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ABOUT CERTAIN SINGULAR KERNELS $K(x, y) = K_1(x - y)K_2(x + y)$

TOMÁS GODOY, LINDA SAAL and MARTA URCIUOLO[1]

§ 1. Introduction.

In this paper we give a solution to a problem about the L^p-boundedness, $1 < p < \infty$, and the weak type 1-1 of certain singular integral operators. Here we study operators of the form

$$Tf(\xi) = \int_{R^n} k_1(\xi - y)k_2(\xi + y)f(y)\,dy \tag{1.1}$$

for a wide class of functions k_1 and k_2.

The case $n = 1, p = 2$, has been solved in [Ri-S] when k_1 is the Hilbert kernel and k_2 satisfies

$$|k_2(x)| \leqq c \quad \text{and} \quad |k_2'(x)| \leqq \frac{c}{|x|}, \quad \text{for some} \quad c > 0 \tag{1.2}$$

The authors used strongly the L^2-boundedness of the Hilbert transform and the local Lipschitz condition (1.2).

Following this approach, we take $k_1(x) = \sum_{j \in Z} 2^{jn}\varphi_j(2^j x)$ where $\{\varphi_j\}_{j \in Z}$ is a family of functions in $L^1(R^n)$ satisfying

$$\int \varphi_j(x)\,dx = 0 \tag{1.3}$$

and for some $0 < \varepsilon < 1$

$$\int |\varphi_j(x + h) - \varphi_j(x)|\,dx \leqq c\,|h|^\varepsilon \tag{1.4}$$

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$$\int (1+|x|^{\varepsilon})\,|\varphi_j(x)|\,dx \leqq c \tag{1.5}$$

with c independent of j. It is known that $k_1(x)$ is a tempered distribution and that the operator of convolution by k_1 is bounded on $L^p(R^n)$, $1 < p < \infty$. See for example [Sa-U]. So we ask for suitable conditions about k_2 in order to obtain the boundedness of the operator given by (1.1), for this kind of kernels k_1.

Condition (1.2) leads us to consider functions k_2 satisfying

$$\|k_2\|_\infty \leqq c \tag{1.6}$$

and for some $0 < \delta < 1$, for all $|h| < \dfrac{|x|}{2}$,

$$|k_2(x+h) - k_2(x)| \leqq c\left(\left|\frac{h}{x}\right|\right)^{\delta} \tag{1.7}$$

The main result we obtain is the following.

THEOREM A. *Let* $\{\varphi_j\}_{j\in Z}$ *be a family of functions in* $L^1(R^n)$ *with compact support contained in* $\{x \in R^n\colon 2^{-1} \leqq |x| \leqq 2\}$ *satisfying* (1.3) *and* (1.4). *Let* k_2 *be a function satisfying* (1.6) *and* (1.7). *Then for* $f \in L^p(R^n)$, $1 < p < \infty$,

$$Tf(\xi) = \lim_{(N,M)\to(-\infty,\infty)} \int \sum_{N\leqq j\leqq M} 2^{jn}\varphi_j(2^j(\xi - y))k_2(\xi + y)f(y)\,dy$$

exists almost everywhere in R^n *and* $\|Tf\|_p \leqq c_p\|f\|_p$. *Moreover, if* $f \in S(R^n)$ $|\{x\colon |Tf(x)| > \lambda\}| \leqq c\lambda^{-1}\|f\|_1$ *for all* $\lambda > 0$ *(weak type* 1–1).

In § 2 we give some preliminaries, in § 3 we prove Theorem A, in § 4 we obtain the same result replacing the hypothesis of compact support of φ_j by (1.5), and in § 5 we show some examples of kernels k_1 and k_2 that give rise to operators Tf as in theorem A.

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§ 2. Preliminaries.

In this section we state some properties about the L^p-boundedness and the weak type 1–1 of certain singular integral operators and the maximal operators associated. We set, for $g\colon R^n \to C$ and $j \in Z$, $g^{(j)}(x) = 2^{jn}_g(2^j x)$.

Let us consider a family of functions $\{\varphi_j\}_{j\in Z}$ in $L^1(R^n)$ satisfying (1.3), (1.4) and (1.5). It is not hard to see that if we take $\{\sigma_k\}_{k\in Z}$ and $\{\mu_k\}_{k\in Z}$ the Borel measures with density $\varphi^{(-k)}_{-k}$ and $|\varphi^{(-k)}_{-k})|$ respectively, then

$$|\hat{\mu}_k(\xi) - \hat{\mu}_k(0)| \leqq c(2^k|\xi|)^\varepsilon, \quad k \in Z$$

$$|\hat{\mu}_k(\xi)| \leqq c(2^k|\xi|)^{-\varepsilon}, \quad k \in Z$$

with ε as in (1.4) and (1.5). Moreover, the same conditions are satisfied by $\{\sigma_k\}_{k\in Z}$. Applying straightforward Theorem F in [D-R] we obtain the following results: For $1 < p < \infty$

$$Mf(x) = \sup_k (|\varphi_k^{(k)}| * |f|)(x) \quad \text{is bounded on} \quad L^p(R^n) \tag{2.1}$$

$$K_1 f(x) = \sum_k (\varphi_k^{(k)} * f)(x) \quad \text{is bounded on} \quad L^p(R^n) \tag{2.2}$$

Moreover if supp $\varphi_k^{(k)} \subseteq \{x: |x| < 2^{k+1}\}$, then

$$K_1^* f(x) = \sup_j \left| \sum_{k \leqq j} (\varphi_k^{(k)} * f)(x) \right| \quad \text{is bounded on} \quad L^p(R^n) \tag{2.3}$$

REMARK 2.4. We also observe that, for $f \in S(R^n)$, the operators M, K_1 and K_1^* above defined, are of weak type 1–1. Indeed, with standard techniques we can see that, for some $c > 0$

$$\sum_{k\in Z} \int_{|x|>|2y|} |\varphi_k^{(k)}(x+y) - \varphi_k^{(k)}(x)|\, dx \leqq c \quad \text{for all } y \in R^n \tag{2.5}$$

See for example [G-St], [Sa-U]. So, the boundedness on $L^2(R^n)$, (2.5) and theorem 2.4 in [C–W] imply the weak type 1–1 of K_1. The proofs of the weak type (1-1) of M and K_1^* follow the same lines than those in the theorem last mentioned.

§ 3. The Main Result.

Before beginning with the proof of Theorem A, we make the following

REMARK 3.1. Let $\{\varphi_j\}_{j\in Z}$ be a family of functions satisfying (1.3), (1.4) and supp $\varphi_j \subseteq \{x: 2^{-1} \leqq |x| \leqq 2\}$. Then, for $r \in Z$, $f \in L^p(R^n)$, $1 < p < \infty$, there exists

$$S_r f(\xi) = \lim_{(N,M)\to(-\infty,\infty)} \sum_{N\leqq j\leqq M} \int_{|x|>2^r} \varphi_j^{(j)}(x) f(\xi - x)\, dx \tag{3.2}$$

for almost every $\xi \in R^n$

Indeed, since supp $\varphi_j^{(j)} \subseteq \{x: 2^{-j-1} \leqq |x| \leqq 2^{-j+1}\}$, we have that

$$S_r f(\xi) = \int_{|x|>2^r} \varphi_{-r}^{(-r)}(x) f(\xi - x)\, dx + \sum_{j\leqq -r-1} \int \varphi_j^{(j)}(x) f(\xi - x)\, dx$$

Thus, for all $r \in Z$,

$$|S_r f(\xi)| \leqq Mf(\xi) + K_1^* f(\xi)$$

where M and K_1^* are defined by (2.1) and (2.3) respectively.

REMARK 3.3. The last inequality of the previous remark, (2.1), (2.3) and remark (2.4) imply that $\sup_{r \in Z} |S_r f(\xi)|$ is bounded on $L^p(R^n)$, $1 < p < \infty$, and for $f \in S(R^n)$ it is of weak type 1–1.

The same results hold for

$$\tilde{S}_r f(\xi) = \lim_{(N,M) \to (-\infty, \infty)} \sum_{N \leqq j \leqq M} \int_{|x| \leqq 2^r} \varphi_j^{(j)}(x) f(\xi - x)\, dx \tag{3.4}$$

Indeed, $\tilde{S}_r f(\xi) = K_1 f(\xi) - S_r f(\xi)$ and we apply (2.2)

PROOF OF THEOREM A. For $M, N \in Z$, $N < M$, $f \in S(R^n)$ and $\xi \in R^n$, we set

$$T_{NM} f(\xi) = \sum_{N \leqq j \leqq M} \int \varphi_j^{(j)}(\xi - y) k_2(\xi + y) f(y)\, dy$$

With a change of variables, we obtain

$$T_{NM} f(\xi) = \sum_{N \leqq j \leqq M} \int \varphi_j^{(j)}(x) k_2(2\xi - x) f(\xi - x)\, dx.$$

We fix $l = l(\xi) \in Z$ such that $2^l \leqq |\xi| < 2^{l+1}$ and we decompose

$$T_{NM} f(\xi) = \sum_{N \leqq j \leqq M} \int_{|x| \leqq 2^l} + \sum_{N \leqq j \leqq M} \int_{2^l < |x| \leqq 2^{l+3}} + \sum_{N \leqq j \leqq M} \int_{|x| > 2^{l+3}} \tag{3.5}$$

Since $\operatorname{supp} \varphi_j^{(j)} \subseteq \{x: 2^{-j-1} \leqq |x| \leqq 2^{-j+1}\}$, the central sum is independent of N and M, for $|N|$ and $|M|$ large enough, and

$$\left| \sum_{N \leqq j \leqq M} \int_{2^l < |x| \leqq 2^{l+3}} \varphi_j^{(j)}(x) k_2(2\xi - x) f(\xi - x)\, dx \right| \leqq 6 \, \|k_2\|_\infty \, Mf(\xi)$$

where Mf is the maximal operator defined in (2.1). Furthermore (2.1) and remark (2.4) imply

$$\lim_{(N,M) \to (-\infty, \infty)} \sum_{N \leqq j \leqq M} \int_{2^l < |x| \leqq 2^{l+3}} \varphi_j^{(j)}(x) k_2(2\xi - x) f(\xi - x)\, dx$$

is bounded on $L^p(R^n)$ and of weak type 1–1.

We now study the first sum of (3.5)

$$\sum_{N\leqq j\leqq M}\int_{|x|\leqq 2^l}\varphi_j^{(j)}(x)k_2(2\xi-x)f(\xi-x)\,dx$$

$$=\sum_{N\leqq j\leqq M}\int_{|x|\leqq 2^l}\varphi_j^{(j)}(x)[k_2(2\xi-x)-k_2(2\xi)]f(\xi-x)\,dx$$

$$+\,k_2(2\xi)\sum_{N\leqq j\leqq M}\int_{|x|\leqq 2^l}\varphi_j^{(j)}(x)f(\xi-x)\,dx$$

Now

$$\sum_{N\leqq j\leqq M}\int_{|x|\leqq 2^l}|\varphi_j^{(j)}(x)|\,|k_2(2\xi-x)-k_2(2\xi)|\,|f(\xi-x)|\,dx\leqq c\,Mf(x)$$

Indeed, since $|x|\leqq 2^l\leqq|\xi|$ we apply (1.7) to obtain

$$\sum_{N\leqq j\leqq M}\int_{|x|\leqq 2^l}|\varphi_j^{(j)}(x)|\,|k_2(2\xi-x)-k_2(2\xi)|\,|f(\xi-x)|\,dx$$

$$\leqq c\sum_{N\leqq j\leqq M}\int_{|x|\leqq 2^l}|\varphi_j^{(j)}(x)|\left(\frac{|x|}{|\xi|}\right)^{\delta}|f(\xi-x)|\,dx$$

$$\leqq c\sum_{j\geqq -l-1}\int_{|x|\leqq 2^l}|\varphi_j^{(j)}(x)|\,2^{-j\delta}\,|f(\xi-x)|\,dx\leqq c\,Mf(\xi)$$

So $\displaystyle\lim_{(N,M)\to(-\infty,\infty)}\sum_{N\leqq j\leqq M}\int_{|x|\leqq 2^l}\varphi_j^{(j)}(x)[k_2(2\xi-x)-k_2(2\xi)]f(\xi-x)\,dx$ exists for all $\xi\in R^n$, moreover (2.1) and remark (2.4) imply that it is bounded on $L^p(R^n)$, $1<p<\infty$, and of weak type 1–1.

Now, if $\tilde{S}_l$ is as in (3.4),

$$\lim_{(N,M)\to(-\infty,\infty)}k_2(2\xi)\sum_{N\leqq j\leqq M}\int_{|x|\leqq 2^l}\varphi_j^{(j)}(x)f(\xi-x)\,dx=k_2(2\xi)\tilde{S}_l f(\xi)$$

and it is absolutely bounded by $\|k_2\|_\infty \sup_{r\in Z}|\tilde{S}_r f(\xi)|$. From this and Remark 3.3 we obtain the L^p-boundedness, $1<p<\infty$, and the weak type 1–1 of the above limit. So the study of the first sum in (3.5) is completed.

We now perform an analogous decomposition for the last sum in (3.5)

$$\sum_{N \leqq j \leqq M} \int_{|x| > 2^{l+3}} \varphi_j^{(j)}(x) k_2(2\xi - x) f(\xi - x)\, dx$$

$$= \sum_{N \leqq j \leqq M} \int_{|x| > 2^{l+3}} \varphi_j^{(j)}(x)[k_2(2\xi - x) - k_2(\xi - x)] f(\xi - x)\, dx$$

$$+ \sum_{N \leqq j \leqq M} \int_{|x| > 2^{l+3}} \varphi_j^{(j)}(x) k_2(\xi - x) f(\xi - x)\, dx$$

But $|x| > 2^{l+3}$ implies $|x - \xi| \geqq |x| - |\xi| \geqq 2|\xi|$ and by (1.7),

$$\sum_{N \leqq j \leqq M} \int_{|x| > 2^{l+3}} |\varphi_j^{(j)}(x)|\, |k_2(2\xi - x) - k_2(\xi - x)|\, |f(\xi - x)|\, dx$$

$$\leqq c \sum_{N \leqq j \leqq M} \int_{|x| > 2^{l+3}} |\varphi_j^{(j)}(x)| \frac{|\xi|^\delta}{|\xi - x|^\delta} |f(\xi - x)|\, dx$$

$$\leqq c \sum_{N \leqq j \leqq M} \int_{|x| > 2^{l+3}} |\varphi_j^{(j)}(x)| \left(\frac{|\xi|}{|x|}\right)^\delta |f(\xi - x)|\, dx$$

In the last inequality we use that $|\xi - x| \geqq \frac{3}{4}|x|$ if $|x| > 2^{l+3}$.

As before, this sum is bounded by

$$c \sum_{m \geqq 0} 2^{-m\delta} \int |\varphi_{-l-m-2}^{(-l-m-2)}(x)|\, |f(\xi - x)|\, dx$$

which, in turn, is bounded by $c\, Mf(\xi)$.

Finally

$$\lim_{(N, M) \to (-\infty, \infty)} \sum_{N \leqq j \leqq M} \int_{|x| > 2^{l+3}} \varphi_j^{(j)}(x) k_2(\xi - x) f(\xi - x)\, dx = S_{l+3}(k_2 f)(\xi)$$

with S_{l+3} as in (3.2). If $f \in L^p(R^n)$ so does $k_2 f$ and thus the L^p-boundedness, $1 < p < \infty$, and the weak type 1–1 of the above limit follow from Remark 3.3.

§4.

In this paragraph we extend the result obtained in §3, asking the family $\{\varphi_j\}_{j \in Z}$ to satisfy (1.3), (1.4) and (1.5). We need the following .

LEMMA 4.1. *Let $\{\varphi_j\}_{j \in Z}$ be a family of functions in $L^1(R^n)$ satisfying* (1.3), (1.4) *and* (1.5). *Then, for* $0 < a < b$, *there exists a finite constant* $c(b/a)$, *depending only on* b/a *and* n *such that*

$$\sum_{j \in Z} \int_{a<|x|<b} |\varphi_j^{(j)}(x)|\, dx \leqq c(b/a)$$

PROOF. As an easy consequence of the Theorem 4′, pag. 153 [St], we note that there exist $q > 1$ and $c > 0$ such that for all $j \in Z$, $\|\varphi_j\|_q \leqq c$. Since $\|\varphi_j^{(j)}\|_q = 2^{jn(1-1/q)} \|\varphi_j\|_q$ we have, by Hölder's inequality that

$$\sum_{2^j < a^{-1}} \int_{a<|x|<b} |\varphi_j^{(j)}(x)|\, dx \leqq c \sum_{2^j < a^{-1}} 2^{jn(1-1/q)}(b^n - a^n)^{1-1/q} = c(b/a).$$

On the other hand,

$$\sum_{2^j > a^{-1}} \int_{a<|x|<b} |\varphi_j^{(j)}(x)|\, dx = \sum_{2^j > a^{-1}} \int_{a<|x|<b} 2^{jn} |\varphi_j(2^j x)|\, (2^j|x|)^{\delta}(2^j|x|)^{-\delta}\, dx$$

$$\leqq a^{-\delta} \sum_{2^j > a-1} 2^{-j\delta} \int |\varphi_j(y)|\, |y|^{\delta}\, dy$$

being the last term bounded independently of a and b.

By personal communication, F. Ricci told us the following result.

LEMMA 4.2. *Let* K_1 *be the tempered distribution given by* $K_1(f) = \sum_{j \in Z} \langle \varphi_j^{(j)}, f \rangle$ *with* φ_j *satisfying* (1.3), (1.4) *and* (1.5), *where, as usual,* $\langle \varphi_j^{(j)}, f \rangle = \int_{R^n} \varphi_j^{(j)}(x) f(x)\, dx$. *Then* K_1 *can be decomposed as* $\sum \langle \psi_j^{(j)}, f \rangle$ *where* $\{\psi_j\}_{j \in Z}$ *is a family of functions with compact support contained in* $\{x\colon 2^{-1} \leqq |x| \leqq 2\}$, *and satisfying* (1.3) *and* (1.4).

A slight modification of the proof of 4.2, gives us the following.

LEMMA 4.3. *Let* $\{\varphi_j\}_{j \in Z}$ *be a family of functions satisfying* (1.3), (1.4) *and* (1.5). *Let* k_2 *be a function satisfying* (1.6) *and* (1.7). *Then there is a family of functions* $\{\beta_j\}_{j \in Z}$ *with compact support contained in* $\{x\colon 2^{-1} \leqq |x| \leqq 2\}$ *satisfying* (1.3) *and* (1.4) *such that for each* $f \in S(R^n)$ *and* $\xi \in R^n - \{0\}$,

$$\sum_{j \in Z} \int \varphi_j^{(j)}(x) k_2(\xi - x) f(\xi - x)\, dx = \sum_{j \in Z} \int \beta_j^{(j)}(x) k_2(\xi - x) f(\xi - x)\, dx$$

PROOF. Before dealing with the proof, we introduce some additional notation.

We set for $k \in Z$, $E_k = \{x \in R^n \colon 2^k < |x| \leqq 2^{k+1}\}$, also for $g \in L^{1,\mathrm{loc}}(R^n)$ we write $m_k(g) = |E_k|^{-1} \int_{E_k} g$ *and we define, for* $x \in R^n$, $\Phi(x) = k_2(2\xi - x)f(\xi - x)$.

We give the proof in several steps.

Step 1. $\sum_{j,1\in Z} \int_{E_1} |\varphi_j^{(j)}(x)|\,|\Phi(x) - m_1(\Phi)|\,dx < \infty.$

Indeed, we pick $r \in R$ such that $2^r = |\xi|/8$. Then

$$\sum_{1\geqq r}\sum_{j\in Z} \int_{E_1} |\varphi_j^{(j)}(x)|\,|\Phi(x) - m_1(\Phi)|\,dx \leqq \sum_{1\geqq r}\sum_{j\in Z} 2\|\Phi\|_{L^\infty(E_1)} \int_{E_1} |\varphi_j^{(j)}(x)|\,dx$$

Now, since $f \in S(R^n)$, we have $\|\Phi\|_{L^\infty(E_1)} \leqq c\,2^{-1}$ for some positive constant c. Then by lemma 4.1 the above sum converges.

On the other hand

$$\sum_{1<r}\sum_{j\in Z} \int_{E_1} |\varphi_j^{(j)}(x)|\,|\Phi(x) - m_1(\Phi)|\,dx$$

$$\leqq \sum_{1<r}\sum_{j\in Z} \int_{E_1} |\varphi_j^{(j)}(x)|\,|E_1|^{-1} \int_{E_1} |\Phi(x) - \Phi(t)|\,dt\,dx$$

$$\leqq \sum_{1<r}\sum_{j\in Z} \sup_{s,t\in E_1} |\Phi(s) - \phi(t)| \int_{E_1} |\varphi_j^{(j)}(x)|\,dx$$

Since $1 < r$ for $s, t \in E_1$ we have $|2\xi - s| \geqq \max\{|\xi|, 2|s-t|\}$. So we can apply (1.7) to obtain

$$|k_2(2\xi - s) - k_2(2\xi - t)| \leqq c\frac{|s-t|^\delta}{|2\xi - s|^\delta} \leqq c\,2^{1\delta}|\xi|^{-\delta} \text{ for some positive constant } c.$$

Then we can write

$$|\Phi(s) - \Phi(t)| = |k_2(2\xi - s)f(\xi - x) - k_2(2\xi - t)f(\xi - t)|$$

$$\leqq |k_2(2\xi - s) - k_2(2\xi - t)|\,|f(\xi - t)| + |f(\xi - s) - f(\xi - t)|\,|k_2(2\xi - t)|$$

$$\leqq c\,2^{1\delta}|\xi|^{-\delta}\|f\|_\infty + 2^{1+2}\|\nabla f\|_\infty \|k_2\|_\infty$$

and we can apply again lemma 4.1 to obtain the statement of step 1.

Step 2. For $j, k \in Z$, let $\varphi_{j,k}$ be the function defined by $\varphi_{j,k} = \varphi_j\chi_{E_k} - |E_k|^{-1}\chi_{E_k} \int_{E_k} \varphi_j$ where χ_{E_k} is the characteristic function of the set E_k. Then there is a family of functions $\{\vartheta_j\}_{j\in Z}$ with compact support contained in $\{x: 2^{-1} \leqq |x| \leqq 2\}$ and satisfying (1.3) and (1.4) such that $\sum_{j,k\in Z} \langle \varphi_{j,k}^{(j)}, \Phi\rangle = \sum_{j\in Z} \langle \vartheta_j^{(j)}, \Phi\rangle$.

Indeed, since

$$\langle \varphi_{j,k}^{(j)}, \Phi\rangle = \int_{E_{k-j}} \varphi_j^{(j)}(x)(\Phi(x) - m_{k-j}(\Phi))\,dx \tag{4.4}$$

the double sum $\sum_{j,k\in Z} \langle \varphi_{j,k}^{(j)}, \Phi\rangle$ is absolutely convergent by step 1 and we can rearrange it to obtain

$$\sum_{j,k\in Z} \langle \varphi_{j,k}^{(j)}, \Phi\rangle = \sum_{1} \sum_{j-k=1} \langle \varphi_{j,k}^{(j)}, \Phi\rangle = \sum_{1} \langle \vartheta_1^{(1)}, \Phi\rangle$$

where $\vartheta_1(x) = \sum_{j-k=1} \varphi_j^{(j-1)}(x)\chi_{E_0}(x)$.

It is not hard to see that the family $\{\vartheta_1\}_{1\in Z}$ satisfies (1.3) and (1.4). This completes the proof of step 2.

Step 3. We define for $j\in Z$ $\lambda_j(x) = \sum_{k\in Z} |E_k|^{-1} \int_{E_k} \varphi_j(t)\,dt\,\chi_{E_k}(x)$ then there is a family of functions $\{\eta_1\}_{1\in Z}$ with compact support contained in $\{x: 2^{-1} \leqq |x| \leqq 2\}$ satisfying (1.3) and (1.4) such that

$$\sum_{j\in Z} \langle \lambda_j^{(j)}, \Phi\rangle = \sum_{j\in Z} \langle \eta_j^{(j)}, \Phi\rangle$$

Indeed, we set $\sigma_k(x) = |E_k|^{-1}\chi_{E_k}(x)$ and $c_{jk} = \int_{E_k} \varphi_j$. Then for each $j\in Z$

$$\left\langle \sum_{N\leqq k\leqq M} c_{jk}\sigma_k^{(j)}, \Phi \right\rangle = \sum_{N+1\leqq k\leqq M} \int_{|x|\leqq 2^k} \varphi_j(x)\,dx\,\langle \sigma_{k-1} - \sigma_k, \Phi\rangle$$

$$+ \int_{|x|\leqq 2^{M+1}} \varphi_j(x)\,dx\,\langle \sigma_M^{(j)}, \Phi\rangle - \int_{|x|\leqq 2^N} \varphi_j(x)\,dx\,\langle \sigma_N^{(j)}, \Phi\rangle$$

Since $\lim_{s\to\infty} \langle \sigma_s, \Phi\rangle = 0$ and $\varphi_j \in L^1(R^n)$, the last two terms go to zero as $M \to +\infty$ and $N \to -\infty$. Now $\sigma_{k-1} - \sigma_k = (\sigma_{-1} - \sigma_0)^{(-k)}$.

So $\langle \lambda_j^{(j)}, \Phi\rangle = \sum_{k\in Z} \int_{|x|\leqq 2^k} \varphi_j(x)\,dx\,\langle (\sigma_{-1} - \sigma_0)^{(j-k)}, \Phi\rangle$.

We observe that $\sum_{k,l}\left|\int_{|x|\leqq 2^k} \varphi_{l+k}(x)\,dx\,\langle(\sigma_{-1}-\sigma_0)^{(l)},\Phi\rangle\right| < \infty$. Indeed

$$\sum_{k\geqq 0}\left|\int_{|x|\leqq 2^k}\varphi_{l+k}(x)\,dx\right| = \sum_{k\geqq 0}\left|\int_{|x|\geqq 2^k}\varphi_{l+k}(x)\,dx\right|$$

$$\leqq \sum_{k\geqq 0} 2^{-k\delta}\int_{|x|\leqq 2^k}|x|^\delta\,|\varphi_{l+k}(x)|\,dx < \infty$$

by (1.5). And, by Hölder's inequality,

$$\sum_{k<0}\int_{|x|\leqq 2^k}|\varphi_{l+k}(x)|\,dx \leqq \sum_{k<0} 2^{kn(1-1/q)}\omega_n^{1-1/q}\,\|\varphi_{l+k}\|_q < \infty$$

where ω_n denotes the measure of the n-dimensional unit sphere. Then

$$\sum_{k,l}\left|\int_{|x|\leqq 2^k}\varphi_{l+k}(x)\,dx\,\langle(\sigma_{-1}-\sigma_0)^{(l)},\Phi\rangle\right| \leqq c\sum_l|\langle(\sigma_{-1}-\sigma_0)^{(l)},\Phi\rangle|$$

$$=\sum_l\left|\left\langle\left(\sum_k\sigma_{-1}-\sigma_0)\chi_{E_k}\right)^{(l)},\Phi\right\rangle\right| \leqq \sum_{k,l}|\langle(\sigma_{-1}-\sigma_0)\chi_{E_k})^{(l)},\Phi\rangle| < \infty.$$

The convergence of the last sum is a consequence of (4.4) and of the statement in Step 1. Then we can write

$$\sum_{j\in Z}\langle\lambda_j^{(j)},\Phi\rangle = \sum_j\sum_k\int_{|x|\leqq 2^k}\varphi_j(x)\,dx\,\langle(\sigma_{-1}-\sigma_0)^{(j-k)},\Phi\rangle = \sum_{j\in Z}\langle\eta_j^{(j)},\Phi\rangle$$

where $\eta_l = \sum_{-k+j=l}\int_{|x|\leqq 2^k}\varphi_j(x)\,dx\,(\sigma_{-1}-\sigma_0)$. A computation shows that $\{\eta_1\}_{1\in Z}$ is a family of functions, with compact support contained in $\{x\colon 2^{-1}\leqq|x|\leqq 2\}$, satisfying (1.3) and (1.4). Two complete the proof we write $\sum_{j\in Z}\langle\varphi_j^{(j)},\Phi\rangle = \sum_{j\in Z}\langle\vartheta_j^{(j)},\Phi\rangle + \sum_{j\in Z}\langle\eta_j^{(j)},\Phi\rangle$.

Then we have the following.

THEOREM B. *Let* $\{\varphi_j\}_{j\in Z}$ *be a family of functions in* $L^1(R^n)$ *satisfying* (1.3), (1.4) *and* (1.5). *Let* k_2 *be a function satisfying* (1.6) *and* (1.7). *Then, for* $f\in L^p(R^n)$, $1<p<\infty$,

$$Tf(\xi) = \lim_{(N,M)\to(-\infty,\infty)} \int \sum_{N\leq j\leq M} 2^{jn}\varphi_j(2^j(\xi - y))k_2(\xi + y)f(y)\,dy$$

exists almost everywhere in R^n *and* $\|Tf\|_p \leq c_p \|f\|_p$. *Moreover, if* $f \in S(R^n)$ $|\{x: |Tf(x)| > \lambda\}| \leq c\lambda^{-1}\|f\|_1$ *for all* $\lambda > 0$ *(weak type* 1–1).

§5.

In this paragraph we give some applications of the results before obtained.

REMARK 5.1. Let $k_1 = \Omega(x)/|x|^n$, with Ω a homogeneous function of degree zero satisfying $\int_{S^{n-1}} \Omega(x)\,dx = 0$, and, for some $\varepsilon > 0$,

$$\int_{S^{n-1}} |\Omega(gx) - \Omega(x)|\,dx \leq c\,|g|^{\varepsilon}$$

for all g in So(n). Here $|\ |$ denotes a smooth distance to the identity. Let k_2 be a function satisfying (1.6) and (1.7). Then the operator given by

$$Tf(\xi) = \text{p.v.} \int_{R^n} k_1(\xi - y)k_2(\xi + y)f(y)\,dy$$

is bounded on $L^p(R^n)$, $1 < p < \infty$, and of weak type 1–1.

Indeed, if we define $\varphi_0(x) = k_1(x)X_{E_0}(x)$, then $k_1(x) = \sum 2^{jn}\varphi_0(2^j x)$, $\operatorname{supp}\varphi_0 \subseteq \{x \in R^n: 2^{-1} \leq |x| \leq 2\}$ and it satisfies (1.3). In order to apply Theorem A it only remains to check the L^1-Hölder condition (1.4). We must estimate

$$\int_{\substack{2^{-1}\leq|x-h|\leq 2\\ 2^{-1}\leq|x|\leq 2}} |\Omega(x-h)/|x-h|^n - \Omega(x)/|x|^n|\,dx$$

$$+ \int_{\substack{2^{-1}\leq|x-h|\leq 2\\ \{x:|x|\leq 2^{-1}\}\cup\{x:|x|\geq 2\}}} |\Omega(x-h)/|x-h|^n|\,dx$$

$$+ \int_{\substack{2^{-1}\leq|x|\leq 2\\ \{x:|x-h|\leq 2^{-1}\}\cup\{x:|x-h|\geq 2\}}} + |\Omega(x)/|x|^n|\,dx$$

The second and third integrals are similar. We study the last one. We can assume $|h| < 1/4$, since for $|h| \geq 1/4$

$$\int |\varphi_0(x+h) - \varphi_0(x)|\, dx \leqq 2\, \|\varphi_0\|_1 \leqq c\, |h|^\varepsilon\, \|\varphi_0\|_1$$

Now, for $|x-h| \leqq 2^{-1}$, we have $|x| \leqq 2^{-1} + |h|$ and for $|x-h| \geqq 2$ we have $|x| \geqq 2 - |h|$. Then

$$\int_{\substack{2^{-1} \leqq |x| \leqq 2 \\ \{x:\, |x-h| \leqq 2^{-1}\} \cup \{x:\, |x-h| \geqq 2\}}} |\Omega(x)| / |x|^n\, dx \leqq \int_{\substack{2^{-1} \leqq |x| \leqq 2 \\ |x-h| \leqq 2}} |\Omega(x)| / |x|^n\, dx$$

$$+ \int_{\substack{2^{-1} \leqq |x| \leqq 2 \\ |x-h| \geqq 2}} |\Omega(x)| / |x|^n\, dx \leqq c\, \|\Omega\|_1\, |h|^\varepsilon.$$

A change to polar coordinates gives us the last bound. It remains to treat the first integral.

$$\int_{\substack{2^{-1} \leqq |x-h| \leqq 2 \\ 2^{-1} \leqq |x| \leqq 2}} |\Omega(x-h)/|x-h|^n - \Omega(x)/|x|^n|\, dx$$

$$\leqq \int_{\substack{2^{-1} \leqq |x-h| \leqq 2 \\ 2^{-1} \leqq |x| \leqq 2}} |\Omega(x-h) - \Omega(x)|\, |x-h|^{-n}\, dx$$

$$+ \int_{\substack{2^{-1} \leqq |x-h| \leqq 2 \\ 2^{-1} \leqq |x| \leqq 2}} |\Omega(x)|\, ||x-h|^{-n} - |x|^{-n}|\, dx$$

We note that $||x-h|^{-n} - |x|^{-n}| \leqq |h| \sum_{0 \leqq k \leqq n-1} \binom{n}{k} |x|^{k-n} |h|^{n-k-1} |x-h|^{-n}$

then $\int_{\substack{2^{-1} \leqq |x-h| \leqq 2 \\ 2^{-1} \leqq |x| \leqq 2}} |\Omega(x)|\, ||x-h|^{-n} - |x|^{-n}|\, dx \leqq c\, |h|\, \|\Omega\|_1$. On the other hand, the change of variable $z = x - h$ gives us

$$\int_{\substack{2^{-1} \leqq |x-h| \leqq 2 \\ 2^{-1} \leqq |x| \leqq 2}} |\Omega(x-h) - \Omega(x)|\, |x-h|^{-n}\, dx \leqq \int_{2^{-1} \leqq |z| \leqq 2} |\Omega(z+h) - \Omega(z)|\, |z|^{-n}\, dz$$

For $z \in R^n$ we set $z = z'r$ with $z' \in S^{n-1}$ and $r \geqq 0$, we also set $\alpha = h/r$. Then the last integral can be written $\int_{2^{-1} \leqq r \leqq 2} r^{-1} \left(\int_{S^{n-1}} |\Omega(z' + \alpha) - \Omega(z')|\, dz' \right) dr$.

Now we apply lemma 5 of [C-W-Z] to obtain, for α small enough,

$$\int_{S^{n-1}} |\Omega(z' + \alpha) - \Omega(z')|\, dz' \leqq \sup_{|g| \leqq |\alpha|} \int_{S^{n-1}} |\Omega(gu) - \Omega(u)|\, dz' \leqq c\,|\alpha|^{\varepsilon} = c\,|h|^{\varepsilon} r^{-\varepsilon}$$

Remark 5.1 follows from this last inequality.

REMARK 5.2. Let $k_2(x)$ be a $C^1(R^n - \{0\})$ function such that, for some constant $c > 0$ and for all $x \in R^n$ $|k_2(x)| \leqq c$ and $|\nabla k_2(x)| \leqq c\,|x|^{-1}$. Then it is easy to see that k_2 satisfies (1.7) for all $\delta \leqq 1$. For example k_2 being homogeneous of degree 0 and smooth out of the origin. A less restrictive condition for K_2 is given by the following remark.

REMARK 5.3. Let $\{\psi_j\}_{j \in Z}$ be a family of measurable functions on R^2 satisfying

(i) Supp $\psi_j \subseteq \{x \in R^n : 2^{-1} \leqq |x| \leqq 2\}$.

(ii) There exist $c > 0$ and $0 < \delta < 1$ such that $|\psi_j(x + h) - \psi_j(x)| \leqq c\,|h|^{\delta}$ for almost all $x \in R^n$.

By (i) and (ii) $\psi_j \in L^\infty(R^n)$ and $\|\psi_j\|_\infty \leqq c$, so if we define $k_2(x) = \sum_{j \in Z} \psi_j(2^j x)$, we have that $k_2 \in L^\infty(R^n)$ and satisfies (1.7). Indeed $|k_2(x + h) - k_2(x)| \leqq c \sum_{j \in Z} 2^{j\delta} |h|^{\delta}$. If $|h| \leqq |x|/2$ and either 2^j or $2^j(x + h)$ belongs to supp ψ_j, then $2^j \leqq c/|x|$. The result follows since for each h and x fixed, at most six terms are involved.

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FACULTAD DE MATEMÁTICA, ASTRONOMÍA Y FISICA
UNIVERSIDAD NATIONAL DE CORDOBA
VALPARAISO Y R. MARTINEZ
5000 CORDOBA
ARGENTINA