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AN EXISTENCE RESULT FOR SIMPLE INDUCTIVE LIMITS OF INTERVAL ALGEBRAS

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Given a C^*-algebra A with unit 1 we define the Elliot triple of A to be

$$(K_0(A), T(A), r_A)$$

where $K_0(A)$ has the usual ordering and $[1]$ as order-unit, $T(A)$ is the tracial state space of A, and with S the state space functor, $r_A: T(A) \to S(K_0(A))$ is given by $r_A(\tau)([p] - [q]) = \tau(p - q)$ for all $\tau \in T(A)$ and projections $p, q \in M_\infty(A)$ where τ is extended to $M_\infty(A)$ by $(a_{ij}) \mapsto \sum_i \tau(a_{ii})$. We identify two such triples (G_i, Δ_i, f_i), $i = 1, 2$ if there are isomorphisms $\phi_0: G_2 \to G_1$, $\phi_T: \Delta_1 \to \Delta_2$ such that the diagram

$$\begin{array}{ccc} \Delta_1 & \xrightarrow{\phi_T} & \Delta_2 \\ {\scriptstyle f_1}\big\downarrow & & \big\downarrow{\scriptstyle f_2} \\ S(G_1) & \xrightarrow[S(\phi_0)]{} & S(G_2) \end{array}$$

commutes. Elliott [3] proved that this triple is a complete invariant for the simple unital C^*-algebras which arise as inductive limits of finite direct sums of matrix algebras over $C([0, 1])$ – AI algebras for short. The project of determining the range of the Elliott triple when applied to simple unital AI algebras was initiated by Thomsen [6]. When A is a simple unital AI algebra $K_0(A)$ is a simple dimension group, $S(K_0(A))$ is a metrizable Choquet simplex, $T(A)$ is another metrizable Choquet simplex and the map r_A is an affine continuous surjection. Furthermore, we know from [6] that the map r_A preserves extreme points i.e. $r_A(\partial_e T(A)) = \partial_e S(K_0(A))$.

In this paper we show that, whenever G is a simple noncyclic dimension group, such that the extreme boundary of the state space is compact and totally disconnected, Δ is a metrizable Choquet simplex and $f: \Delta \to S(G)$ an affine continuous map with $f(\partial_e \Delta) = \partial_e S(G)$, then (G, Δ, f) is the Elliott triple of some simple unital AI algebra.

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NOTATION 1. For K a compact convex subset of a linear topological space, we denote by $\mathrm{Aff}(K)$ the complete order-unit space of affine continuous real-valued functions on K with pointwise ordering and 1 as the order-unit. For $\lambda: K \to L$ a continuous affine map between compact convex sets, let $\mathrm{Aff}(\lambda): \mathrm{Aff}(L) \to \mathrm{Aff}(K)$ be the positive order-unit-preserving homomorphism given by $\mathrm{Aff}(\lambda)(h) = h \circ \lambda$ for all $h \in \mathrm{Aff}(L)$. It is well-known that for K a compact convex subset of a locally convex Hausdorff space, the state space of $\mathrm{Aff}(K)$ is naturally isomorphic to K via evaluation.

For convenience we write $s^{\#}$ for $\mathrm{Aff}S(s)$ when s is a homomorphism of ordered groups, and write "homomorphism" instead of "positive order-unit-preserving homomorphism" when dealing with homomorphisms of order-unit spaces.

DEFINITIONS 2. Let Δ be a Choquet simplex. A partition of unity $v_1, \dots, v_k$ in $\mathrm{Aff}(\Delta)$ is said to be *extreme* if there are closed non-empty faces $\Delta_1, \dots, \Delta_k$ in Δ with $\Delta = \mathrm{hull}\{\Delta_1, \dots, \Delta_k\}$ such that $v_i|_{\Delta_j} \equiv \delta_{ij}$ for $1 \leqq i,j \leqq k$. A partition of unity $v_1, \dots, v_k$ in $\mathrm{Aff}(\Delta)$ is *peaked* if $\|v_j\| = 1, j = 1, \dots, k$. Note that in this case $\mathrm{span}\{v_1, \dots, v_k\} \cong l_k^\infty$ with $v_1, \dots, v_k$ as the standard basis.

For $v, 1 - v$ an extreme partition of unity in $\mathrm{Aff}(\Delta)$ with corresponding faces E, E^c we let $\mathrm{Aff}(\Delta)_v = \{f \in \mathrm{Aff}(\Delta): f|_{E^c} \equiv 0\}$, which is an order-unit space with order-unit v, and define a homomorphism $\pi_v: \mathrm{Aff}(\Delta) \to \mathrm{Aff}(\Delta)_v$ by $\pi_v(f)|_E = f|_E$ and $\pi_v(f)|_{E^c} \equiv 0$ for $f \in \mathrm{Aff}(\Delta)$.

LEMMA 3. *Suppose that Δ is a Choquet simplex and $(l_{n_i}^\infty, v_i)$ an inductive system with $\varinjlim (l_{n_i}^\infty, v_i) = \mathrm{Aff}(\Delta)$. Let $v_1, \dots, v_k$ be a peaked partition of unity in $\mathrm{Aff}(\Delta)$ and let $\varepsilon > 0$. There is an $i \in \mathbb{N}$ and a homomorphism $\rho: \mathrm{span}\{v_1, \dots v_k\} \to l_{n_i}^\infty$ such that $\|v_{\infty i} \circ \rho - id\| < \varepsilon$.*

PROOF. Since $\cup_{i=1}^\infty v_{\infty i}((l_{n_i}^\infty)^+)$ is dense in $\mathrm{Aff}(\Delta)^+$ there are $x_1, \dots, x_{k-1} \in (l_{n_{i_0}}^\infty)^+$ with $\left\|\left(1 - \frac{\varepsilon}{2k}\right)v_j - v_{\infty i_0}(x_j)\right\| < \frac{\varepsilon}{2k^2}, j = 1, \dots, k-1$. Now

$$v_{\infty i_0}\left(\sum_{j=1}^{k-1} x_j\right) < \left(1 - \frac{\varepsilon}{2k}\right)\sum_{j=1}^{k-1} v_j + \frac{\varepsilon}{2k} \leqq 1$$

and there is an $i \geqq i_0$ so that $v_{ii_0}(\sum_{j=1}^{k-1} x_j) < 1$. Let $y_j = v_{ii_0}(x_j), j = 1, \dots, k-1$ and $y_k = 1 - \sum_{j=1}^{k-1} y_j$. Then $(y_j)_{j=1}^k$ is a partition of unity in $l_{n_i}^\infty$. Since

$$1 - \frac{\varepsilon}{k} - \left(1 - \frac{\varepsilon}{2k}\right)v_k < \sum_{j=1}^{k-1} v_{\infty i}(y_j) < 1 - \left(1 - \frac{\varepsilon}{2k}\right)v_k$$

we have that

$$\left(1-\frac{\varepsilon}{2k}\right)v_k < \nu_{\infty i}(y_k) < \frac{\varepsilon}{k} + \left(1-\frac{\varepsilon}{2k}\right)v_k.$$

Hence, $\|v_j - \nu_{\infty i}(y_j)\| < \frac{\varepsilon}{k}, j = 1,\ldots,k$.

Let $\rho: \operatorname{span}\{v_1,\ldots,v_k\} \to l^\infty_{n_i}$ be the homomorphism given by $\rho(v_j) = y_j$ for $j = 1,\ldots,k$. Clearly, $\|\nu_{\infty i}\circ\rho - id\| < \varepsilon$.

LEMMA 4. *Let Δ be a metrizable Choquet simplex and $w, 1-w$ an extreme partition of unity in* $\mathrm{Aff}(\Delta)$. *Let V be a subspace of* $\mathrm{Aff}(\Delta)$ *with $1 \in V$ and $V \cong l^\infty_m$ for some $m \in \mathbb{N}$. Let $F \subseteq \mathrm{Aff}(\Delta)_w$ be a finite subset and let $\varepsilon > 0$. There is a subspace W of $\mathrm{Aff}(\Delta)_w$ with $w \in W \cong l^\infty_n$ for some $n \in \mathbb{N}$ such that* $\operatorname{dist}(f, W) < \varepsilon$ *for all $f \in F$ and a homomorphism $\eta: V \to W$ with $\|\eta - \pi_w|_V\| < \varepsilon$.*

PROOF. There is a $\delta > 0$ such that if $x_1,\ldots,x_l \in l^{\infty+}_k$ with $\|\sum_{i=1}^l x_i - 1\| < \delta$ then there are $y_1,\ldots,y_l \in l^{\infty+}_k$ with $\sum_{i=1}^l y_i = 1$ and $\|x_i - y_i\| < \frac{\varepsilon}{2m}$ for $1 \leqq i \leqq l$. It follows from Theorem 2.7.2 of [1] that there is a subspace W of $\mathrm{Aff}(\Delta)_w$ with $w \in W$ and $W \cong l^\infty_n$ for some $n \in \mathbb{N}$ such that $\operatorname{dist}(f, W) < \varepsilon$ for all $f \in F$ and $\operatorname{dist}(\pi_w(e_i), W) < \frac{\delta \wedge \varepsilon}{4m}$ for $1 \leqq i \leqq m$ where $e_1,\ldots,e_m$ is the standard basis for $V \cong l^\infty_m$. Since $\operatorname{dist}(\pi_w(e_i), W^+) \leqq 2\operatorname{dist}(\pi_w(e_i), W)$ there are $x_1,\ldots,x_m \in W^+$ with $\|x_i - \pi_w(e_i)\| < \frac{\delta\wedge\varepsilon}{2m}$ for $1 \leqq i \leqq m$. So $\|\sum_{i=1}^m x_i - w\| < \delta$ and there are $y_1,\ldots,y_m \in W^+$ with $\sum_{i=1}^m y_i = w$ and $\|x_i - y_i\| < \frac{\varepsilon}{2m}$ for $1 \leqq i \leqq m$. Let η: $V \to W$ be the homomorphism given by $\eta(e_i) = y_i$ for $1 \leqq i \leqq m$. Let $x \in V$ with $\|x\| \leqq 1$. Then $x = \sum_{i=1}^m \alpha_i e_i$ for some $\alpha_1,\ldots,\alpha_m \in [-1, 1]$ and

$$\|\eta(x) - \pi_w(x)\| \leqq \sum_{i=1}^m |\alpha_i|\,\|y_i - \pi_w(e_i)\| \leqq \sum_{i=1}^m (\|y_i - x_i\| + \|x_i - \pi_w(e_i)\|) < \varepsilon.$$

DEFINITION 5. A *tree* is a triple $(X, \leqq, x_0)$ where $(X, \leqq)$ is a partially ordered set with a maximal element x_0 such that $s(x) = \{y \in X : y < x, \forall z < x : y \nless z\}$ is finite for every $x \in X$, $s(x) \cap s(y) = \emptyset$ when $x \neq y$ and X is the union of the level sets $\mathscr{L}^i$ given by $\mathscr{L}^1 = \{x_0\}$ and $\mathscr{L}^{i+1} = \cup_{y\in\mathscr{L}^i} s(y)$ for $i \in \mathbb{N}$. For $i \in \mathbb{N}$ we let $L^i = \mathscr{L}^i \cup \{y \in \cup_{j=1}^i \mathscr{L}^j : \forall x \in X : x \nless y\}$ – *the leaves* of $\cup_{j=1}^i \mathscr{L}^j$ – and $c(x) = \{y \in L^{i+1} : y \leqq x\}$ for $x \in L^i$.

REMARK 6. Let Δ be a metrizable Choquet simplex with compact and totally disconnected extreme boundary. Since every locally compact, totally disconnected topological space has a basis of sets being both open and closed, there is a basis Y for the compact, totally disconnected metric space $\partial_e \Delta$ such that $\partial_e \Delta \in Y$ and $(Y, \subseteq, \partial_e \Delta)$ is a tree, where $s(y)$ is either empty or consists of mutually disjoint sets with union y for every $y \in Y$. Now put $X = \{g \in \mathrm{Aff}(\Delta) : \exists y \in Y : g|_{\partial_e \Delta} = 1_y\}$. Then $(X, \leqq, 1)$ is a tree where each L^i is an extreme partition of unity in $\mathrm{Aff}(\Delta)$ and the span of X is dense in $\mathrm{Aff}(\Delta)$.

PROPOSITION 7. *Let G be a simple noncyclic dimension group, such that the extreme boundary of the state space is compact and totally disconnected. Let Δ be a metrizable Choquet simplex and let $f : \Delta \to S(G)$ be a continuous affine map with $f(\partial_e \Delta) = \partial_e S(G)$. There is a system $(\mathbf{Z}^{n_i}, s_i)$ with positive order-unit-preserving connecting homomorphisms and inductive limit $\varinjlim(\mathbf{Z}^{n_i}, s_i) = G$, a collection $1 \in X \subseteq \mathrm{AffS}(G)$ with dense span such that $(X, \leqq, 1)$ is a tree where each L^i is an extreme partition of unity in $\mathrm{AffS}(G)$ and homomorphisms $\rho_i : W_i \to \mathrm{AffS}(\mathbf{Z}^{n_i})$, δ_i: $\mathrm{AffS}(\mathbf{Z}^{n_i}) \to W_{i+1}$ where $W_i = \mathrm{span}(L^i)$ such that*

$$\|\delta_i \circ \rho_i - \mathrm{id}\| < 2^{-i},$$

$$\|\rho_{i+1} \circ \delta_i - s_i^{\#}\| < 2^{-i} n_i^{-1},$$

$$\|s_{\infty i}^{\#} - \delta_i\| < 2^{-i}$$

*for all $i \in \mathbf{N}$ and moreover there are Markov operators θ_h: $C_{\mathbf{R}}([0,1]) \to C_{\mathbf{R}}([0,1])$ of the form $\theta_h = N_i^{-1}(\theta_h^1 + \ldots + \theta_h^{N_i})$ where $\theta_h^1, \ldots, \theta_h^{N_i}$ are restrictions of unital *-endomorphisms of $C([0,1])$ for $h \in L^i$, $i \geqq 2$ such that*

$$\mathrm{Aff}(\Delta) = \varinjlim(W_i \otimes C_{\mathbf{R}}([0,1]), \theta_i),$$

$$\theta_{\infty i}(v \otimes 1) = \mathrm{Aff}(f)(v),\ v \in W_i,\ i \in \mathbf{N}$$

where $\theta_i : W_i \otimes C_{\mathbf{R}}([0,1]) \to W_{i+1} \otimes C_{\mathbf{R}}([0,1]$ is the homomorphism given by

$$\theta_i\Big(\sum_{g \in L^i} g \otimes x_g\Big) = \sum_{\substack{g \in L^i \\ h \in c(g)}} h \otimes \theta_h(x_g)$$

and

$$\mathrm{mult}(s_i) \geqq 2^i N_i \# L^{i+1}.$$

PROOF. Let $(\mathbf{Z}^{n_i}, s_i)$ be a system with positive order-unit-preserving connecting homomorphisms and inductive limit $\varinjlim(\mathbf{Z}^{n_i}, s_i) = G$. Since G is simple and noncyclic we may assume that $\mathrm{mult}(s_i) \to \infty$ as $i \to \infty$ where mult denotes the smallest entry of a given matrix. We may therefore also assume that $\mathrm{mult}(s_i) \geqq 1$ for all $i \in \mathbf{N}$.

By the above remark there is a collection $1 \in X \subseteq \mathrm{AffS}(G)$ with dense span such that $(X, \leqq, 1)$ is a tree where each L^i is an extreme partition of unity in $\mathrm{AffS}(G)$.

Note that $\mathrm{Aff}(f)$ takes extreme partitions of unity in $\mathrm{AffS}(G)$ to extreme partitions of unity in $\mathrm{Aff}(\Delta)$ because $f(\partial_e\Delta) = \partial_e S(G)$.

For convenience we write $\mathrm{Aff}(\Delta)_g$ and π_g for $\mathrm{Aff}(\Delta)_{\mathrm{Aff}(f)(g)}$ and $\pi_{\mathrm{Aff}(f)(g)}$ respectively for $g \in X$.

For $m \in \mathbb{N}$ a partition of unity $\zeta_1, \ldots, \zeta_m$ in $C_{\mathbb{R}}([0,1])$ is chosen such that $\zeta_i\left(\dfrac{i-1}{m}\right) = 1$, $1 \leqq i \leqq m$, and we let $\iota_m : l^\infty_m \to C_{\mathbb{R}}([0,1])$, $\kappa_m : C_{\mathbb{R}}([0,1]) \to l^\infty_m$ be the homomorphisms given by $\iota_m(e_i) = \zeta_i$, $1 \leqq i \leqq m$ and $\kappa_m(x) = \sum_{i=1}^m x\left(\dfrac{i-1}{m}\right)e_i$ for $x \in C_{\mathbb{R}}([0,1])$. Note that $\kappa_m \circ \iota_m = \mathrm{id}$.

Let $(d_i)_{i=1}^\infty$ and $(a_i)_{i=1}^\infty$ be dense sequences in $C_{\mathbb{R}}([0,1])$ and $\mathrm{Aff}(\Delta)$ respectively. We show that there are increasing sequences $(i_p)_{p=1}^\infty$, $(j_p)_{p=1}^\infty$ in $\mathbb{N}$ ($j_1 = 1$) and

- subspaces $\mathrm{Aff}(f)(g) \in Z_g \subseteq \mathrm{Aff}(\Delta)_g$ such that $Z_g \cong l^\infty_{m_g}$ for some $m_g \in \mathbb{N}$ and $\mathrm{dist}(\pi_g(a_q), Z_g) < 2^{-p}$ for all $1 \leqq q \leqq p$, $g \in L^{j_p}$, $p \in \mathbb{N}$,
- homomorphisms $\eta_h : Z_g \to Z_h$ such that $\|\eta_h - \pi_h|_{Z_g}\| < 2^{-p}$ for all $g \in L^{j_p}$, $h \in L^{j_{p+1}}$, $h \leqq g$, $p \in \mathbb{N}$,
- Markov operators $\theta_h : C_{\mathbb{R}}([0,1]) \to C_{\mathbb{R}}([0,1])$ which are of the form $\theta_h = N_p^{-1}(\theta_h^1 + \ldots + \theta_h^{N_p})$ where $\theta_h^1, \ldots, \theta_h^{N_p}$ are restrictions of unital $*$-endomorphisms of $C([0,1])$ and $\|\theta_h(f) - \iota_h \circ \eta_h \circ \kappa_g(f)\| < 2^{-p}$ for all $f \in F_g$, $g \in L^{j_p}$, $h \in L^{j_{p+1}}$, $h \leqq g$, $p \in \mathbb{N}$ where $\iota_g = \iota_{m_g} \circ \phi_g : Z_g \to C_{\mathbb{R}}([0,1])$, $\kappa_g = \phi_g^{-1} \circ \kappa_{m_g}$: $C_{\mathbb{R}}([0,1]) \to Z_g$ for some isomorphism $\phi_g : Z_g \to l^\infty_{m_g}$ and
$$F_g = \bigcup_{q=1}^{p} (\theta_{gg_q}(\{d_1, \ldots, d_p\}) \cup \iota_g \circ \eta_{gg_q} \circ \kappa_{g_q}(\{d_1, \ldots, d_p\}))$$
where $g_q \in L^{j_q}$ is the unique function $g_q \geqq g$,
- homomorphisms $\rho_p : W_p \to \mathrm{AffS}(\mathbb{Z}^{n_{i_p}})$ and $\delta_p : \mathrm{AffS}(\mathbb{Z}^{n_{i_p}}) \to W_{p+1}$ where $W_p = \mathrm{span}(L^{j_p})$ such that
$$\|s^{\#}_{\infty i_p} \circ \rho_p - \mathrm{id}\| < 2^{-p-1},$$
$$\|s^{\#}_{\infty i_p} - \delta_p\| < 2^{-p-1} n_{i_p}^{-1},$$
$$\|\rho_{p+1} \circ \delta_p - s^{\#}_{i_{p+1} i_p}\| < 2^{-p} n_{i_p}^{-1}$$
for all $p \in \mathbb{N}$ and such that
- $\mathrm{mult}(s_{i_{p+1} i_p}) \geqq 2^p N_p \,\#\, L^{j_{p+1}}$ for all $p \in \mathbb{N}$.

By Lemma 3, there is an $i_1 \in \mathbb{N}$ and a homomorphism $\rho_1 : W_1 \to \mathrm{AffS}(\mathbb{Z}^{n_{i_1}})$ such that $\|s^{\#}_{\infty i_1} \circ \rho_1 - \mathrm{id}\| < 2^{-2}$. Since the span of X is dense in $\mathrm{AffS}(G)$ there is a $j_2 > 1$ and a homomorphism $\delta_1 : \mathrm{AffS}(\mathbb{Z}^{n_{i_1}}) \to W_2$ such that $\|s^{\#}_{\infty i_1} - \delta_1\| < 2^{-2} n_{i_1}^{-1}$. It follows from Theorem 2.7.2 of [1] that there is a subspace $1 \in Z_1 \subseteq \mathrm{Aff}(\Delta)$ such that $Z_1 \cong l^\infty_{m_1} \in \mathbb{N}$ and $\mathrm{dist}(a_1, Z_1) < 2^{-1}$.

Suppose that $(i_p)_{p=1}^P$, $(j_p)_{p=1}^{P+1}$ are increasing sequences in $\mathbb{N}$ ($j_1 = 1$) and that Z_g,

$g \in L^{j_p}$, $1 \leqq p \leqq P$, η_h, θ_h, $h \in L^{j_p}$, $2 \leqq p \leqq P$, δ_p, ρ_p, $1 \leqq p \leqq P$ and $\operatorname{mult}(s_{i_{p+1}i_p})$, $1 \leqq p \leqq P-1$ satisfies the above conditions. It follows from Lemma 4 that for every $g \in L^{j_P}$, $h \in L^{j_{P+1}}$, $h \leqq g$ there is a subspace $\operatorname{Aff}(f)(h) \in Z_h \subseteq \operatorname{Aff}(\Delta)_h$ such that $Z_h \cong l^\infty_{m_h}$ for some $m_h \in \mathbb{N}$ and a homomorphism $\eta_h : Z_g \to Z_h$ such that

$$\operatorname{dist}(\pi_h(a_q), Z_h) < 2^{-(P+1)},\ 1 \leqq q \leqq P+1,$$

$$\|\eta_h - \pi_h|_{Z_g}\| < 2^{-P}.$$

It follows from the Krein-Milman theorem for Markov operators of [5] that there is a Markov operator $\theta_h : C_{\mathbb{R}}([0,1])$ of the form $\theta_h = N_{Ph}^{-1}(\theta_h^1 + \ldots + \theta_h^{N_{Ph}})$ where $\theta_h^1, \ldots, \theta_h^{N_{Ph}}$ are restrictions of unital $*$-endomorphisms of $C([0,1])$ such that $\|\theta_h(f) - \iota_h \circ \eta_h \circ \kappa_g(f)\| < 2^{-P}$ for all $f \in F_g$. We may assume that $N_{Ph} = N_P$ for all $h \in L^{j_{P+1}}$. There is a $k_1 > i_P$ such that $\operatorname{mult}(s_{k_1 i_P}) \geqq 2^P N_P \# L^{j_{P+1}}$. It follows from Lemma 3 that there is a $k_2 > k_1$ and a homomorphism $\rho : W_{P+1} \to \operatorname{AffS}(\mathbb{Z}^{n_{k_2}})$ such that $\|s^\#_{\infty k_2} \circ \rho - \mathrm{id}_{W_{P+1}}\| < 2^{-P-2} n_{i_P}^{-1}$. Since

$$\|s^\#_{\infty k_2} \circ \rho \circ \delta_P - s^\#_{\infty i_P}\| \leqq \|s^\#_{\infty k_2} \circ \rho \circ \delta_P - \delta_P\| + \|\delta_P - s^\#_{\infty i_P}\| < 2^{-P} n_{i_P}^{-1}$$

and the unit ball in $\operatorname{AffS}(\mathbb{Z}^{n_{i_P}})$ is compact, it follows that there is an $i_{P+1} \geqq k_2$ such that $\|s^\#_{i_{P+1} i_P} \circ \rho \circ \delta_P - s^\#_{i_{P+1} i_P}\| < 2^{-P} n_{i_P}^{-1}$. Note that $\operatorname{mult}(s_{i_{P+1} i_P}) \geqq 2^P N_P \# L^{j_{P+1}}$ and put $\rho_{P+1} = s^\#_{i_{P+1} i_P} \circ \rho$. Since the span of X is dense in $\operatorname{AffS}(G)$ there is a $j_{P+2} > j_{P+1}$ and a homomorphism $\delta_{P+1} : \operatorname{AffS}(\mathbb{Z}^{n_{i_{P+1}}}) \to W_{P+2}$ such that $\|s^\#_{\infty i_{P+1}} - \delta_{P+1}\| < 2^{-(P+1)-1} n_{i_{P+1}}^{-1}$. The result follows from Zorn's lemma.

For $p \in \mathbb{N}$ let Z_p be the subspace of $\operatorname{Aff}(\Delta)$ spanned by $(Z_g)_{g \in L^{j_p}}$. Note that Z_p is isomorphic to the order-unit-space direct sum of $(Z_g)_{g \in L^{j_p}}$. Moreover $\operatorname{dist}(a_q, Z_p) < 2^{-p}$ for all $1 \leqq q \leqq p$. Let $\eta_p : Z_p \to Z_{p+1}$ be the homomorphism given by

$$\eta_p\Bigl(\sum_{g \in L^{j_p}} v_g\Bigr) = \sum_{\substack{g \in L^{j_p} \\ h \in L^{j_{p+1}} \\ h \leqq g}} \eta_h(v_g)$$

for $v_g \in Z_g$, $g \in L^{j_p}$. Then η_p is a homomorphism with $\eta_p(\operatorname{Aff}(f)(g)) = \operatorname{Aff}(f)(g)$ for all $g \in W_p$. Let $v \in Z_p$ with $\|v\| \leqq 1$. Then v is of the form $v = \sum_{g \in L^{j_p}} v_g$ where $v_g \in Z_g$, $\|v_g\| \leqq 1$ and

$$\|\eta_p(v) - v\| = \Biggl\| \sum_{\substack{g \in L^{j_p} \\ h \in L^{j_{p+1}} \\ h \leqq g}} (\eta_h(v_g) - \pi_h(v_g)) \Biggr\| = \max_{\substack{g \in L^{j_p} \\ h \in L^{j_{p+1}} \\ h \leqq g}} \|\eta_h(v_g) - \pi_h(v_g)\| < 2^{-p}.$$

Therefore the sequence $(\eta_{qp}(v))_{q=p}^\infty$ in $\operatorname{Aff}(\Delta)$ is Cauchy for every $v \in Z_p$ – let $\alpha_p(v)$ denote the limit. Then $\alpha_p : Z_p \to \operatorname{Aff}(\Delta)$ is a homomorphism with $\alpha_{p+1} \circ \eta_p = \alpha_p$ and $\alpha_p(\operatorname{Aff}(f)(g)) = \operatorname{Aff}(f)(g)$ for all $g \in W_p$. Thus there is a homomorphism $\alpha : \varinjlim (Z_p, \eta_p) \to \operatorname{Aff}(\Delta)$ with $\alpha \circ \eta_{\infty p} = \alpha_p$. By $\|\eta_p(v) - v\| < 2^{-p}\|v\|$ the homomorphism α is seen to be isometric. Using that $(a_i)_{i=1}^{\infty}$ is dense in $\mathrm{Aff}(\Delta)$, $\mathrm{dist}(a_q, Z_p) < 2^{-p}$ for all $1 \leqq q \leqq p$ and $1 \in Z_p$ for all $p \in \mathbb{N}$ and the fact that α is isometric, one shows that α is an isomorphism.

Define $I_p: Z_p \to W_p \otimes C_{\mathbb{R}}([0,1])$ and $K_p: W_p \otimes C_{\mathbb{R}}([0,1]) \to Z_p$ by

$$I_p\left(\sum_{g \in L^{j_p}} v_g\right) = \sum_{g \in L^{j_p}} g \otimes \iota_g(v_g),$$

$$K_p\left(\sum_{g \in L^{j_p}} g \otimes x_g\right) = \sum_{g \in L^{j_p}} \kappa_g(x_g)$$

for $v_g \in Z_g$, $x_g \in C_{\mathbb{R}}([0,1])$, $g \in L^{j_p}$. The maps I_p, K_p are homomorphisms and $K_p \circ I_p = \mathrm{id}$. Letting $\omega_p = I_{p+1} \circ \eta_p \circ K_p$ and $\beta_p = \eta_{\infty p} \circ K_p$ we have $\beta_{p+1} \circ \omega_p = \beta_p$ and so there is a homomorphism $\beta: \varinjlim (W_p \otimes C_{\mathbb{R}}([0,1]), \omega_p) \to \varinjlim (Z_p, \eta_p)$ with $\beta \circ \omega_{\infty p} = \beta_p$, $p \in \mathbb{N}$. It is easy to see that this is an isomophism. In addition $\alpha \circ \beta_p(g \otimes 1) = \alpha \circ \eta_{\infty p} \circ K_p(g \otimes 1) = \alpha \circ \eta_{\infty p}(\mathrm{Aff}(f)(g)) = \alpha_p(\mathrm{Aff}(f)(g)) = \mathrm{Aff}(f)(g)$ for all $g \in W_p$.

Let $\theta_p: W_p \otimes C_{\mathbb{R}}([0,1]) \to W_{p+1} \otimes C_{\mathbb{R}}([0,1])$ be the homomorphism given by

$$\theta_p\left(\sum_{g \in L^{j_p}} g \otimes x_g\right) = \sum_{\substack{g \in L^{j_p} \\ h \in L^{j_{p+1}} \\ h \leqq g}} h \otimes \theta_h(x_g).$$

The set $\{\sum_{g \in L^{j_p}} g \otimes x_g : x_g \in \{d_i : i \in \mathbb{N}\}\}$ is dense in $W_p \otimes C_{\mathbb{R}}([0,1])$ and

$$\bigcup_{q=1}^{p} \left(\theta_{pq}\left\{ \sum_{h \in L^{j_q}} h \otimes x_h : x_h \in \{d_1, \ldots, d_p\} \right\} \cup \omega_{pq}\left\{ \sum_{h \in L^{j_q}} h \otimes x_h : x_h \in \{d_1, \ldots, d_p\} \right\} \right)$$

is equal to $\{\sum_{g \in L^{j_p}} g \otimes x_g : x_g \in F_g\}$. For $z = \sum_{g \in L^{j_p}} g \otimes x_g$ where $x_g \in F_g$ we have

$$\|\theta_p(z) - \omega_p(z)\| = \sum_{\substack{g \in L^{j_p} \\ h \in L^{j_{p+1}} \\ h \leqq g}} \|\theta_h(x_g) - \iota_h \circ \eta_h \circ \kappa_g(x_g)\| \leqq 2^{-p}$$

and it follows from (a slight modification of) Lemma 3.4 of [5] that there is an isomorphism $\gamma: \varinjlim (W_p \otimes C_{\mathbb{R}}([0,1]), \theta_p) \to \varinjlim (W_p \otimes C_{\mathbb{R}}([0,1]), \omega_p)$ such that $\gamma \circ \theta_{\infty p} = \gamma_p$ where $\gamma_p: W_p \otimes C_{\mathbb{R}}([0,1]) \to \varinjlim (W_p \otimes C_{\mathbb{R}}([0,1]), \omega_p)$ is the homomorphism given by $\gamma_p(x) = \lim_{q \to \infty} \omega_{\infty q} \circ \theta_{qp}(x)$ for $x \in W_p \otimes C_{\mathbb{R}}([0,1])$. Note that $\gamma_p(g \otimes 1) = \omega_{\infty p}(g \otimes 1)$ for all $g \in W_p$.

To sum up, there is an isomorphism $\alpha \circ \beta \circ \gamma: \varinjlim (W_p \otimes C_{\mathbb{R}}([0,1]), \theta_p) \to \mathrm{Aff}(\Delta)$ with $\alpha \circ \beta \circ \gamma \circ \theta_{\infty i}(g \otimes 1) = \mathrm{Aff}(f)(g)$ for all $g \in W_i$ and $i \in \mathbb{N}$.

LEMMA 8. *Let* $M, N \in \mathbb{N}$ *and* $\lambda_1, \ldots, \lambda_m \geqq 0$ *with* $\sum_{j=1}^{m} \lambda_j > 0$. *There are* $k_1, \ldots, k_m \in \mathbb{N}_0$ *such that*

$$M - N < N \sum_{j=1}^{m} k_j \leqq M,$$

$$\sum_{j=1}^{m} \left| \frac{Nk_j}{M} - \lambda_j \right| < \frac{2Nm}{M} + \left| 1 - \sum_{j=1}^{m} \lambda_j \right|.$$

PROOF. Put $\lambda = \sum_{j=1}^{m} \lambda_j$ and $\mu_j = \frac{M}{\lambda N} \sum_{l=1}^{j} \lambda_l$ for $1 \leqq j \leqq m$. There is an $h_j \in \mathbf{N}_0$ with $\mu_j - 1 < h_j \leqq \mu_j$ for $1 \leqq j \leqq m$. Put $k_1 = h_1$ and $k_{j+1} = h_{j+1} - h_j$ for $1 \leqq j \leqq m - 1$. We have

$$M - N = N(\mu_m - 1) < Nh_m = N \sum_{j=1}^{m} k_j \leqq N\mu_m = M,$$

$$\left| \frac{Nk_1}{M} - \frac{\lambda_1}{\lambda} \right| = \frac{N}{M} \left| k_1 - \frac{M\lambda_1}{N\lambda} \right| = \frac{N}{M} |k_1 - \mu_1| < \frac{N}{M}$$

and

$$\left| \frac{Nk_j}{M} - \frac{\lambda_j}{\lambda} \right| = \frac{N}{M} \left| k_j - \frac{M\lambda_j}{N\lambda} \right| = \frac{N}{M} |(h_j - h_{j-1}) - (\mu_j - \mu_{j-1})| < 2\frac{N}{M}$$

for $2 \leqq j \leqq m$. Therefore

$$\sum_{j=1}^{m} \left| \frac{Nk_j}{M} - \lambda_j \right| < \frac{2Nm}{M} + |1 - \lambda|.$$

THEOREM 9. *Let G be a simple noncyclic dimension group, such that the extreme boundary of the state space is compact and totally disconnected. Let Δ be a metrizable Choquet simplex and let $f: \Delta \to S(G)$ be a continuous affine map with $f(\partial_e \Delta) = \partial_e S(G)$. Then (G, Δ, f) is the Elliott triple of some simple unital* AI *algebra.*

PROOF. Let $(\mathbf{Z}^{n_i}, s_i), X, (\rho_i)_{i=1}^{\infty}, (\delta_i)_{i=1}^{\infty}, (\theta_h)_{h \in X - \{1\}}$ and mult (s_i) be as in Proposition 7. Let $(a^i_{pq}), (\lambda^i_{hq}), (\mu^i_{pq})$ and (v^i_{ph}) be the matrices for $s_i, \delta_i, s_i^{\#}$ and ρ_i respectively and put $Z_i = \{(p,q): 1 \leqq p \leqq n_{i+1}, 1 \leqq q \leqq n_i, \sum_{h \in L^{i+1}} v^{i+1}_{ph} \lambda^i_{hq} = 0\}$ – the zero entries of the matrix for $\rho_{i+1} \circ \delta_i$ for $i \in \mathbf{N}$.

It follows from Lemma 8 that for $(p,q) \notin Z_i$ there are $(k^h_{pq})_{h \in L^{i+1}}$ in $\mathbf{N}_0$ such that

$$a^i_{pq} - N_i < N_i \sum_{h \in L^{i+1}} k^h_{pq} \leqq a^i_{pq},$$

$$\sum_{h \in L^{i+1}} \left| \frac{N_i k^h_{pq}}{a^i_{pq}} - \frac{v^{i+1}_{ph} \lambda^i_{hq}}{\mu^i_{pq}} \right| < \frac{2N_i \# L^{i+1}}{a^i_{pq}} + \left| 1 - \sum_{h \in L^{i+1}} \frac{v^{i+1}_{ph} \lambda^i_{hq}}{\mu^i_{pq}} \right|.$$

For $(p,q) \in Z_i$ choose $(k^h_{pq})_{h \in L^{i+1}} \subseteq \mathbf{N}_0$ such that $a^i_{pq} - N_i < N_i \sum_{h \in L^{i+1}} k^h_{pq} \leqq a^i_{pq}$.

Let $r^i_{pq} = a^i_{pq} - N_i \sum_{h \in L^{i+1}} k^h_{pq}$ for $1 \leqq p \leqq n_{i+1}$, $1 \leqq q \leqq n_i$ and $i \in \mathbf{N}$. Let $m_i =$

$(m_1^i, \ldots, m_{n_i}^i)$ denote the order-unit in $\mathbf{Z}^{n_i}$. Then $A_i = \oplus_{p=1}^{n_i} M_{m_p^i} \otimes C([0,1])$ is the interval algebra with $K_0(A_i) = \mathbf{Z}^{n_i}$. For all $h \in L^{i+1}$, $1 \leqq r \leqq N_i$ and $i \in \mathbf{N}$ there is a continuous function $\omega_h^r : [0,1] \to [0,1]$ such that $\theta_h^r(x) = x \circ \omega_h^r$ for all $x \in C([0,1])$. Let $\psi_i : A_i \to A_{i+1}$ be the $*$-homomorphism with characteristic functions ω_h^r repeated k_{pq}^h times, $h \in L^{i+1}$, $1 \leqq r \leqq N_i$ and $\mathrm{id}_{[0,1]}$ repeated r_{pq}^i times from the qth summand of A_i to the pth summand of A_{i+1} for $1 \leqq p \leqq n_{i+1}$, $1 \leqq q \leqq n_i$ and $i \in \mathbf{N}$. Let $B = \varinjlim (A_i, \psi_i)$ and note that $K_0(B) = \varinjlim (\mathbf{Z}^{n_i}, s_i) = G$.

Let $\iota_i : W_i \hookrightarrow W_{i+1}$ be inclusion and let $\vartheta_i : W_{i+1} \otimes C_R([0,1]) \to W_{i+1} \otimes C_R([0,1])$ be the homomorphism given by

$$\vartheta_i\left(\sum_{h \in L^{i+1}} h \otimes x_h\right) = \sum_{h \in L^{i+1}} h \otimes \theta_h(x_h).$$

Observe that $\theta_i \circ (\iota_i \otimes \mathrm{id}_{C_{\mathrm{R}}([0,1])})$. Define

$$\zeta_i = \vartheta_i \circ (\delta_i \otimes \mathrm{id}_{C_{\mathrm{R}}([0,1])}) : \mathrm{AffT}(A_i) \to W_{i+1} \otimes C_{\mathrm{R}}([0,1]),$$

$$\eta_i = \rho_i \otimes \mathrm{id}_{C_{\mathrm{R}}([0,1])} : W_i \otimes C_{\mathrm{R}}([0,1]) \to \mathrm{AffT}(A_i)$$

for $i \in \mathbf{N}$.

We show that the triangles of the following diagram commutes up to an error which is summable.

$$\begin{array}{ccc}
W_i \otimes C_{\mathrm{R}}([0,1]) & \xrightarrow{\theta_i} & W_{i+1} \otimes C_{\mathrm{R}}([0,1]) \\
\downarrow \eta_i & \nearrow \zeta_i & \downarrow \eta_{i+1} \\
\mathrm{AffT}(A_i) & \xrightarrow[\mathrm{AffT}(\psi_i)]{} & \mathrm{AffT}(A_{i+1})
\end{array}$$

As for the upper triangle we have

$$\begin{aligned}
\|\theta_i - \zeta_i \circ \eta_i\| &= \|\vartheta_i \circ (\iota_i \otimes \mathrm{id}_{C_{\mathrm{R}}([0,1])}) - \vartheta_i \circ (\delta_i \otimes \mathrm{id}_{C_{\mathrm{R}}([0,1])}) \circ (\rho_i \otimes \mathrm{id}_{C_{\mathrm{R}}([0,1])})\| \\
&\leqq \|\iota_i - \delta_i \circ \rho_i\| \\
&< 2^{-i}.
\end{aligned}$$

Let $x = (x_1, \ldots, x_{n_i}) \in \mathrm{AffT}(A_i)$ and $1 \leqq p \leqq n_{i+1}$. The pth coordinate of AffT $(\psi_i)(x)$ is

$$\sum_{q=1}^{n_i} \frac{m_q^i}{m_p^{i+1}} \left(\sum_{h \in L^{i+1}} k_{pq}^h \sum_{r=1}^{N_i} x_q \circ \omega_h^r + r_{pq}^i x_q \right)$$

and the pth coordinate of $\eta_{i+1} \circ \zeta_i(x)$ is

$$\sum_{q=1}^{n_i} \sum_{h \in L^{i+1}} v_{ph}^{i+1} \lambda_{hq}^i \theta_h(x_q).$$

Assume that $\|x\| \leqq 1$. Then

$$\|\mathrm{AffT}(\psi_i)(x) - \eta_{i+1}\circ\zeta_i(x)\|$$

$$\leqq \max_{1\leqq p\leqq n_{i+1}} \left\{ \sum_{q=1}^{n_i} \sum_{h\in L^{i+1}} \left| \frac{m_q^i k_{pq}^h}{m_p^{i+1}} - v_{ph}^{i+1}\lambda_{hq}^i \frac{1}{N_i} \right| \left\| \sum_{r=1}^{N_i} x_q\circ\omega_h^r \right\| + \sum_{q=1}^{n_i} \frac{m_q^i r_{pq}^i}{m_p^{i+1}} \|x_q\| \right\}$$

$$\leqq \max_{1\leqq p\leqq n_{i+1}} \left\{ \sum_{q=1}^{n_i} \sum_{h\in L^{i+1}} \left| \frac{m_q^i k_{pq}^h N_i}{m_p^{i+1}} - v_{ph}^{i+1}\lambda_{hq}^i \right| + \sum_{q=1}^{n_i} \frac{m_q^i a_{pq}^i 2^{-i}}{m_p^{i+1}} \right\}$$

$$\leqq \max_{1\leqq p\leqq n_{i+1}} \left\{ \sum_{q=1}^{n_i} \mu_{pq}^i \sum_{h\in L^{i+1}} \left| \frac{k_{pq}^h N_i}{a_{pq}^i} - \frac{v_{ph}^{i+1}\lambda_{hq}^i}{\mu_{pq}^i} \right| + 2^{-i} \right\}$$

$$\leqq \max_{1\leqq p\leqq n_{i+1}} \left\{ \sum_{\substack{1\leqq q\leqq n_i\\(p,q)\in Z_i}} \mu_{pq}^i \sum_{h\in L^{i+1}} \frac{k_{pq}^h N_i}{a_{pq}^i} + \sum_{\substack{1\leqq q\leqq n_i\\(p,q)\notin Z_i}} \mu_{pq}^i \right.$$
$$\left. \times \left(\frac{2N_i \,\#\, L^{i+1}}{a_{pq}^i} + \left| 1 - \sum_{h\in L^{i+1}} \frac{v_{ph}^{i+1}\lambda_{hq}^i}{\mu_{pq}^i} \right| \right) + 2^{-i} \right\}$$

$$\leqq \max_{1\leqq p\leqq n_{i+1}} \left\{ \sum_{\substack{1\leqq q\leqq n_i\\(p,q)\in Z_i}} \mu_{pq}^i + 2^{1-i} + \sum_{\substack{1\leqq q\leqq n_i\\(p,q)\notin Z_i}} \left| \mu_{pq}^i - \sum_{h\in L^{i+1}} v_{ph}^{i+1}\lambda_{hq}^i \right| + 2^{-i} \right\}$$

$$\leqq \max_{1\leqq p\leqq n_{i+1}} \left\{ \sum_{\substack{1\leqq q\leqq n_i\\(p,q)\in Z_i}} \|\rho_{i+1}\circ\delta_i - s_i^{\#}\| + 2^{1-i} + \sum_{\substack{1\leqq q\leqq n_i\\(p,q)\notin Z_i}} \|\rho_{i+1}\circ\delta_i - s_i^{\#}\| + 2^{-i} \right\}$$

$$= n_i\|\rho_{i+1}\circ\delta_i - s_i^{\#}\| + 3\cdot 2^{-i}$$

$$\leqq 4\cdot 2^{-i}.$$

Since

$$\|\theta_{i+1}\circ\zeta_i - \zeta_{i+1}\circ\mathrm{AffT}(\psi_i)\|$$
$$\leqq \|\theta_{i+1}\circ\zeta_i - \zeta_{i+1}\circ\eta_{i+1}\circ\zeta_i\| + \|\zeta_{i+1}\circ\eta_{i+1}\circ\zeta_i - \zeta_{i+1}\circ\mathrm{AffT}(\psi_i)\|$$
$$\leqq \|\theta_{i+1} - \zeta_{i+1}\circ\eta_{i+1}\| + \|\eta_{i+1}\circ\zeta_i - \mathrm{AffT}(\psi_i)\|$$
$$\leqq 5\cdot 2^{-i}$$

the sequence $(\theta_{\infty k+1}\circ\zeta_k\circ\mathrm{AffT}(\psi_{ki})(x))_{k=i}^{\infty}$ is Cauchy for all $x\in\mathrm{AffT}(A_i)$ – let $\alpha_i(x)$ denote the limit. Then $\alpha_i:\mathrm{AffT}(A_i)\to\mathrm{Aff}(\Delta)$ is a homomorphism with $\alpha_{i+1}\circ\mathrm{AffT}(\psi_i) = \alpha_i$. Thus there is a homomorphism $\alpha:\mathrm{AffT}(B)\to\mathrm{Aff}(\Delta)$ with $\alpha\circ\mathrm{AffT}(\psi_{\infty i}) = \alpha_i$. Using that $(\|\mathrm{AffT}(\psi_i) - \eta_{i+1}\circ\zeta_i\|)_{i=1}^{\infty}$ and $(\|\zeta_i\circ\eta_i - \theta_i\|)_{i=1}^{\infty}$ are summable one shows that α is an isomorphism.

Note that $T(B)\cong\Delta$ via $S(\alpha):\Delta\to T(B)$. We now show that $r_B\circ S(\alpha) = f$. Let κ_i and χ_i be the inclusions

$$\kappa_i : W_i \hookrightarrow W_i \otimes C_{\mathbb{R}}([0,1]),\ g \mapsto g \otimes 1,$$

$$\chi_i : \mathrm{Aff}\,\mathrm{S}(\mathbb{Z}^{n_i}) \hookrightarrow \mathrm{Aff}\,\mathrm{S}(\mathbb{Z}^{n_i}) \otimes C_{\mathbb{R}}([0,1]) = \mathrm{Aff}\,\mathrm{T}(A_i),\ g \mapsto g \otimes 1.$$

With these definitions we have that

$$\theta_{\infty i} \circ \kappa_i = \mathrm{Aff}(f)|_{W_i},$$

$$\zeta_i \circ \chi_i = \kappa_{i+1} \circ \delta_i$$

and $p = \chi_i[p]$ for every projection $p \in A_i \subset \mathrm{Aff}\,\mathrm{T}(A_i)$. Let $w \in \Delta$ and $p \in A_i$ be a projection.

$$\begin{aligned}
&r_B \circ S(\alpha)(w)[\psi_{\infty i}(p)] \\
&= S(\alpha)(w)(\psi_{\infty i}(p)) \\
&= w \circ \alpha \circ \psi_{\infty i}(p) \\
&= w \circ \alpha_i(p) \\
&= \lim_{k\to\infty} w \circ \theta_{\infty k+1} \circ \zeta_k \circ \psi_{ki}(p) \\
&= \lim_{k\to\infty} w \circ \theta_{\infty k+1} \circ \zeta_k \circ \chi_k[\psi_{ki}(p)] \\
&= \lim_{k\to\infty} w \circ \theta_{\infty k+1} \circ \kappa_{k+1} \circ \delta_k[\psi_{ki}(p)] \\
&= f(w)\Big(\lim_{k\to\infty} \delta_k[\psi_{ki}(p)]\Big) \\
&= f(w)[\psi_{\infty i}(p)].
\end{aligned}$$

Hence (G, Δ, f) is the Elliott triple of B.

The final step of the proof consists of replacing B by a simple AI algebra with the same Elliott triple. Let ϕ_i be the $*$-homomorphism obtained from ψ_i by replacing two of the characteristic functions in each entry of ψ_i by h_0 and h_1 where $h_0(t) = \frac{t}{2}$ and $h_1(t) = \frac{t+1}{2}$ for $t \in [0,1]$ and all $i \in \mathbb{N}$. It follows from [2] that the C^*-algebra $A = \varinjlim (A_i, \phi_i)$ is simple. Note that $K_0(\phi_i) = K_0(\psi_i) = s_i$. Since

$$\|\mathrm{Aff}\mathrm{T}(\phi_i) - \mathrm{Aff}\,\mathrm{T}(\psi_i)\| < 2\mathrm{mult}(s_i)^{-1} \leqq 2^{1-i}$$

the sequence $(\mathrm{Aff}\mathrm{T}(\psi_{\infty k} \circ \phi_{ki})(x))_{k=i}^{\infty}$ is Cauchy for every $x \in \mathrm{Aff}\,\mathrm{T}(A_i)$, $i \in \mathbb{N}$ – let $\gamma_i(x)$ denote the limit. Then $\gamma_i : \mathrm{Aff}\,\mathrm{T}(A_i) \to \mathrm{Aff}\,\mathrm{T}(B)$ is a homomorphism with $\gamma_{i+1} \circ \mathrm{Aff}\,\mathrm{T}(\phi_i) = \gamma_i$ for all $i \in \mathbb{N}$. There is an isomorphism $\gamma : \mathrm{Aff}\,\mathrm{T}(A) \to \mathrm{Aff}\,\mathrm{T}(B)$ such that $\gamma \circ \mathrm{Aff}\,\mathrm{T}(\phi_{\infty i}) = \gamma_i$ for all $i \in \mathbb{N}$. Let $\tau \in T(B)$ and $p \in A_i$ be a projection.

Then

$$\begin{aligned} r_A \circ S(\gamma)(\tau)[\phi_{\infty i}(p)] &= S(\gamma)(\tau)(\phi_{\infty i}(p)) \\ &= \gamma \circ \mathrm{AffT}\,(\phi_{\infty i})(p)(\tau) \\ &= \gamma_i(p)(\tau) \\ &= \lim_{k\to\infty} \mathrm{AffT}\,(\psi_{\infty k} \circ \phi_{ki})(p)(\tau) \\ &= \lim_{k\to\infty} \tau(\psi_{\infty k} \circ \phi_{ki}(p)) \\ &= \tau(\psi_{\infty i}(p)) \\ &= r_B(\tau)[\psi_{\infty i}(p)]. \end{aligned}$$

We conclude that $r_A \circ S(\gamma) \circ S(\alpha) = f$.

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