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ON APPROXIMATION IN WEIGHTED SOBOLEV SPACES AND SELF-ADJOINTNESS

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Abstract.

Necessary and sufficient conditions for approximation by test functions in a type of weighted Sobolev spaces are given. As an application a necessary condition for essential self-adjointness of a perturbed Laplacian is proved. A lemma on the equivalence of two capacities is proved and used to obtain criteria for closability, continuity and compactness of certain embeddings.

1. Introduction.

Let $\rho \in L^1_{\rm loc}(\mathsf{R}^N)$ be a positive function, locally bounded away from zero. In [6] the spectral properties of the operator $A = -\dfrac{1}{\rho}\Delta$ on $L^2(\rho)$ with domain $C_0^\infty(\mathsf{R}^N)$ are investigated. In particular it is proved that for $N \geqq 3$ a necessary condition for A to be essentially self-adjoint is that $\int \rho(x)dx = \infty$. In the present paper we sharpen this result. This is done by giving necessary and sufficient conditions for the density of test functions in a weighted Sobolev space.

For $p \geqq 1$ we define $L^{m,p}(\mathsf{R}^N)$ as the set of distributions u on R^N such that

$$\|u\|_{m,p} = \left(\sum_{k=1}^{m} \int |\nabla^k u(x)|^p dx\right)^{1/p} < \infty;$$

here $\nabla^k u$ denotes the vector $(D^\alpha u)_{|\alpha|=k}$. $L_0^{m,p}(\mathsf{R}^N)$ is the completion of $C_0^\infty(\mathsf{R}^N)$ with respect to the $L^{m,p}$ norm. Now let $\Omega \subset \mathsf{R}^N$ be an open set and let μ be a nontrivial positive Radon measure on Ω. We will study the space $H_\mu^{m,p}(\Omega)$, defined as the completion of $L^p(\mu) \cap L^{m,p}(\mathsf{R}^N) \cap C_\Omega^\infty$ with respect to the norm

$$\|u\|_{m,p;\mu} = \|u\|_{L^p(\mu)} + \|u\|_{m,p};$$

here $C_\Omega^\infty = \{u \in C^\infty(\mathsf{R}^N) : \operatorname{supp} u \subset \Omega\}$. The closure of $C_0^\infty(\Omega)$ in $H_\mu^{m,p}(\Omega)$ is denoted $\mathring{H}_\mu^{m,p}(\Omega)$. Note that if $p < N$ then by Sobolev's inequality the elements in

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$L_0^{m,p}$ can be identified with functions in L^{p^*}, where $p^* = \dfrac{Np}{N-p}$. The elements in $\mathring{H}_\mu^{m,p}(\Omega)$ are naturally identified with elements in $L_0^{m,p}$. Note also that $L^{m,p} \subset L^p_{\text{loc}}$.

The theorem to be proved is the following. (See Section 2 for definitions of the capacitites $B_{m,p}$ and H_1^{N-m} used.)

THEOREM 1. *Let μ be a nontrivial positive Radon measure concentrated on $\Omega \subset \mathsf{R}^N$ and let C denote $B_{m,p}$ for $p > 1$ and H_1^{N-m} for $p = 1$. Then*

(i) *$\mathring{H}_\mu^{m,p}(\Omega) = H_\mu^{m,p}(\Omega)$ if either $p \geqq N$ or $p < N$ and $C(\Omega^c) = \infty$.*

Suppose now that $p < N$ and $C(\Omega^c) < \infty$. Then $\mathring{H}_\mu^{m,p}(\Omega) = H_\mu^{m,p}(\Omega)$ if and only if

(ii) *$\mu(\Omega) = \infty$, when either $m \geqq N$, $p = 1$ or $mp > N$, $p > 1$.*

(iii) *$\mu(F^c) = \infty$ for every closed set $F \subset \mathsf{R}^N$ satisfying $C(F) < \infty$, when either $1 < p \leqq \dfrac{N}{m}$ or $m < N, p = 1$.*

REMARK. When $1 < p < N$, $mp > N$ or $1 = p < N$, $m \geqq N$ the condition $C(\Omega^c) < \infty$ is just a complicated way of saying that Ω^c should be bounded; see [2].

Letting A be the operator above we can now prove the following theorem.

THEOREM 2. *A necessary condition for the operator A to be essentially self-adjoint is that $\int_{F^c} \rho(x)dx = \infty$ whenever F is a closed set such that $B_{1,2}(F) < \infty$.*

PROOF. Suppose ρ does not satisfy the condition in the theorem. Then by Theorem 1 and the Hahn-Banach theorem there is a function $0 \neq u \in H_\rho^{1,2}(\mathsf{R}^N)$ such that

$$\int u(x)v(x)\rho(x)dx + \int \nabla u(x)\cdot\nabla v(x)\,dx = 0$$

for all $v \in \mathring{H}_\rho^{1,2}(\mathsf{R}^N) \supset D(\bar{A})$, where $\bar{A}$ is the closure of A. Thus

$$(u, v + \bar{A}v)_{L^2(\rho)} = 0,$$

so $u \in D((I + A)^*) = D(A^*)$ and $u + A^*u = 0$. Now suppose $\bar{A} = A^*$. Then $u \in D(\bar{A})$ and hence

$$\int |u(x)|^2 \rho(x)\,dx + \int |\nabla u(x)|^2\,dx = 0$$

which implies that $u = 0$. This contradiction shows that $\bar{A}$ is not self-adjoint.

2. Capacities.

We will denote different constants, not depending on the essential functions or variables considered, by A. The ball with radius r and centered at x will be denoted by $B(x,r)$. If $x = 0$ we will write only $B(r)$. The annulus $B(R)\backslash B(r)$ is denoted by $A(R,r)$.

We start by defining some convolution kernels needed.

DEFINITION 1. The *Bessel kernels* G_α, the *Riesz kernels* I_α and the *truncated Bessel kernels* $G_{\alpha;1}$ are defined by

$$\hat{G}_\alpha(\xi) = (1 + |\xi|^2)^{-\alpha/2},$$

$$I_\alpha(x) = |x|^{\alpha - N},\ 0 < \alpha < N$$

and

$$G_{\alpha;1}(x) = \theta(x) G_\alpha(x)$$

where $\theta \in C_0^\infty(B(1))^+$ is an arbitrary but fixed function such that $\theta = 1$ on $B(\frac{1}{2})$. For $\alpha \geqq 1$ we *define*

$$K_\alpha = I_1 * G_{\alpha-1}.$$

It is easy to see, analogously to the cases with Riesz or Bessel potentials, that, for $1 < p < N$, $L_0^{m,p} = \{K_m * f : f \in L^p(\mathrm{R}^N)\}$.

Each of these kernels gives rise to corresponding capacities as follows.

DEFINITION 2. For $E \subset \mathrm{R}^N$ and $p > 1$ we *define*

$$B_{\alpha,p}(E) = \inf\{\|f\|_p^p : f \in L_+^p,\ G_\alpha * f \geqq 1 \text{ on } E\},$$

$$R_{\alpha,p}(E) = \inf\{\|f\|_p^p : f \in L_+^p,\ I_\alpha * f \geqq 1 \text{ on } E\}$$

$$C_{\alpha,p}(E) = \inf\{\|f\|_p^p : f \in L_+^p,\ K_\alpha * f \geqq 1 \text{ on } E\}$$

and

$$B_{\alpha,p;1}(E) = \inf\{\|f\|_p^p : f \in L_+^p,\ G_{\alpha;1} * f \geqq 1 \text{ on } E\}.$$

It is proved in [2] that $B_{\alpha,p}$ and $R_{\alpha,p}$ are finite simultaneously, although not comparable, for $1 < p < \dfrac{N}{\alpha}$. It is not hard to see from that proof that $C_{\alpha,p}$ and $B_{\alpha,p}$ are finite at the same time, for $1 < p < N$. The set functions $B_{\alpha,p}$ and $B_{\alpha,p;1}$ are comparable. See [4] for a proof of this fact. We will need the following lemma, which, for technical convenience, is the reason for introducing the truncated Bessel kernel.

LEMMA 1. *Let $p > 1$. If $F \subset \mathrm{R}^N$ is closed then*

$$B_{\alpha,p;1}(F) = \inf\{\|f\|_p^p : f \in L_+^p,\ C^\infty \ni G_{\alpha;1} * f \geqq 1 \text{ on } F\}.$$

PROOF. We may assume $B_{\alpha,p;1}(F)$ finite. Let $A_0 = \overline{B(1)}$ and $A_j = \overline{B(j+1)} \setminus B(j)$ for $j \geqq 1$. It is proved in [2] that $\sum_{j=1}^{\infty} B_{\alpha,p}(F \cap A_j) \leqq AB_{\alpha,p}(F)$. Hence

$$\sum_{j=1}^{\infty} B_{\alpha,p;1}(F \cap A_j) < \infty. \tag{1}$$

Now choose $f_j \in L^p_+$ such that $\|f_j\|_p^p \leqq 2^{-j} + B_{\alpha,p;1}(F \cap A_j)$ and $G_{\alpha;1} * f_j \geqq 1$ on a neighbourhood of $F \cap A_j$. This can be done since $G_{\alpha;1} * f_j$ is lower semicontinuous. It is seen from the definition of $B_{\alpha,p;1}$ that we may assume that $\operatorname{supp} f_j \subset A'_j$ where we denote $E' = \{x: \operatorname{dist}(x, E) \leqq 1\}$. By a standard regularization we obtain functions $g_j \in C_0^\infty(A''_j)^+$ such that

$$\|g_j\|_p^p \leqq A(2^{-j} + B_{\alpha,p;1}(F \cap A_j))$$

and $G_{\alpha;1} * g_j \geqq 1$ on $F \cap A_j$. Consequently, if we set $g = \sum_{j=M}^{\infty} g_j$ for some M to be specified later we get $G_{\alpha;1} * g \geqq 1$ on $F \setminus B(M)$. Also, since the sum defining g is uniformly locally finite, $G_{\alpha;1} * g \in C^\infty$ and

$$\|g\|_p^p \leqq A \sum_{j=M}^{\infty} \|g_j\|_p^p \leqq A \sum_{j=M}^{\infty} (2^{-j} + B_{\alpha,p;1}(F \cap A_j)). \tag{2}$$

Now let $\varepsilon > 0$. By (1) and (2) we get $\|g\|_p < \varepsilon$ if M is large enough. By the same argument as before with lower semicontinuity and regularization we can find a function $h \in C_0^\infty(\mathrm{R}^N)^+$ such that $\|h\|_p \leqq B_{\alpha,p;1}(F \cap \overline{B(M)})^{1/p} + \varepsilon$ and $G_{\alpha;1} * h \geqq 1$ on $\mathrm{F} \cap \overline{B(M)}$. Setting $f = g + h$ we obtain $\|f\|_p \leqq B_{\alpha,p;1}(F)^{1/p} + 2\varepsilon$ and $C^\infty \ni G_{\alpha;1} * f \geqq 1$ on F. Since ε was arbitrary the lemma follows.

For $p = 1$ the appropriate capacities are Hausdorff capacities.

DEFINITION 3. Let $0 \leqq d < N$. Then for subsets E of R^N we *define*

$$H_\rho^d(E) = \inf \sum_{i=1}^{\infty} r_i^d$$

where the infimum is taken over all countable coverings $\bigcup_{i=1}^{\infty} B(x_i, r_i) \supset E$, with $r_i \leqq \rho$. For $d < 0$ we *define* $H_\rho^d(E) = H_\rho^0(E)$.

The following lemma is immediate except for (iii) which is (a variant of) the well-known Frostman lemma.

LEMMA 2. *Let* $0 < d < N$. *Then*

(i) $H_\infty^d(E) \leqq H_1^d(E) \leqq AH_\infty^d(E) + A(H_\infty^d(E))^{N/d}$. *In particular* H_∞^d *and* H_1^d *are finite at the same time.*

(ii) $\sum_{j=0}^{\infty} H_1^d(E \cap A_j) \leqq AH_1^d(E)$, *where* A_j *is as in the proof of Lemma 1.*

(iii) $H_1^d(E)$ *is comparable to* $\sup\{\mu(E): |\mu|(B(x,r)) \leqq r^d, r \leqq 1, x \in \mathrm{R}^N\}$, *for Borel sets* E.

The following lemma, partly proved by Adams [3], will give a substitute when $p = 1$ for the potentials used when $p > 1$.

LEMMA 3. *Let m be an integer, $0 < m < N$. Then for closed sets $F \subset \mathbf{R}^N$, $H_1^{N-m}(F)$ is comparable to*

$$\inf\{\|\varphi\|_1 + \|\nabla^m\varphi\|_1 : \varphi \in C^\infty, 0 \leqq \varphi \leqq 1, \varphi = 1 \text{ on a neighbourhood of } F\}.$$

PROOF. Suppose $\varphi \in C^\infty$, $\varphi = 1$ on a neighbourhood of F and $\|\varphi\|_1 + \|\nabla^m\varphi\|_1 < \infty$. If μ is a positive measure supported by F then by [9, Sec. 1.4] we have

$$\mu(F) \leqq \int \varphi\, d\mu \leqq A \sup_{x;0<r\leqq 1} \frac{\mu(B(x,r))}{r^{N-m}} (\|\varphi\|_1 + \|\nabla^m\varphi\|_1).$$

Taking the supremum over μ with

$$\sup_{x;0<r\leqq 1} \frac{\mu(B(x,r))}{r^{N-m}} \leqq 1$$

we get by Lemma 2 (iii) that $H_1^{M-m}(F) \leqq A(\|\varphi\|_1 + \|\nabla^m\varphi\|_1)$ which proves one direction of the lemma.

To prove the other direction suppose first that F is compact. Cover F by balls $B(x_i, r_i)$, $r_i \leqq 1$, $1 \leqq i \leqq s$, such that

$$\sum_{i=1}^{s} r_i^{N-m} \leqq H_1^{N-m}(F) + \varepsilon,$$

where $\varepsilon > 0$. By Lemma 3.1 of [7] there are functions $\psi_i \in C_0^\infty(B(x_i, 2r_i))$, $1 \leqq i \leqq s$, such that $|D^\alpha\psi_i| \leqq A_\alpha r_i^{-|\alpha|}$ and such that $\varphi = \sum_{i=1}^{s}\psi_i$ satisfies $\varphi = 1$ on a neighbourhood of F. We get

$$\|\varphi\|_1 + \|\nabla^m\varphi\|_1 \leqq \sum_{i=1}^{s} \int_{B(x_i,2r_i)} (|\psi_i(x)| + |\nabla^m\psi_i(x)|)\,dx$$

$$\leqq A \sum_{i=1}^{s} (r_i^N + r_i^{N-m}) \leqq AH_1^{N-m}(F) + A\varepsilon.$$

Since ε was arbitrary we are done in case F is compact.

For the general case we introduce a partition of unity $1 = \sum_{n=0}^{\infty}\zeta_n$, where $0 \leqq \zeta_n \leqq 1$, $\zeta_n = 1$ on a neighbourhood of A_{2n}, $\operatorname{supp}\zeta_n \subset A'_{2n}$ and $|\nabla^k\zeta_n| \leqq A$ for $1 \leqq k \leqq m$. Here A_j and A'_j are as in the proof of Lemma 1. Now choose functions φ_n corresponding to the sets $F \cap A'_{2n}$ according to the construction for compact sets in a way that

$$\|\varphi_n\|_1 + \|\nabla^m\varphi_n\|_1 \leqq AH_1^{N-m}(F \cap A'_{2n}) + 2^{-n}\varepsilon,$$

where $\varepsilon > 0$. Letting $\varphi = \sum_{n=0}^{\infty} \zeta_n \varphi_n$ we obtain, using Leibniz' rule, interpolation and Lemma 2(ii),

$$\begin{aligned}\|\varphi\|_1 + \|\nabla^m \varphi\|_1 &\leqq A \sum_{n=0}^{\infty} \sum_{k=0}^{m} \int_{A'_{2n}} |\nabla^k \varphi(x)|\, dx \\ &\leqq A \sum_{n=0}^{\infty} \int (|\varphi_n(x)| + |\nabla^m \varphi_n(x)|)\, dx \\ &\leqq \sum_{n=0}^{\infty} (H_1^{N-m}(F \cap A'_{2n}) + 2^{-n}\varepsilon) \\ &\leqq 2\varepsilon + A \sum_{n=0}^{\infty} H_1^{N-m}(F \cap A_n) \leqq 2\varepsilon + AH_1^{N-m}(F).\end{aligned}$$

Since ε was arbitrary the lemma follows.

3. Some Applications of Lemma 3.

We record here some generalizations, depending on Lemma 3, to the case $p = 1$ of some results in [9, Ch. 12]. As before let μ be a positive Radon measure concentrated on $\Omega \subset \mathsf{R}^N$ and let W, X and Y be the completions of $C_0^\infty(\Omega)$ with respect to the norms

$$\|u\|_W = \int |u(x)|\, dx + \int |\nabla^m u(x)|\, dx$$

$$\|u\|_X = \int |u|\, d\mu + \int |\nabla^m u(x)|\, dx,$$

and

$$\|u\|_Y = \int |\nabla^m u(x)|\, dx,$$

respectively. Then we have the following theorems.

THEOREM 3. *The identity operator defined on $C_0^\infty(\Omega)$ and mapping $L^1(\Omega)$ into X is closable if and only if μ is absolutely continuous with respect to H_1^{N-m}.*

THEOREM 4. *The identity operator defined on $C_0^\infty(\Omega)$ and mapping W into $L^1(\mu)$ is closable if and only if μ is absolutely continuous with respect to H_1^{N-m}.*

THEOREM 5. *Let $m \leqq N$. Then the identity operator defined on $C_0^\infty(\Omega)$ and mapping Y into $L^1(\mu)$ is closable if and only if μ is absolutely continuous with respect to H_1^{N-m}.*

REMARK. Note that for $m \geqq N$ and for $m = N$ respectively the above condition on absolute continuity is empty so the operators are always closable. In the proof of Theorem 3 below "quasieverywhere" can be read "everywhere" in this case.

For the proofs we will need two lemmas, proved in [5]. We give the proofs here for the convenience of the reader. Recall that a function u is called H_1^d-quasicontinuous if it is defined H_1^d-quasieverywhere and if for every $\varepsilon > 0$ there is an open set G such that $u|_{G^c}$ is continuous and $H_1^d(G) < \varepsilon$.

LEMMA 4. *Suppose that $u_n \in C_0^\infty(\Omega)$ and that $\|u_{n+1} - u_n\|_W < 4^{-n}$. Then u_n converges H_1^{N-m}-quasieverywhere to an H_1^{N-m}-quasicontinuous function.*

PROOF. Let μ be a positive Radon measure such that $\mu(B(x,r)) \leqq r^{N-m}$ for all $x \in \mathrm{R}^N$ and all $r \leqq 1$. Then by [9, Sec. 1.4] we have

$$\int |u_{n+1} - u_n| \, d\mu \leqq A 4^{-n}.$$

By monotone convergence, $\tilde{u}(x) = \lim_{n\to\infty} u_n(x)$ exists a.e. $[\mu]$. Now, let

$$F = \{x : \tilde{u}(x) \text{ is defined}\}$$

and

$$E_n = \{x \in F : |\tilde{u}(x) - u_n(x)| \geqq 2^{-n}\}.$$

Then, by part (iii) of Lemma 2, $H_1^{N-m}(F^c) = 0$. Also,

$$\mu(E_n) \leqq A\, 2^n \int |\tilde{u} - u_n| \, d\mu \leqq A\, 2^{-n},$$

so $H_1^{N-m}(E_n) \leqq A\, 2^{-n}$. Let $F_k = \cup_{n=k}^\infty E_n$. Then $H_1^{N-m}(F_k) \leqq A\, 2^{-k}$. Thus, given $\varepsilon > 0$, we may choose k and an open set G_k with $H_1^{N-m}(G_k) < \varepsilon$ such that $F_k \cup F^c \subset G_k$. Since $u_n \to \tilde{u}$ uniformly on G_k^c, the lemma is proved.

LEMMA 5. *Suppose that u is H_1^d-quasicontinuous and that $E = \{x : u(x) \neq 0\}$ is a Borel set with $|E| = 0$. Then $H_1^d(E) = 0$.*

PROOF. Suppose that $H_1^d(E) = c > 0$. There is an open set G with $H_1^d(G) < \varepsilon$ such that $u|_{G^c}$ is continuous. We do not specify ε here because the choice of it depends on a certain constant, appearing later in the proof. However, ε is a fixed positive number less than c.

Let $K \subset \mathrm{E} \setminus G$ be a compact set such that $H_1^d(K) > c - \varepsilon$ and set $K_n = \left\{x : \operatorname{dist}(x, K) \leqq \frac{1}{n}\right\}$. By Lemma 2, part (iii), we can choose measures μ_n supported by K_n such that

$$\sup_{x;0<r\leqq 1} \frac{\mu_n(B(x,r))}{r^d} \leqq A$$

and $\mu_n(K_n) = H_1^d(K_n)$.

Define $\phi_n(x) = n^N \phi(nx)$, where $\phi \in C_0^\infty(B(1))$ is a function such that $0 \leqq \phi \leqq A$ and $\int \phi = 1$. Set $\nu_n = \phi_n * \mu_n$. Then we have

$$\phi_n * \mu_n(y) = \int \phi_n(y-t)\,d\mu_n(t) \leqq A n^N \mu_n\left(B\left(y, \frac{1}{n}\right)\right) \leqq A n^{N-d}.$$

Thus, for $r \leqq \dfrac{1}{n}$

$$\frac{\nu_n(B_r(x))}{r^d} \leqq A(nr)^{N-d} \leqq A.$$

For $\dfrac{1}{n} \leqq r \leqq 1$ we have

$$\begin{aligned}
\frac{\nu_n(B(x,r))}{r^d} &\leqq \frac{1}{r^d}\int \chi_{B(x,r)}(y) \int \phi_n(y-t)\,d\mu_n(t)\,dy \\
&= \frac{1}{r^d}\int \chi_{B(x,r)} * \phi_n(t)\,d\mu_n(t) \\
&\leqq \frac{1}{r^d}\int \chi_{B(x,r+\frac{1}{n})}(t)\,d\mu_n(t) \\
&\leqq A\,\frac{\mu_n\left(B\left(x, r+\frac{1}{n}\right)\right)}{\left(r+\frac{1}{n}\right)^d} \leqq A.
\end{aligned}$$

Thus we obtain

$$\sup_{x;0<r\leqq 1} \frac{\nu_n(B(x,r))}{r^d} \leqq A.$$

Also,

$$\begin{aligned}
\nu_n(\mathrm{R}^N) &= \int\int \phi_n(x-y)\,d\mu_n(y)\,dx \\
&= \int\int \phi_n(x-y)\,dx\,d\mu_n(y) \\
&= \mu_n(K_n) = H_1^d(K_n) \\
&\geqq H_1^d(K) \geqq c - \varepsilon.
\end{aligned}$$

Let $K_n^* = \operatorname{supp} \nu_n$. Then, since $|E| = 0$, we get

$$H_1^d(K_n^* \setminus E) \geqq A^{-1} \nu_n(E^c) = A^{-1} \nu_n(\mathbb{R}^N) \geqq A^{-1}(c - \varepsilon).$$

Now we are in the position to specify ε: take any positive ε satisfying $A^{-1}(c - \varepsilon) > \varepsilon$. Then we obtain

$$H_1^d(K_n^* \setminus E) > H_1^d(G).$$

Hence there are points $x_n \in K_n^* \cap E^c \cap G^c$. We may assume that x_n converges to some point x_0. Since $x_n \in K_n^*$ there are points $y_n \in K$ such that $|x_n - y_n| < \dfrac{2}{n}$. Then $y_n \to x_0$, so in particular $x_0 \in K$. By the continuity of u on G^c we obtain that $0 = u(x_n) \to u(x_0) \neq 0$. From this contradiction we conclude that $H_1^d(E) = 0$ and the lemma is proved.

PROOF OF THEOREM 3. We start with the sufficiency part. Suppose $\{u_n\} \subset C_0^\infty(\Omega)$ is a Cauchy sequence in X, converging to zero in $L^1(\Omega)$. Then $D^\alpha u_n$ converges in $L^1(\Omega)$ for $|\alpha| = m$ and since obviously $D^\alpha u_n \to 0$ as distributions we get $D^\alpha u_n \to 0$ in $L^1(\Omega)$. Hence, passing to a subsequence, we may assume that u_n converges H_1^{N-m}-quasieverywhere by Lemma 4. Also, by Lemma 5 we get that, H_1^{N-m}-quasieverywhere, $u_n \to 0$. Thus $u_n \to 0$ a.e $[\mu]$, and since $\{u_n\}$ is a Cauchy sequence in $L^1(\mu)$ we obtain $u_n \to 0$ in X.

For the necessity part suppose $F \subset \Omega$ is a compact set satisfying $H_1^{N-m}(F) = 0$ and $\mu(F) > 0$. Let $G_n \supset F$ be shrinking open sets such that $H_1^{N-m}(\bar{G}_n) \to 0$ and $\mu(G_n \setminus F) \to 0$ as $n \to \infty$. Then, by Lemma 3, we can find $\varphi_n \in C_0^\infty(\Omega)$ such that $\varphi_n = 1$ on G_n and

$$\|\varphi_n\|_1 + \|\nabla^m \varphi_n\|_1 \to 0.$$

Moreover, by construction, $\varphi_n \to 0$ uniformly outside every neighbourhood of F and there is a compact set $K \subset \Omega$ such that $\operatorname{supp} \varphi_n \subset K$ for all n. Now let $\varepsilon > 0$ and choose n so that $\mu(G_n \setminus F) < \varepsilon/4$. Then, for j and k large enough,

$$\int_\Omega |\varphi_j - \varphi_k| \, d\mu \leqq \int_{K \setminus G_n} |\varphi_j - \varphi_k| \, d\mu + 2\mu(G_n \setminus F) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence $\{\varphi_n\}$ is a Cauchy sequence in X, converging to zero in $L^1(\Omega)$. However, since

$$\|\varphi_n\|_{L^1(\mu)} \geqq \mu(F) > 0,$$

we cannot have $\varphi_n \to 0$ in X. This proves the necessity part.

The proofs of Theorems 4 and 5 follow the same lines as in [9, Sec. 12.4], making use of the proof of Theorem 3.

Using Lemma 3 we can also obtain necessary and sufficient conditions for continuity and compactness of the embedding of X into the Sobolev space $W^{k,q}(\mathrm{R}^N)$. To state these theorems we need first some definitions. Let $m < N$. Then a set $F \subset B(x, r)$ is called $(m, 1)$-unessential if

$$H_\infty^{N-m}(F) \leqq \gamma r^{N-m},$$

where γ is a sufficiently small constant, depending only on m and N. For $m \geqq N$ only the empty set is called $(m, 1)$-unessential.

Let $\mathscr{F}(\Omega)$ be the family of all balls $B(x, r)$ such that $B(x, r)\backslash\Omega$ is $(m, 1)$-unessential. Then we define

$$D_{m,1}(\mu, \Omega) = \sup\{r; B(x, r) \in \mathscr{F}(\Omega), \inf \mu(B(x, r)\backslash F) \leqq r^{N-m}\},$$

where the infimum is taken over all $(m, 1)$-unessential closed sets $F \subset B(x, r)$. We then have the following theorems.

THEOREM 6. *Let $0 \leqq k \leqq m$, $1 \leqq q < \infty$ and $m - k > N(1 - 1/q)$. Then*

$$\|u\|_q + \|\nabla^k u\|_q \leqq A \|u\|_X$$

for all $u \in C_0^\infty(\Omega)$ if and only if there are positive constants r and a such that

$$\mu(B(x, r)\backslash F) \geqq a$$

for all balls $B(x, r) \in \mathscr{F}(\Omega)$ and all $(m, 1)$-unessential sets $F \subset B(x, r)$. The best constant A is comparable to

$$D^{m-N(1-1/q)} \max\{D^{-k}, 1\},$$

where $D = D_{m,1}(\mu, \Omega)$.

THEOREM 7. *Let $0 \leqq k \leqq m$, $1 \leqq q < \infty$ and $m - k > N(1 - 1/q)$. Then X is compactly embedded into $W^{k,q}(\mathrm{R}^N)$ if and only if $D_{m,1}(\mu, \Omega) < \infty$ and*

$$\lim_{R \to \infty} D_{m,1}(\mu, \Omega\backslash\overline{B(R)}) = 0.$$

The proofs of these theorems are the same as those of the corresponding theorems for $p > 1$ in [9, Sec. 12.2–12.3], relying now on Lemma 3 of the present paper.

4. Proof of Theorem 1.

We will need some basic results on the function spaces $L^{m,p}$ and $L_0^{m,p}$. We state first a well-known lemma of Hardy type.

LEMMA 6. (i) *If* $1 \leqq p < N$ *and* $u \in L_0^{1,p}(\mathsf{R}^N)$ *then*

$$\int \frac{|u(x)|^p}{|x|^p}\,dx \leqq A \int |\nabla u(x)|^p\,dx.$$

(ii) *If* $p > N$ *and* $u \in L^{1,p}(\mathsf{R}^N)$ *then*

$$\int_{B(1)^c} \frac{|u(x)|^p}{|x|^p}\,dx \leqq A\left(\int_{B(1)} |u(x)|^p\,dx + \int |\nabla u(x)|^p\,dx\right).$$

(iii) *If* $p = N$ *and* $u \in L^{1,p}(\mathsf{R}^N)$ *then*

$$\int_{B(2)^c} \frac{|u(x)|^p}{(|x|\log|x|)^p}\,dx \leqq A\left(\int_{B(2)} |u(x)|^p\,dx + \int |\nabla u(x)|^p\,dx\right).$$

A proof of essentially the following decomposition lemma can be found in Lizorkin [8].

LEMMA 7. *Let* $1 \leqq p < N$. *Then for each* $u \in L^{m,p}(\mathsf{R}^N)$ *there is a unique constant* c *such that* $u - c \in L_0^{m,p}(\mathsf{R}^N)$.

We turn now to the proof of Theorem 1, divided into four cases starting with the main one.

The case $1 < p \leqq \dfrac{N}{m}$ *or* $p = 1, m < N$. We start by proving the sufficiency part. Suppose that μ satisfies the condition in the theorem. We will show then that $C_\Omega^\infty \cap H_\mu^{m,p} \subset L_0^{m,p}$. Let $u \in C_\Omega^\infty \cap H_\mu^{m,p}$ and suppose that $c \neq 0$, where $u - c \in L_0^{m,p}$. We can assume that $c > 0$. Let

$$F = \left\{x : |u(x) - c| \geqq \frac{c}{2}\right\}$$

and suppose first that $p > 1$. Then $u - c$ can be written $u - c = K_m * f$ where $f \in L^p$ and we get

$$C_{m,p}(F) \leqq C_{m,p}\left(\left\{x : K_m * |f| \geqq \frac{c}{2}\right\}\right) \leqq \frac{2^p}{c^p}\,\|f\|_p^p < \infty.$$

If $p = 1$ then in the same way as in the proof of Lemma 3

$$H_\infty^{N-m}(F) \leqq A \int |\nabla^m(u(x) - c)|\,dx < \infty.$$

On the other hand, since $|u| \geqq \dfrac{c}{2}$ on F^c, we have

$$\mu(F^c) \leqq \left(\frac{2}{c}\right)^p \int |u|^p \, d\mu < \infty.$$

This contradicts the condition on μ so $c = 0$. It follows that $u \in L_0^{m,p}$. Note that since $\Omega^c \subset F$ we must have $c = 0$ if $C(\Omega^c) = \infty$, without using any condition on μ.

Now let $\eta_R \in C_0^\infty(B(2R))$ satisfy $0 \leqq \eta_R \leqq 1, \eta_R = 1$ on $B(R)$ and $|\nabla^k \eta_R| \leqq AR^{-k}$ for $k \leqq m$. Then for $R \geqq 1$ we get

$$\|u - u\eta_R\|_{m,p}^p \leqq A \sum_{1 \leqq l+k \leqq m} \int |\nabla^k u(x)|^p |\nabla^l (1 - \eta_R)(x)|^p \, dx$$

$$\leqq A \sum_{k=1}^m \int_{B(R)^c} |\nabla^k u(x)|^p \, dx + A \sum_{\substack{1 \leqq l+k \leqq m \\ l \geqq 1}} R^{-lp} \int_{A(2R,R)} |\nabla^k u(x)|^p \, dx$$

$$\leqq A \sum_{k=1}^m \int_{B(R)^c} |\nabla^k u(x)|^p \, dx + AR^{-p} \int_{A(2R,R)} |u(x)|^p \, dx.$$

Thus $\eta_R u \to u$ in $L^{m,p}$ as $R \to \infty$ by Lemma 6 (i). Also

$$\int |\eta_R u - u|^p \, d\mu \leqq \int_{B(R)^c} |u|^p \, d\mu \to 0$$

as $R \to \infty$ since $u \in H_\mu^{m,p}(\Omega)$. It follows that $u \in \mathring{H}_\mu^{m,p}(\Omega)$, i.e. $C_\Omega^\infty \cap H_\mu^{m,p}(\Omega) \subset \mathring{H}_\mu^{m,p}(\Omega)$ and hence $H_\mu^{m,p}(\Omega) = \mathring{H}_\mu^{m,p}(\Omega)$.

We now turn to the necessity part. Suppose that $H_\mu^{m,p}(\Omega) = \mathring{H}_\mu^{m,p}(\Omega)$ and that F is a closed set such that $B_{m,p}(F) < \infty$ and $\mu(F^c) < \infty$ where now $p > 1$. Assume also that $B_{m,p}(\Omega^c) < \infty$. Then there is an open set $G \supset \Omega^c$ such that $B_{m,p}(\bar{G}) < \infty$. By Lemma 1 we can find $f \in L_+^p$ such that $G_{m;1} * f \in C^\infty$ and $G_{m;1} * f \geqq 1$ on $F \cup \bar{G}$. Now let T be a smooth function on $\mathbf{R}_+$ such that $T(t) = 1$ if $t \geqq 1$ and $\sup_{t>0} |t^{k-1} T^{(k)}(t)| < \infty$ for $0 \leqq k \leqq m$. Then by the truncation theorem in [1] (it works also for the truncated kernel) there is a function $g \in L^p$ such that $T \circ (G_{m;1} * f) = G_{m;1} * g$ and $\|g\|_p \leqq A \|f\|_p$. We set $u = 1 - G_{m;1} * g$. Then $u \in C_\Omega^\infty \cap L^{m,p}$ but, by Lemma 7, $u \notin L_0^{m,p}$ since $G_{m;1} * g \in L_0^{m,p}$. Moreover

$$\int |u|^p \, d\mu = \int_{F^c} |u|^p \, d\mu \leqq \mu(F^c) < \infty$$

by the assumption on F, so $u \in H_\mu^{m,p}(\Omega) \setminus \mathring{H}_\mu^{m,p}(\Omega)$. This contradiction shows that μ must satisfy the condition in the theorem.

For $p = 1$ we use instead $u = 1 - \varphi$ where φ is the function constructed in

Lemma 3, satisfying $\varphi = 1$ on a neighbourhood of $F \cup \bar{G}$. Since $\int |\varphi(x)| dx + \int |\nabla^m \varphi(x)| dx < \infty$ it follows easily, using the multiplier η_R above, that $\varphi \in L_0^{m,1}$. Hence $u \notin \mathring{H}_\mu^{m,1}(\Omega)$. But $\int |u| \, d\mu \leqq \mu(F^c) < \infty$ so $u \in H_\mu^{m,1}$ and we have again obtained a contradiction.

The case $1 < p < N$, $mp > N$ *or* $1 = p < N$, $m \geqq N$. For the sufficiency we again decompose $u = v + c$ where $v \in L_0^{m,p}$. Using Sobolev's inequality

$$\sup_{B(x,1)} |v| \leqq A\left(\int_{B(x,1)} |v(y)|^{p^*} dy\right)^{1/p^*} + A\left(\int_{B(x,1)} |\nabla^m v(y)|^p \, dy\right)^{1/p}$$

for every $x \in \mathbf{R}^N$, we see that $v(x) \to 0$ uniformly as $|x| \to \infty$. Thus, if $c \neq 0$, we can find R such that $|v| \leqq \frac{|c|}{2}$ on $B(R)^c$. Then, if μ is not finite,

$$\int |u|^p \, d\mu \geqq \left(\frac{|c|}{2}\right)^p \mu(B(R)^c) = \infty$$

and we have a contradiction. Thus $c = 0$ and we can proceed as in the first case, again using Lemma 6(i). Note that if Ω^c is unbounded it follows immediately that $c = 0$, without any condition on μ, since $v = c$ on Ω^c.

For the necessity we observe that if Ω^c is bounded then there is $u \in C_\Omega^\infty$ such that $u(x) = 1$ for large $|x|$. Then $u \in L^{m,p} \setminus L_0^{m,p}$ since $u \notin L^{p^*}$. If μ is finite we get $u \in H_\mu^{m,p}(\Omega) \setminus \mathring{H}_\mu^{m,p}(\Omega)$ and we are done.

The case $p > N$ is proved as above, now invoking Lemma 6 (ii).

The case $p = N$ is proved by Lemma 6 (iii), this time using the multiplier

$$\eta_R(x) = \chi\left(\frac{1}{\log R} \log \frac{R^2}{|x|}\right),$$

where χ is a smooth function satisfying $\chi(t) = 0$ for $t \leqq \frac{1}{4}$ and $\chi(t) = 1$ for $t \geqq \frac{3}{4}$. This completes the proof.

Remark. The same question of density of test functions can be asked about the more general norm

$$\|u\| = \|u\|_{L^p(\mu)} + \sum_{k=l}^{m} \|\nabla^k u\|_p,$$

where $m \geqq l \geqq 2$. In case $lp \geqq N$ approximation is always possible. This is proved in the same way as in this paper, using only somewhat different Hardy inequalitites. In the general case it is easy to give an implicit necessary and sufficient condition. Namely, with obvious notation, $H_\mu^{l,m,p} = \mathring{H}_\mu^{l,m,p}$ if and only if

$$u \in \mathring{H}_{\mu}^{l,m,p},\ P \in \mathsf{P}_{l-1},\ \int |u - P|^p \, d\mu < \infty \Rightarrow P = 0.$$

However, it is not clear whether this condition can be stated in a more transparent way in the spirit of Theorem 1, for example in terms of polynomial capacities; cfr [9, Ch. 10].

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