimates above.

In a similar manner, we get $\operatorname{order}(g : h(\cdot, 1)) = \operatorname{order}(h(\cdot, 1) : g) = 1$, when G is not a polynomial. Otherwise 0.

(c) implies (a). We use Lemma 6.1 on the convex function $h(\cdot, \cdot)$. The lemma implies, since $h(0, \rho' - \delta) < +\infty$ for all $\delta > 0$, that

$$\operatorname{order}(h(\cdot, 1) : h(\cdot, 0)) \leqq \frac{\rho'}{\rho' - 1} = \rho. \tag{6.31}$$

With a change of variables $s \mapsto 1 - s$, we get from the other side

$$\operatorname{order}(h(\cdot, 0) : h(\cdot, 1)) \leqq \frac{1 - \lambda'}{(1 - \lambda') - 1} = 1/\lambda. \tag{6.32}$$

By submultiplicativity

$$\operatorname{order}(g : f) \leqq \operatorname{order}(g : h(\cdot, 1)) \operatorname{order}(h(\cdot, 1) : h(\cdot, 0)) \operatorname{order}(h(\cdot, 0) : f) \leqq \rho;$$

$$\operatorname{order}(f : g) \leqq \operatorname{order}(f : h(\cdot, 0)) \operatorname{order}(h(\cdot, 0) : h(\cdot, 1)) \operatorname{order}(h(\cdot, 1) : g) \leqq \frac{1}{\lambda}.$$

If F and G are polynomials both orders will be zero.

(c′) implies (c) and (b′) implies (b) obviously, so we are done.

REMARK. We see that condition (b) actually is two conditions. We will refer to these as condition (b) holds for every $\varepsilon > 0$ or condition (b) holds for some $\varepsilon > 0$

respectively. Note also that the conditions in Theorem 6.2 are independent of α and β, for if some of the conditions in Theorem 6.2 holds for $\alpha, \beta \in \mathbf{R}^m$, with $\sum \alpha_i = \sum \beta_i = 1$, then condition (a) holds. Since condition (a) is independent of α, β the condition also holds for some other α', β', $\sum \alpha_i' = \sum \beta_i' = 1$.

THEOREM 6.3. *Let $F, G \in \mathcal{O}(\mathbf{C}^n)$ be two entire functions and $\Omega' \subset \mathbf{C}^m$ a logarithmically convex Reinhardt domain such that condition* (c) *holds in Theorem* 6.2 *for $\Omega = \mathbf{C}^n \times \Omega'$. If* order$(F:G) > 1$, *then $\Omega' \subset \Omega^{\alpha}_{\lambda'} \cap \Omega^{\beta}_{\rho'}$ for some $\alpha, \beta \in \mathbf{R}^m$ and for some $\lambda' = \dfrac{\lambda}{\lambda - 1}$ and $\rho' = \dfrac{\rho}{\rho - 1}$ such that ρ and λ satisfies condition* (a). *If* order$(F:G) \leqq 1$ *then $\Omega' \subset \Omega^{\beta}_{\rho'}$.*

Proof. Put

$$S' = \{x \in \mathbf{R}^m;\ x = (\log|w_1|, \ldots, \log|w_m|) \text{ for some } w \in \Omega'\}. \tag{6.33}$$

Then S' is an open convex set and its intersection with the line $t \mapsto (t, \ldots, t)$ is the line segment $(t, \ldots, t)$, $\lambda' < t < \rho'$ for some λ' and ρ' satisfying $-\infty \leqq \lambda' < 0$ and $1 < \rho' \leqq +\infty$. Thus by convexity and since S' contains the origin $S' \subset \operatorname{int} S^{\alpha}_{\lambda'} \cap S^{\beta}_{\rho'}$ for some $\alpha, \beta \in \mathbf{R}^m$ satisfying $\sum \alpha_i = \sum \beta_i = 1$, with $S^{\alpha}_{\lambda'}$, $S^{\beta}_{\rho'}$ defined by (6.2). Moreover if $H \in \mathcal{O}(\Omega) = \mathcal{O}(\mathbf{C}^n \times \Omega')$ is a holomorphic function, then the function $h(\cdot,\cdot)$ defined by (3.3) satisfies

$$\operatorname{order}(h(\cdot,1):h(\cdot,0)) \leqq \frac{\rho'}{\rho' - 1} \tag{6.34}$$

and

$$\operatorname{order}(h(\cdot,0):h(\cdot,1)) \leqq \frac{\lambda' - 1}{\lambda'}. \tag{6.35}$$

If H also satisfies condition (c) then by submultiplicativity

$$\operatorname{order}(g:f) \leqq \operatorname{order}(g:h(\cdot,1))\operatorname{order}(h(\cdot,1):h(\cdot,0))\operatorname{order}(h(\cdot,0):f) \leqq \frac{\rho'}{\rho' - 1};$$

$$\operatorname{order}(f:g) \leqq \operatorname{order}(f:h(\cdot,0))\operatorname{order}(h(\cdot,0):h(\cdot,1))\operatorname{order}(h(\cdot,1):g) \leqq \frac{1 - \lambda'}{\lambda'}.$$

Thus

$$\operatorname{order}(G:F) \leqq \frac{\rho'}{\rho' - 1} = \rho, \tag{6.36}$$

$$\operatorname{order}(F:G) \leqq \frac{\lambda' - 1}{\lambda'} = \frac{1}{\lambda},$$

for some ρ and λ satisfying $0 < \lambda \leqq 1$, $1 \leqq \rho < +\infty$, $\rho' = \dfrac{\rho}{\rho - 1}$, $\lambda' = \dfrac{\lambda}{\lambda - 1}$ and also as we see condition (a) in Theorem 6.2. But if condition (a) holds for ρ and λ we can extend F and G to $\mathbf{C}^n \times \Omega^{\alpha}_{\lambda'} \cap \Omega^{\beta}_{\rho'}$. Thus since S' is contained in $S^{\alpha}_{\lambda'} \cap S^{\beta}_{\rho'}$ all we have to do now is to check the points in Ω' for which some component $w_i = 0$. The set of i for which this can happen is the set of i for which the component x_i in $x \in S^{\alpha}_{\lambda'} \cap S^{\beta}_{\rho'}$ with the other components given fixed values is unbounded from below. If λ' and ρ' are finite this is exactly the set of i for which $\lambda'\alpha_i \geqq 0$ and $\rho'\beta_i \geqq 0$. By the definition of $\Omega^{\alpha}_{\lambda'}$ and $\Omega^{\beta}_{\rho'}$ we conclude that $\Omega' \subset \Omega^{\alpha}_{\lambda'} \cap \Omega^{\beta}_{\rho'}$ if λ' and ρ' are finite. If $\rho' = +\infty$ we just choose all β_i positive so that $\Omega^{\beta}_{+\infty} = \mathbf{C}^m$, but if $\lambda' = -\infty$ there is no α such that $\Omega^{\alpha}_{-\infty} = \mathbf{C}^m$. The best we can do is to choose $\alpha_i = \delta_{ij}$ for some j. Then $\Omega^{\alpha}_{-\infty} = \{w \in \mathbf{C}^m; w_j \neq 0\}$. This is why we need the condition order$(F\colon G) > 1$, hence $\lambda' > -\infty$, to conclude that there exist λ', ρ', α and β such that $\Omega' \subset \Omega^{\alpha}_{\lambda'} \cap \Omega^{\beta}_{\rho'}$ for $\lambda' = \dfrac{\lambda}{\lambda - 1}$, $\rho' = \dfrac{\rho}{\rho - 1}$ with λ and ρ satisfying condition (a).

The obvious counterexample of Theorem 6.3 for order$(F\!:\!G) \leqq 1$ is when $F = G$. However if $f(t) > g \,\square\, K(t)$ for some t then condition (b′) cannot be satisfied for all $\varepsilon > 0$ using a function H holomorphic in $\mathbf{C}^n \times \Omega^{\beta}_{\rho'}$ with all $\beta_i \geqq 0$ by the maximum modulus principle. This follows since then the polydisk with radii $e^{\rho'}$ is contained in $\Omega^{\beta}_{\rho'}$, so that $(t,s) \mapsto h(t,s)$ is increasing in $s \in]-\infty, \rho'[$. If condition (b′) is fulfilled for some $\varepsilon > 0$ then $h(t,0) \geqq f(t)$ and $h(t,1) \leqq g \,\square\, K(t) + \varepsilon$, which leads to a contradiction if $0 < \varepsilon < f(t) - g \,\square\, K(t)$. Thus even if order$(F\!:\!G) \leqq 1$ it might happen that the function H must have a pole for some $w_i = 0$. To be precise either Ω' is contained in $\Omega^{\alpha}_{\lambda'} \cap \Omega^{\beta}_{\rho'}$ or there is a constant A such that AF, G cannot be extended to $\mathbf{C}^n \times \Omega'$ fulfilling condition (b) or (b′).

PROPOSITION 6.4. *Let F and G be two entire functions and let p, q be their coefficient functions respectively. For each $\varepsilon > 0$ in condition* (b) *or* (b′) *in Theorem* 6.2, *we can choose a holomorphic function H in $\Omega \subset \mathbf{C}^n \times \mathbf{C}^m$ which is rational in the $w \in \mathbf{C}^m$* variable if *$\tilde{\tilde{p}}$ and $\tilde{\tilde{q}}$ are finite in the same set and $\tilde{\tilde{p}} - \tilde{\tilde{q}}$ is bounded there. This is not possible iff $f - g \,\square\, K$ or $g - f \,\square\, K$ is unbounded from above.*

PROOF. Recall the definition of H in (6.4). We can approximate the Weierstrass functions by polynomials, since we made estimates only on compact sets. Recalling the definitions of μ and ν in (6.4) we conclude that we can choose H rational in the w variable if $\tilde{\tilde{p}}$ and $\tilde{\tilde{q}}$ are finite in the same set and $\tilde{\tilde{p}} - \tilde{\tilde{q}}$ is bounded there. We will see later in Proposition 6.6 that they must be finite in the same set if condition (b) is to hold. If on the other hand H is rational in the $w \in \mathbf{C}^m$ variable then there exists a polynomial P *in* $\mathbf{C}^m$ such that PH can be extended to an entire function in $\mathbf{C}^n \times \mathbf{C}^m$. (We regard P as constant in the first n variables). But $|P|$ is bounded on the set of $w \in \mathbf{C}^m$ such that $|w_i| = e$ and we can choose $P(1)$ nonzero. Thus if H satisfies condition (b′) for some $\varepsilon > 0$ and $f - g \square K$ is unbounded from above we get a contradiction by the maximum modulus theorem in the same manner as in the paragraph proceeding Theorem 6.3. After a change of variables $(z_1, \ldots, z_n, w_1, \ldots, w_m) \mapsto \left(z_1, \ldots, z_n, \frac{1}{w_1}, \ldots, \frac{1}{w_m}\right)$ we can apply the result on the case when $g - f \square K$ is unbounded.

REMARK. The first part of the proposition is also true for condition (c) and (c′) since if H satisfies condition (b) or (b′) for some $\varepsilon > 0$ then it also satisfies condition (c) or (c′) respectively.

We can get bounds for the coefficient functions from the growth functions. Assume that $f \square K \leqq g + C$ or $g \square K \leqq f + D$. Then the following estimates hold by Theorem 5.1.

$$\tilde{q} \square K \geqq g \geqq f \square K - C \geqq \tilde{p} \square K - C \tag{6.37}$$

or

$$\tilde{p} \square K \geqq f \geqq g \square K - D \geqq \tilde{q} \square K - D \tag{6.38}$$

Using (5.1) and since none of the functions $\tilde{\tilde{p}}, \tilde{\tilde{q}}$ or $\tilde{K}$ attains $-\infty$ we get $\tilde{\tilde{p}} + C \geqq \tilde{\tilde{q}}$ or $\tilde{\tilde{q}} + D \geqq \tilde{\tilde{p}}$. That is, we can estimate $\tilde{\tilde{p}} - \tilde{\tilde{q}}$ from above or below. However we cannot have both $f \square K \leqq g + C$ and $g \square K \leqq f + D$ at the same time if F and G are transcendental. On the other hand if $|f - g| \leqq C$ then we can make estimates as above to get $|\tilde{\tilde{p}} - \tilde{\tilde{q}}| \leqq C - \tilde{K}$. It is easy to calculate $\tilde{K}$ and it is done in Kiselman [2], where also the estimates $\log(\tau + 1) \leqq -\tilde{K}(\tau) \leqq \log(\tau + 1) + 1$ are obtained. We see that in this case the function H can have logarithmic growth in the powers of the $w \in \mathbf{C}^m$ variable. Taking $C = 0$ we see that this should be possible to improve.

Let us call a logarithmically convex Reinhardt domain $\Omega' \subset \mathbf{C}^m$ a λ', ρ'*-domain* if $\inf[t; (e^t, \ldots, e^t) \in \Omega'] = \lambda' < 0$, $\sup[t; (e^t, \ldots, e^t) \in \Omega'] = \rho' > 0$ and the complement of Ω' contains $\{w \in \mathbf{C}^m; w_i = 0\}$ for some $i = 1, \ldots, m$, which is always the case when $\lambda' > -\infty$. Then $\Omega' \subset \Omega^{\alpha}_{\lambda'} \cap \Omega^{\beta}_{\rho'}$ for some $\alpha, \beta \in \mathbf{R}^m$ and the following proposition holds.

PROPOSITION 6.5. *Each condition in Theorem 6.2 is equivalent with the same condition using* $\omega = \mathbf{C}^n \times \Omega'$, $\Omega' \subset \mathbf{C}^m$ *a* (λ', ρ')*-domain instead of* $\Omega = \{(z, w) \in \mathbf{C}^n \times \mathbf{C}^m; w \in \Omega^{\alpha}_{\lambda'} \cap \Omega^{\beta}_{\rho'}\}$.

PROOF. If we examine the proof that (b), (b′) implies (c), (c′) implies (a) in Theorem 6.2 we see that this works also for a $(\lambda' \rho')$-domain. Thus if some condition holds for ω then condition (a) will hold and from this we can conclude that the same condition holds for Ω. The other implication is trivial.

Thus we have found all finite-dimensional logarithmically convex Reinhardt domains to which we can extend in the way of Theorem 6.2 without putting any extra conditions on F and G.

REMARK. It is clear that it is enough for ω to contain a (λ', ρ')-domain and be contained in one to satisfy Proposition 6.5.

A theorem like Theorem 6.2 can not hold if F is a polynomial when G is not, or vice versa, as seen by the following proposition.

PROPOSITION 6.6. *Let $F, G \in \mathcal{O}(\mathbf{C}^n)$ satisfy condition* (a) *or* (b) *in Theorem* 6.2. *Expand F and G in homogeneous polynomials. Then $F(z) = \sum_{j=M}^{N} P_j(z)$, where $P_M, P_N \neq 0$, if and only if $G(z) = \sum_{j=M}^{N} Q_j(z)$, $Q_M, Q_N \neq 0$. Also $F(z) = \sum_{j=M}^{\infty} P_j(z)$, where $P_M \neq 0$, if and only if $G(z) = \sum_{j=M}^{\infty} Q_j(z)$, $Q_M \neq 0$.*

PROOF. We have already noted in the proof of Theorem 6.2 that condition (a) would be violated otherwise. If we examine the proof that (b′) implies (c′) implies (a), we see that (b) implies (a) regardless of if F and G are both polynomials or not.

7. Extension of entire functions, refined case.

If we instead expand $F \in \mathcal{O}(\mathbf{C}^n)$ in a Taylor series

$$F(z) = \sum_k A_k z^k, \qquad z \in \mathbf{C}^n,\ k \in \mathbf{N}^n, \tag{7.1}$$

where k is a multi-index, we define *the refined coefficient function of F* as

$$a(k) = \begin{cases} -\log|A_k|, & k \in \mathbf{N}^n; \\ +\infty & k \in \mathbf{R}^n \setminus \mathbf{N}^n; \end{cases} \tag{7.2}$$

Similar connections hold between the refined growth and coefficient functions, as between the ordinary growth and coefficient functions. Define K_n by

$$K_n(\xi) = K(\xi_1) + \dots + K(\xi_n), \qquad \xi \in \mathbf{R}^n, \tag{7.3}$$

with K defined by (5.2). Then we have the following theorem.

THEOREM 7.1 (Kiselman [2], Theorem 6.6). *Let F be an entire function in $\mathbf{C}^n$. Define a, f_r by* (7.2), (3.5) *respectively and K_n by* (7.3). *Then*

$$\tilde{a} \leqq f_r \leqq \tilde{a} \,\square\, K_n \qquad on\ \mathbf{R}^n.$$

COROLLARY 7.2. *Let F, G be two entire functions in $\mathbf{C}^n$. Let f_r, g_r be defined by* (3.5) *and a, b by* (7.2), *with F, G respectively. Then*

(7.4) $$\operatorname{order}(f_r : g_r) = \operatorname{order}(\tilde{a} : \tilde{b}) = \operatorname{type}(\tilde{\tilde{b}} : \tilde{\tilde{a}}).$$

PROOF. This follows from Lemma 3.1 and Theorem 2.3 in the same way as in Corollary 5.2.

LEMMA 7.3. *Let $F \in \mathcal{O}(\mathbf{C}^n)$ be an entire function and a be its refined coefficient function defined by* (7.2). *Then*

$$\frac{a(k)}{|k|} \to +\infty \qquad \text{as } |k| \to +\infty,$$

where $|k| = \sum_{j=1}^n |k_j| = \sum_{j=1}^n k_j$.

PROOF. Since F is entire

$$|A_k| \prod_{l=1}^n R_l^{k_l} \to 0 \qquad \text{as } |k| \to +\infty,$$

for all $R \in \mathbf{R}_+^n$. Hence taking logarithms

$$\sum_{l=1}^n k_l \log R_l - a(k) \to -\infty \qquad \text{as } |k| \to +\infty.$$

From the definition of a this actually holds for $k \in \mathbf{R}^n$. Thus we can apply Lemma 5.4 on a refined coefficient function since by definition it is obviously lower semicontinuous.

Define the convex functions

(7.5) $$h_w(t) = \sup_z [\log |H(z, w)|;\, z \in \mathbf{C}^n,\, |z_i| \leqq e^{t_i}], \qquad t \in \mathbf{R}^n,\, w \in \Omega' \subset \mathbf{C}^m;$$

and

(7.6) $$\begin{aligned} h(t, s) &= \sup_{z,w} [\log |H(z, w)|;\, z \in \mathbf{C}^n,\, w \in \Omega',\, |z_i| \leqq e^{t_i},\, |w_i| = e^s,\, \forall i] \\ &= \sup_w [h_w(t);\, |w_i| = e^s,\, \forall i = 1, \ldots, m], \qquad t \in \mathbf{R}^n,\, s \in \mathbf{R}; \end{aligned}$$

Then in complete analogy with Theorem 6.2, we have

THEOREM 7.4. *Let $F, G \in \mathcal{O}(\mathbf{C}^n)$ be two transcendental entire functions. Define their refined growth functions f_r, g_r by* (3.5) *respectively. Let $\lambda, \rho \in \mathbf{R}$, $\alpha, \beta \in \mathbf{R}^m$ satisfy $0 < \lambda \leqq 1 \leqq \rho < +\infty$ and $\sum \alpha_i = \sum \beta_i = 1$. Define the domain*

$$\Omega = \{(z, w) \in \mathbf{C}^n \times \mathbf{C}^m;\, w \in \Omega_{\lambda'}^{\alpha} \cap \Omega_{\rho'}^{\beta}\}$$

where $\lambda' = \dfrac{\lambda}{\lambda - 1}$, $(-\infty \leqq \lambda' < 0)$, $\rho' = \dfrac{\rho}{\rho - 1}$, $(1 < \rho' \leqq +\infty)$ *and* $\Omega^{\alpha}_{\lambda'}$, $\Omega^{\beta}_{\rho'}$ *are defined by* (6.3′). *Then the following conditions are equivalent:*

(a)
$$\operatorname{order}(g_r : f_r) \leqq \rho;$$
$$\operatorname{order}(f_r : g_r) \leqq \frac{1}{\lambda};$$

(b) *For each* $\varepsilon > 0$ *(or equivalently some* $\varepsilon > 0$*) there exists an* $H \in \mathcal{O}(\Omega)$, *such that*

$$\begin{cases} f_r \leqq h_w \,\square\, K_n + \varepsilon, \; h_w \leqq f_r \,\square\, K_n + \varepsilon, & |w_i| = 1, \; \forall i = 1, \dots, m; \\ g_r \leqq h_w \,\square\, K_n + \varepsilon, \; h_w \leqq g_r \,\square\, K_n + \varepsilon, & |w_i| = e, \; \forall i = 1, \dots, m, \end{cases}$$

where h_w *is defined by* (7.5) *and* K_n *by* (7.3),

(b′) *The condition* (*b*) *holds with the extra assumption*

$$H(z, \mathbf{1}) = F(z) \text{ and } H(z, \boldsymbol{e}) = G(z);$$

where $\mathbf{1} = (1, \dots, 1)$ *and* $\boldsymbol{e} = (e, \dots, e)$. (*m times*).

(c) *There exists an* $H \in \mathcal{O}(\Omega)$, *such that*

$$\operatorname{order}(f_r : h(\cdot, 0)) = \operatorname{order}(h(\cdot, 0) : f_r) = 1;$$
$$\operatorname{order}(g_r : h(\cdot, 1)) = \operatorname{order}(h(\cdot, 1) : g_r) = 1,$$

where $h(\cdot, \cdot)$ *is defined by* (7.6).

(c′) *The condition* (c) *holds with the extra assumption*

$$H(z, \mathbf{1}) = F(z) \text{ and } H(z, \boldsymbol{e}) = G(z).$$

(*If F and G are polynomials the theorem still holds if condition* (c) *is altered to*

$$\operatorname{order}(f_r : h(\cdot, 0)) = \operatorname{order}(h(\cdot, 0) : f_r) = 0;$$
$$\operatorname{order}(g_r : h(\cdot, 1)) = \operatorname{order}(h(\cdot, 1) : g_r) = 0.$$

In this case condition (a) *is equivalent with* $\operatorname{order}(f_r : g_r) = \operatorname{order}(g_r : f_r) = 0$.)

If a holomorphic function $H \in \mathcal{O}(\Omega)$ *satisfies condition* (b) *or* (b′) *for some* $\varepsilon > 0$ *then it also satisfies condition* (c) *or* (c′) *respectively.*

Proof. Make the expansions

$$F(z) = \sum_{k \in N^n} A_k z^k, \qquad G(z) = \sum_{k \in N^n} B_k z^k.$$

Let a and b be the refined coefficient functions of F and G respectively. This time in the proof that (a) implies (b′) we use the function H defined by

(7.7) $$H(z,w) = \frac{1}{E_N\left(\frac{1}{e}\right)}\left[E_N\left(\frac{w^\gamma}{e}\right)\sum_{k\in N^n} A_k z^k w^{\mu_k} + E_N\left(\frac{1}{w^{\gamma'}}\right)\sum_{k\in N^n} B_k z^k\left(\frac{w}{e}\right)^{\nu_k}\right],$$

with the same definitions of N, E_N, γ and γ' as in the proof of Theorem 6.2. The multi-indices $\mu_k, \nu_k \in \mathbf{Z}^m$ are chosen such that $\mu_{ki} = \nu_{ki} = 0$, if $\tilde{\tilde{a}}(k) = +\infty$, which occurs if and only if $\tilde{\tilde{b}}(k) = +\infty$, since we are demanding finite order. (Recall Corollary 7.2). Otherwise we take μ_{ki} as the integer part of $\alpha_i \min(\tilde{\tilde{a}}(k) - \tilde{\tilde{b}}(k) - N, 0)$ and ν_{ki} as the integer part of $\beta_i \max(\tilde{\tilde{a}}(k) - \tilde{\tilde{b}}(k) + N + m, 0)$. We get the partial refined coefficient function c_w of H as

(7.8) $$c_w(k) = -\log\left|\frac{1}{E_N(1/e)}E_N\left(\frac{w^\gamma}{e}\right)A_k w^{\mu_k} + E_N\left(\frac{1}{w^{\gamma'}}\right)B_k\left(\frac{w}{e}\right)^{\nu_k}\right]\right|.$$

By the triangle inequality and (6.5), we have

(7.9) $$-\log(1+e^{-N-1}) - \log\left(\left|E_N\left(\frac{w^\gamma}{e}\right)A_k w^{\mu_k}\right| + \left|E_N\left(\frac{1}{w^{\gamma'}}\right)B_k\left(\frac{w}{e}\right)^{\nu_k}\right|\right) \leqq c_w(k)$$

and

(7.10) $$c_w(k) \leqq -\log(1-e^{-N-1}) - \log\left|\left|E_N\left(\frac{w^\gamma}{e}\right)A_k w^{\mu_k}\right| - \left|E_N\left(\frac{1}{w^{\gamma'}}\right)B_k\left(\frac{w}{e}\right)^{\nu_k}\right|\right|.$$

The estimates are now the same as those in Theorem 6.2. When $|w_i| = 1$, we get $\tilde{a} - e^{2-N} \leqq c_w$ by (7.9) and estimates like (6.6) and (6.7), which implies $\tilde{c}_w \leqq \tilde{a} + e^{2-N}$. For $k \in \mathbf{N}^n$ such that $a(k) = \tilde{\tilde{a}}(k)$, we get by (7.10) $c_w(k) \leqq \tilde{\tilde{a}}(k) + e^{2-N}$. Hence using Lemma 5.4 we get $\tilde{a} \leqq \tilde{c}_w + e^{2-N}$, so we have $\tilde{a} - \varepsilon \leqq \tilde{c}_w \leqq \tilde{a} + \varepsilon$ using $e^{2-N} < \varepsilon$. Finally by Theorem 7.1

$$f_r \leqq \tilde{a} \,\square\, K_n \leqq \tilde{c}_w \,\square\, K_n + \varepsilon \leqq h_w \,\square\, K_n + \varepsilon, \qquad |w_i| = 1;$$

$$h_w \leqq \tilde{c}_w \,\square\, K_n \leqq \tilde{a} \,\square\, K_n + \varepsilon \leqq f_r \,\square\, K_n + \varepsilon, \qquad |w_i| = 1;$$

When $|w_i| = e$ we get using (7.9), (7.10) and similar estimates as in (6.8), (6.9), $\tilde{b} - \varepsilon \leqq \tilde{c}_w \leqq \tilde{b} + \varepsilon$. Hence

$$g_r \leqq \tilde{b} \,\square\, K_n \leqq \tilde{c}_w \,\square\, K_n + \varepsilon \leqq h_w \,\square\, K_n + \varepsilon, \qquad |w_i| = e;$$

$$h_w \leqq \tilde{c}_w \,\square\, K_n \leqq \tilde{b} \,\square\, K_n + \varepsilon \leqq g_r \,\square\, K_n + \varepsilon, \qquad |w_i| = e;$$

To show that H is holomorphic in Ω, we now show that

$$|A_k| R^k r^{\mu_k} = |A_k| \prod_{l=1}^{n} R_l^{k_l} \prod_{i=1}^{m} r_i^{\mu_{ki}} \to 0 \qquad \text{as } |k| \to +\infty,$$

uniformly for all $R \in \mathrm{R}^n_+$, $0 \leqq R_l \leqq R' < +\infty$ and for all $r \in \mathrm{R}^m_+$ such that

$$(\log r_1, \ldots, \log r_m) \in S, \ S \text{ is compact and } S \subset \operatorname{int} S^{\alpha}_{\lambda'}.$$

Taking logarithms, this is equivalent to

$$\text{(7.11)} \qquad \frac{\sum_{i=1}^m \mu_{ki} \log r_i - a(k)}{|k|} \to -\infty \qquad \text{as } |k| \to +\infty.$$

By Corollary 7.2 we have $\tilde{\tilde{b}} \leqq d\tilde{\tilde{a}} + c_d$, for $d > 1/\lambda$, so as in the proof of Theorem 6.2 we can make estimates

$$\text{(7.12)} \qquad \frac{\sum_{i=1}^m \mu_{ki} \log r_i - a(k)}{|k|}$$

$$\leqq \max\left(-\frac{a(k)}{|k|}, \frac{\sum_{i=1}^m \alpha_i(\tilde{\tilde{a}}(k) - d\tilde{\tilde{a}}(k) + O(1)) \log r_i - \tilde{\tilde{a}}(k)}{|k|}\right)$$

$$= \max\left(-\frac{a(k)}{|k|}, \frac{[(1-d)\sum_{i=1}^m \alpha_i \log r_i - 1]\tilde{\tilde{a}}(k) + O(\max|\log r_i|)}{|k|}\right)$$

and the last expression tends to $-\infty$ locally uniformly on $S^{\alpha}_{d'}$ as $|k| \to +\infty$ for $d' > \dfrac{1}{1-d} > \lambda'$ by Lemma 7.3 and Lemma 5.4.

Also we can show that

$$|B_k| R^k \left(\frac{r}{e}\right)^{\nu_k} = |B_k| \prod_{l=1}^n R_l^{k_l} \prod_{i=1}^m \left(\frac{r_i}{e}\right)^{\nu_{ki}} \to 0 \qquad \text{as } |k| \to +\infty,$$

uniformly for all $R \in \mathrm{R}^n_+$, $0 \leqq R_l \leqq R' < +\infty$ and for all $r \in \mathrm{R}^m_+$ such that

$$(\log r_1, \ldots, \log r_m) \in S, \ S \text{ is compact and } S \subset \operatorname{int} S^{\beta}_{\rho'}.$$

The proofs that (b) implies (c) and (b′) implies (c′) are similar to those of Theorem 6.2 and omitted. Also the proof that (c) implies (a) is similar and omitted. We have the trivial implications (c′) implies (c) and (b′) implies (c′), so we are done.

We have some similar statements as those following Theorem 6.2.

COROLLARY 7.5. *The conditions in Theorem* 7.4 *are independent of* α *and* β.

THEOREM 7.6. *Let* $F, G \in \mathcal{O}(\mathrm{C}^n)$ *be two entire functions and* $\Omega' \subset \mathrm{C}^m$ *be a logarithmically convex Reinhardt domain such that condition* (c) *holds in Theorem* 7.4 *for* $\Omega = \mathrm{C}^n \times \Omega'$. *If* $\operatorname{order}(f_r : g_r) \geqq 1$, *then* $\Omega' \subset \Omega^{\alpha}_{\lambda'} \cap \Omega^{\beta}_{\rho'}$ *for some* $\alpha, \beta \in \mathrm{R}^m$ *and* *for some* $\lambda' = \dfrac{\lambda}{\lambda - 1}$ *and* $\rho' = \dfrac{\rho}{\rho - 1}$ *such that* ρ *and* λ *satisfies condition* (a). *If* $\operatorname{order}(f_r : g_r) \leqq 1$ *then* $\Omega' \subset \Omega^{\beta}_{\rho'}$.

Also in the refined case it is necessary to have $\Omega' \subset \Omega^{\alpha}_{\lambda'} \cap \Omega^{\beta}_{\rho'}$ to ensure the possibility to extend all functions with specified orders to $\mathbf{C}^n \times \Omega'$.

PROPOSITION 7.7. *Let F and G be two entire functions and let a, b be their refined coefficient functions respectively. For each* $\varepsilon > 0$ *in condition* (b) *or* (b′) *in Theorem* 7.4, *we can choose a holomorphic function H in* $\Omega \subset \mathbf{C}^n \times \mathbf{C}^m$ *which is rational in the* $w \in \mathbf{C}^m$ *variable if* $\tilde{\tilde{a}}$ *and* $\tilde{\tilde{b}}$ *are finite in the same set and* $\tilde{\tilde{a}} - \tilde{\tilde{b}}$ *is bounded there. This is not possible if* $f_r - g_r \,\square\, K_n$ *or* $g_r - f_r \,\square\, K_n$ *is unbounded from above.*

PROPOSITION 7.8. *Each condition in Theorem* 7.4 *is equivalent with the same condition using* $\omega = \mathbf{C}^m \times \Omega'$, $\Omega' \subset \mathbf{C}^m$ *a* (λ', ρ')*-domain instead of* $\Omega = \{(z, w) \in \mathbf{C}^n \times \mathbf{C}^m;\, w \in \Omega^{\alpha}_{\lambda'} \cap \Omega^{\beta}_{\rho'}\}$.

PROPOSITION 7.9. *Let* $F, G \in \mathcal{O}(\mathbf{C}^n)$ *satisfy condition* (a) *or* (b) *in Theorem* 7.4. *Expand F, G in Taylor series*

$$F(z) = \sum_{k \in N^n} A_k z^k, \qquad G(z) = \sum_{k \in N^n} B_k z^k.$$

Let $C(F)$ *denote the convex hull of those* $k \in \mathbf{N}^n$ *for which* $A_k \neq 0$ *and define* $C(G)$ *similarly. Then* $C(F) = C(G)$.

PROOF. Let $\operatorname{dom}(\tilde{\tilde{a}}) = \{x \in \mathbf{R}^n;\, \tilde{\tilde{a}}(x) < +\infty\}$ be the *effective domain* of $\tilde{\tilde{a}}$, where a is the refined coefficient function of F. If b is the refined coefficient function of G then by Corollary 7.2 we must have $\operatorname{dom}(\tilde{\tilde{a}}) = \operatorname{dom}(\tilde{\tilde{b}})$, if condition (a) is to hold. Now $C(F) \subset \operatorname{dom}(\tilde{\tilde{a}}) \subset \operatorname{cl} C(F)$ holds in general and we will see that actually $\operatorname{dom}(\tilde{\tilde{a}}) = C(F)$. Define for $j \in \mathbf{N}$

$$a_j(k) = \begin{cases} a(k), & k_i \leqq j, \forall i; \\ +\infty & \text{otherwise}; \end{cases} \tag{7.13}$$

and let $C(a_j)$ denote the convex hull of $\operatorname{dom}(a_j)$. Then $C(a_j)$ is a subset of $C(F)$ and is closed since it is finitely generated. Now take a point $x \in \operatorname{cl} C(F) \backslash C(F)$ and an arbitrary number $M \in \mathbf{R}$. Then $x \notin C(a_j)$. Since $C(a_j)$ is closed and a_j is bounded from below there exists a non-vertical hyperplane $P_j = \{(y, z) \in \mathbf{R}^n \times \mathbf{R};\, z = \xi_j \cdot y + c_j\}$ passing through (x, M) and also satisfying $\xi_j \cdot k + c_j \leqq a_j(k)$ for each k. Since by Lemma 7.3 a has faster growth than any linear function we can take $P_{j+1} = P_j = P$ for all $j \geqq N$, for some number N and some hyperplane P. Now $a_j \to a$ pointwise, so by (2.3) and since M was arbitrary we conclude that

$\tilde{\tilde{a}}(x) = +\infty$. This shows that condition (a) implies $C(F) = C(G)$. Also (b) implies (a) regardless of the expansions of F and G.

8. Transformation of the relative order.

In Theorem 6.2 the correspondence between the maximum size of the domain Ω and the relative orders is complete when $\operatorname{order}(F:G)$, $\operatorname{order}(G:F) > 1$ or if F, G are polynomials. This is not always the case, but we will see that it is possible to transform the orders. Assume that $F, G \in \mathcal{O}(\mathbf{C}^n)$ are transcendental and that l, k are integers. We define the mapping $\sigma_l\colon \mathbf{C}^n \to \mathbf{C}^n$ by $\sigma_l(z) = (z_1^l, \dots, z_n^l)$ and make the transformations

$$F_{lk}(z) = F(\sigma_l(z))^k; \tag{8.1}$$

$$G_{kl}(z) = G(\sigma_k(z))^l.$$

We have $|\sigma_l(z)| = |z|^l$, with the norm $|z| = \max_i |z_i|$, which is the norm we will now use to determine the growth function of F_{lk}

$$\begin{aligned} f_{lk}(t) &= \sup[\log|F_{lk}(z)|;\ z \in \mathbf{C}^n,\ |z| \leqq e^t] \\ &= \sup[\log|F(\sigma_l(z))^k|;\ z \in \mathbf{C}^n,\ |z| \leqq e^t] \\ &= \sup[k\log|F(z')|;\ z' \in \mathbf{C}^n,\ |z'| \leqq e^{lt}] \\ &= kf(lt), \end{aligned} \tag{8.2}$$

f being the growth function of F. Similarily $g_{kl}(t) = lg(kt)$. We can now see the effect of the transformation on the relative order:

$$f_{lk}(t) \leqq \frac{1}{a} g_{kl}(at) + c_a,$$

$$kf(lt) \leqq \frac{l}{a} g(akt) + c_a,$$

$$f(s) \leqq \frac{l}{ka} g\left(\frac{ka}{l} s\right) + c_a.$$

Thus

$$\operatorname{order}(F_{lk} : G_{kl}) = \frac{l}{k} \operatorname{order}(F : G) \tag{8.3}$$

and similarly

$$\operatorname{order}(G_{kl} : F_{lk}) = \frac{k}{l} \operatorname{order}(G : F). \tag{8.4}$$

This gives the invariance

$$\text{order}(F_{lk} : G_{kl})\,\text{order}(G_{kl} : F_{lk}) = \text{order}(F : G)\,\text{order}(G : F) \geqq 1. \tag{8.5}$$

If we have strict inequality there are numbers $k, l \in \mathbf{N}$ such that both

$$\text{order}(F_{lk} : G_{kl}), \text{order}(G_{kl} : F_{lk}) > 1.$$

Otherwise we can get the orders arbitrarily close to one.

These transformations also work on the refined growth functions.

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