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ISOMETRIES OF BANACH ALGEBRAS SATISFYING THE VON NEUMANN INEQUALITY

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Introduction.

A famous classical result of R. Kadison [K] says that every isometry of one C^*-algebra onto another is given by a Jordan isomorphism followed by a unitary multiplication. This result is generalized in the recent works [AS], [MT1] and [MT2] on isometries of certain non self-adjoint algebras of operators on Hilbert space. In the context of these works it is even possible to describe explicitly the Jordan isomorphisms. In this paper we generalize Kadison's theorem even further to the context of Banach algebras satisfying the von Neumann inequality.

In what follows Z denotes a complex, unital Banach algebra (i.e. Z has a unit e and $\|e\| = 1$) with an open unit ball D. The term "Banach algebra" will always mean "complex, unital Banach algebra". We say that the *von Neumann inequality* hold in Z if

$$\|f(z)\| \leqq \|f\|_\infty := \max\{|f(\lambda)|; |\lambda| = 1\}$$

for every $z \in D$ and every polynomial f. It is well known that the von Neumann inequality holds in $B(H)$, the algebra of all bounded operators on the Hilbert space H, see [N], [FSN, Chapter I.8]. Therefore, it holds in every subalgebra of a C^*-algebra. The following generalization of Kadison's theorem is our main result.

THEOREM. *Let Z be a Banach algebra satisfying the von Neumann inequality, and let φ be a surjective isometry of Z. Then there exists a Jordan automorphism ψ of Z and a unitary element u of Z so that $\varphi(z) = u\psi(z)$ for all $z \in Z$.*

The notions of Jordan automorphisms and unitary elements in general Banach algebras will be explained latter.

Our approach is similar to that of [AS] and uses the notion of the partial Jordan triple product and its connection to the holomorphic structure. We show

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that under the validity of the von Neumann inequality, the symmetric part Z_s of Z (in the sense of holomorphy) is the space $H + iH$ of decomposable elements (where H is the space of the Hermitean elements of Z), and that it is a C^*-algebra. Moreover, the abstract partial Jordan triple product (defined in terms of the holomorphy) coincides with the algebraic one: $\{x, y, z\} = (xy^{\#}z + zy^{\#}x)/2$ for all $x, z \in Z$ and $y \in Z_s$. A key result (Proposition 2.2) is the fact that the validity of the von Neumann inequality in Z is equivalent to the complete integrability of the vector field $h(z) = e - z^2$ in D. We also use the well-known fact that isometries preserve the partial Jordan triple product in general complex Banach spaces.

The paper is organized as follows. Section 1 below contains the background material on Hermitean elements of Banach algebras and on the partial Jordan triple product in Banach spaces and its connection to the holomorphic structure. Section 2 contains the connection between the von Neumann inequality and the vector field $h(z) = e - z^2$, as well as the proof that $Z_s = H + iH$. Section 3 contains the result on isometries generalizing Kadison's theorem mentioned above, as well as its Lie algebraic counterpart which describe the bounded Hermitean operators. Finally, in section 4 we collect some known facts and raise some questions on Banach algebras satisfying the von Neumann inequality.

1. Background.

Hermitean elements of a Banach algebra.

We begin with a quick review of some basic facts concerning Hermitean elements in Banach algebras which will be needed later. See [BD3], [BD1], [BD2] and [Do] for more details.

Let Z be a Banach algebra with a dual space Z^*. Let $S := \{\varphi \in Z^*; \varphi(e) = \|\varphi\| = 1\}$ be the set of *states*. The *numerical range* of $z \in Z$ is $V(z) := \{\varphi(z); \varphi \in S\}$. An element $z \in Z$ is *Hermitean* if $V(z) \subset \mathsf{R}$. It is known that z is Hermitean if and only if $\|\exp(itz)\| = 1$ for every $t \in \mathsf{R}$. Let H denote the set of all Hermitean elements in Z. H is a real-linear, closed subspace of Z containing e, with the property that $a, b \in H$ implies $i(ab - ba) \in H$. The subspace $H + iH$ of *decomposable* elements is also closed, and $H \cap iH = \{0\}$. The *involution* on $H + iH$ is defined by $(a + ib)^{\#} := a - ib, a, b \in H$. It is continuous, but need not be isometric. In general not much more can be said about H and $H + iH$; there are many examples where $H = \mathsf{R}e$ (this hold for instance in the disk algebra and in H^∞), and in general, H and $H + iH$ are not algebras. A fundamental result concerning $H + iH$ is the *Vidav-Palmer Theorem*, see [BD1, Chap. 2, Th. 9, p. 65], [BD3] and the references therein.

1.1 THEOREM. *Let Z be a Banach algebra so that $Z = H + iH$, namely every element in Z is decomposable. Then Z is a C^*-algebra with respect to the involution $\#$ and the given algebraic operations and norm.*

The partial Jordan triple product and the symmetric part of Banach space.

We survey here the notions of the *partial Jordan triple product* and the *symmetric part* of a general complex Banach space. For more details see the survey article [A] or the original papers [Vi] and [U1]. The books [L] and [U2] are general references to Jordan triples and the associated bounded symmetric domains.

Let Z be a complex Banach space with an open unit ball D. $\mathrm{Aut}(D)$ denotes the real Banach Lie group of all biholomorphic automorphisms of D. Its Lie algebra, $\mathrm{aut}(D)$, is a real Banach Lie algebra, and it is identified with the completely integrable holomorphic vector fields on D. Namely, a holomorphic function $h := D \to Z$ belongs to $\mathrm{aut}(D)$ if and only if there exists a one-parameter subgroup $\{\varphi_t\}_{t\in \mathrm{R}}$ of $\mathrm{Aut}(D)$ so that $\frac{\partial}{\partial t}\varphi_t(z) = h(\varphi_t(z))$ for every $z \in D$ and $t \in \mathrm{R}$. One denotes $\varphi_t = \exp(th)$. The Lie brackets in $\mathrm{aut}(D)$ are given by $[h,k](z) = h'(z)k(z) - k'(z)h(z)$, $z \in D$. The *symmetric part* of Z is

$$Z_s := \mathrm{aut}(D)(0) = \{h(0); h \in \mathrm{aut}(D)\}.$$

It is known that Z_s is closed *complex* linear subspace of Z whose open unit ball is $D_s := \mathrm{Aut}(D)(0) = D \cap Z_s$, the *symmetric part of* D. $\mathrm{Aut}(D)$ admits a *Cartan decomposition* $\mathrm{aut}(D) = k \oplus p$, where

$$k = \mathrm{aut}(D) \cap B(Z) = \{h \in \mathrm{aut}(D); h(0) = 0\}$$

is the subspace of *skew-Hermitean* bounded operators on Z, and

$$p = \{h \in \mathrm{aut}(D); h'(0) = 0\}$$

is a subspace of even polynomials of degree $\leqq 2$. Precisely, $p = \{h_a; a \in Z_s\}$, where $h_a(z) := a - q_a(z)$, and q_a is a continuous, homogeneous polynomial of degree 2. q_a extends to a continuous, symmetric bilinear form by polarization: $q_a(z,w) = (q_a(z+w) - q_a(z) - q_a(w))/2$. The partial *Jordan triple product* is the map $\{\ldots\}: Z \times Z_s \times Z \to Z$, defined by $\{z,a,w\} := q_a(z,w)$. Z is a JB*-triple if $Z_s = Z$, i.e. the Jordan triple product is defined everywhere. This is the case precisely when $D_s = D$, i.e. when $\mathrm{Aut}(D)$ acts transitively on D, and D is a *bounded symmetric domain.* For example, every C^*-algebra is a JB*-triple with respect to the triple product $\{z,a,w\} := (za^*w + wa^*z)/2$.

Let $s_0(z) := -z$ be the symmetry at 0. The corresponding *Cartan involution* on $\mathrm{aut}(D)$, $\theta(h) := s_0 h s_0$, is a Lie-automorphism and $\vartheta^2 = I$. It is easy to check that $k = \{h \in \mathrm{aut}(D); \vartheta(h) = h\}$ and $p = \{h \in \mathrm{aut}(D); \vartheta(h) = -h\}$. From these facts it follows that $[p,p] \subset k$, $[k,p] \subset p$, and $[k,k] \subset k$. Explicitly, for $a, b \in Z_s$ and $u \in k$,

$$[h_a, h_b](z) = 2\{a,b,z\} - 2\{b,a,z\}, [u, h_a] = h_{u(a)}.$$

In particular, for any $a \in Z_s$ the operator $D(a,a) := \{a, a, \cdot\}$ is Hermitean. Notice also that the previous formula implies that $D(a,a)b = \{a,a,b\} \in Z_s$. It follows that *$Z_s$ is closed under the triple product and it is therefore a* JB*-*triple.* Basic facts in JB*-triples yield that *for every $a \in Z_s$, the restriction of $D(a,a)$ to Z_s has a positive spectrum* and the following identity (called the "*C*-axiom*") holds: $\|D(a,a)\|_{B(Z_s)} = \|a\|^2$.

The group of *linear isometries* of Z is identified naturally with

$$K := \{\varphi \in \operatorname{Aut}(D); \varphi(0) = 0\} = \operatorname{Aut}(D) \cap \operatorname{GL}(Z).$$

The Lie algebra of K is clearly k. The Isometries are automorphisms and the skew-Hermitean operators are derivations of the partial Jordan triple product. Precisely,

1.2. PROPOSITION. (i) *Let Z_1, Z_2 be complex Banach spaces with open unit balls D_1, D_2 respectively, and let φ be an isometry of Z_1 onto Z_2. Let* $\operatorname{aut}(D_j) = k_j \oplus p_j$, $(j = 1,2)$, *be the Cartan decompositions. Then* $\varphi k_1 \varphi^{-1} = k_2$, $\varphi p_1 \varphi^{-1} = p_2$, and $\varphi h_a \varphi^{-1} = h_{\varphi(a)}$ *for every* $a \in (Z_1)_s$. *In particular* $\varphi((Z_1)_s) = (Z_2)_s$, *and*

$$\varphi\{z,a,w\} = \{\varphi(z), \varphi(a), \varphi(w)\}, \quad a \in (Z_1)_s, \quad z, w \in Z_1.$$

(ii) *Let Z be a complex Banach space with an open unit ball D and let $h \in k$. Then $h(Z_s) \subseteq Z_s$, and for every $a \in Z_s$ and $z, w \in Z$,*

$$h\{z,a,w\} = \{h(z), a, w\} + \{z, h(a), w\} + \{z, a, h(w)\}.$$

PROOF. (i) The fact that $\varphi(\operatorname{aut}(D_1))\varphi^{-1} = \operatorname{aut}(D_2)$ is obvious. $\varphi k_1 \varphi^{-1} = k_2$ follows from this and from $\varphi(0) = 0$. Let $a \in (Z_1)_s$, and let $h := \varphi h_a \varphi^{-1}$. Then $h \in \operatorname{aut}(D_2)$, $h'(0) = 0$ and $h(0) = \varphi(a)$. Thus, $\varphi(a) \in (Z_2)_s$ and $h = h_{\varphi(a)} \in p_2$. This completes the proof of the first three identities as well as the inclusion $\varphi((Z_1)_s) \subseteq (Z_2)_s$. The reverse inclusion follows by using φ^{-1} instead of φ; this yields the fourth identity. Applying both sides of the third identity on the element $\varphi(z)$, we get $\varphi\{z,a,z\} = \{\varphi(z), \varphi(a), \varphi(z)\}$. By polarization, φ preserves the partial triple product.

(ii) Let $\varphi_t = \exp(th)$, $t \in \mathbb{R}$. Then $\varphi_t \in K$ and by (i) $\varphi_t((Z_1)_s) = (Z_2)_s$. Differentiating with respect to t at 0, we get $h((Z_1)_s) \subseteq (Z_2)_s$. Let $a \in (Z_1)_s$, and let $z, w \in Z_1$. Then by the last formula in (i) $\varphi_t\{z,a,w\} = \{\varphi_t(z), \varphi_t(a), \varphi_t(w)\}$. Differentiating this at $t = 0$, we get the desired result.

Linear operators h on Z which satisfy the identity in (ii) are called derivations of the partial Jordan triple product, or *triple derivations* for short. Notice that Proposition 1.2 implies that for any $a \in Z_s$ the operator $iD(a,a)$ is a triple derivation.

The completely integrable holomorphic vector fields are characterized by tangency to the unit sphere.

1.3. PROPOSITION. *Let Z be a complex Banach space with an open unit ball D. Let $h: Z \to Z$ be a holomorphic polynomial. Then the following conditions are equivalent:*

(i) $h_{|D} \in \mathrm{aut}(D)$;

(ii) *h is tangent to the unit sphere ∂D. Namely, if $z \in Z$, $f \in Z^*$ are so that* $\|z\| = \|f\| = f(z) = 1$ *then* $\mathrm{Re}(f(h(z))) = 0$.

See [Ka] and [U2, Lemma 4.4] for a proof.

2. The symmetric part of Banach algebras which satisfy the von Neumann inequality.

Let Z be a Banach algebra with a unit e, an open unit ball D and a symmetric part $Z_s = \mathrm{aut}(D)(0)$. Let H denote the space of all Hermitean elements in Z.

2.1. PROPOSITION. (i) $Z_s \subseteq H + iH$. (ii) *If $a \in Z_s$, then* $a^{\#} = \{e, a, e\}$.

PROOF. Let $a \in Z_s$, and let $h_a := a - q_a \in p \subset \mathrm{aut}(D)$ be the corresponding vector field on D. Set $b := q_a(e) = \{e, a, e\}$. By Proposition 1.3 we have for every state $\varphi \in S, 0 = \mathrm{Re}(\varphi(h_a(e)) = \mathrm{Re}(\varphi(a - b))$. It follows that $a - b \in iH$. Since Z_s is C-linear, we get $ia \in Z_s$ and $q_{ia}(e) = ib$. Thus by the same arguments, $ia - q_{ia}(e) = i(a + b) \in iH$. Set $a_1 := (a + b)/2$ and $a_2 := (a - b)/2i$. Then $a_1, a_2 \in H$ and $a = a_1 + ia_2 \in H + iH$. Also, $b = a^{\#} = a_1 - ia_2$, and so the restriction of the involution of $H + iH$ to Z_s is given by $a^{\#} = \{e, a, e\}$.

2.2. PROPOSITION. (i) *The integral curves of the vector field $h(z) = h_e(z) := e - z^2$ on D are given by*

$$\varphi_t(z) = \exp(th)(z) := (r(t) + z)(e + r(t)z)^{-1}, \quad t \in \mathrm{R}, \quad r(t) := \tanh(t);$$

(ii) *h is completely integrable in D if and only if Z satisfies the von Neumann inequality.*

PROOF. For any $z \in Z$ let J_z be the maximal open interval containing 0 in which the initial value problem: $\partial\varphi_t(z)/\partial t = h(\varphi_t(z))$, $t \in J_z$; $\varphi_0(z) = z$; has a solution $\varphi_t(z) \in D$. Let I_z be the maximal open interval containing 0 of those $t \in \mathrm{R}$ for which $(e + r(t)z)^{-1}$ exists and $\sigma_t := (r(t) + z)(e + r(t)z)^{-1}$ belongs to D. The meaning of (i) is that $J_z = I_z$ and $\varphi_t(z) = \sigma_t(z)$, $t \in J_z$. Indeed, since $(\partial/\partial t)r(t) = 1/\cosh^2(t) = 1 - r(t)^2$, and $(\partial/\partial r)(e + rz)^{-1} = -(e + rz)^{-2}z$, we get by a direct differentiation $(\partial/\partial t)\sigma_t(z) = e - \sigma_t(z)^2$. This establishes (i). To prove (ii), we observe first that the completeness of h is equivalent to the fact that the Mobius transformations

$$\psi_r(z) := (r + z)(e + rz)^{-1}, \quad -1 < r < 1, \quad z \in D,$$

map D into itself. Let $\mathrm{D} := \{\lambda \in \mathrm{C}; |\lambda| < 1\}$. Since D is circular and $\mathrm{Aut}(\mathrm{D})$ is generated by the ψ_r, $-1 < r < 1$, and the rotations $\rho_t(z) := e^{it}z$, we get that every $\psi \in \mathrm{Aut}(\mathrm{D})$ maps D into itself. The submultiplicativity of the norm of Z implies that D is a multiplicative semigroup. Thus, the finite products of members of $\mathrm{Aut}(\mathrm{D})$ (namely, the finite Blaschke products) map D into itself. By [F], the convex combinations of finite Blaschke products are norm-dense in the closed unit ball of the disk algebra A. It follows that for every $f \in A$ with $\|f\|_\infty \leqq 1$ and $z \in D$ we have $\|f(z)\| \leqq 1$. This is equivalent to the validity of the von Neumann inequality in Z. Conversely, if the von Neumann inequality holds in Z then ψ_r maps D into itself for all $-1 < r < 1$. Since $\psi_r = \exp(th)$, $r = \tanh(t)$, we see that h is completely integrable in D.

2.3. Remark. Proposition 2.2 yields the von Neumann inequality in C^*-algebras. Indeed, it is well known that if Z is a C^*-algebra, then for every $a \in D$ the Potapov-Mobius transformation

$$\Phi_a(z) := (e - aa^*)^{-1/2}(a + z)(e + a^*z)^{-1}(e - a^*a)^{1/2}$$

belongs to $\mathrm{Aut}(D)$, see [IS]. Applying this with $a = re$, $-1 < r < 1$, we see that $\psi_r \in \mathrm{Aut}(\mathrm{D})$ becomes a member of $\mathrm{Aut}(D)$ in the natural way. By the proof of Proposition 2.2, this implies the validity of the von Neumann inequality in Z. The original proof of the von Neumann inequality (see [N] and [RSN, Section 135]) uses also similar analytic tools. Another proof is given in [FSN], and is based on much heavier tools (unitary dilations of contractions and the spectral theorem for unitary operators).

2.4. Proposition. (i) *If* $a \in H + iH$ *and* $b \in Z_s$ *then* $ab, ba \in Z_s$. *In particular,* Z_s *is an algebra,* $H + iH$ *is a module over* Z_s, *and* $(H + iH)Z_s = Z_s(H + iH) = Z_s$.
(ii) $H + iH = Z_s$ *if and only if* $e \in Z_s$.

Proof. For $a \in Z$ consider the multiplication operators $L_a(z) := az$ and $R_a(z) := za$. Then

$$\exp(itL_a) = L_{\exp(ita)} \quad \text{and} \quad \exp(itR_a) = R_{\exp(ita)}, \quad t \in \mathrm{R}.$$

Since $\|L_z\| = \|R_z\| = \|z\|$, it follows that $a \in H \Leftrightarrow L_a \in H(B(Z)) \Leftrightarrow R_a \in H(B(Z))$. Here $H(B(Z))$ denotes the space of Hermitean elements of the Banach algebra $B(Z)$ of all bounded operators on Z. Let $a \in H$ and $b \in Z_s$. Then, by Proposition 1.3, $\exp(itL_a)(b) = \exp(ita) \cdot b$, and $\exp(itR_a)(b) = b \cdot \exp(ita)$ belong to Z_s for all $t \in \mathrm{R}$. Differentiating with respect to t at 0, we get $iab, iba \in Z_s$. Since Z_s is C-linear, this implies (i). Next, if $H + iH = Z_s$ then $e \in H \subseteq iH = Z_s$. Conversely, assume that $e \in Z_s$ and let $a \in H$. By (i) with $b = e$, we get $a \in Z_s$. Thus $H \subseteq Z_s$. Since Z_s is C-linear, this implies $H + iH \subseteq Z_s$. Using Proposition 2.1 we get $H + iH = Z_s$.

2.5. REMARK. If $e \in Z_s$, and in particular – if the von Neumann inequality holds in Z (Proposition 2.2), then $Z_s = H + iH$ is a Banach algebra (Proposition 2.3) in which every element is decomposable. By Theorem 1.1, Z_s is a C^*-algebra with respect to the involution $\#$ and the given algebraic operations and norm. We do not know whether $e \in Z_s$ by itself implies the von Neumann inequality in Z.

The following theorem generalizes the result discussed in Remark 2.5. We prefer to avoid Theorem 1.1 and to give an almost self contained proof, in order to illustrate the power of the Jordan theoretic techniques.

2.6. THEOREM. *Let Z be a Banach algebra with a unit e, satisfying the von Neumann inequality. Then $Z_s = H + iH$ is a C^*-algebra with respect to the given multiplication, norm, and the involution $(a_1 + ia_2)^\# = a_1 - ia_2, a_1, a_2 \in H$. Moreover, the partial Jordan triple product $\{\dots\}: Z \times Z_s \times Z \to Z$ constructed via the holomorphy coincides with the algebraic partial triple product, namely:*

$$\{x, y, z\} = (xy^\# z + zy^\# x)/2, \quad x, z \in Z, \quad y \in Z_s.$$

PROOF. By Proposition 2.4, $Z_s = H + iH$ is a closed subalgebra of Z with a unit e and involution $\#$ (which at the moment is known only to be an anti-linear homeomorphism of Z_s). By Proposition 2.2 we have $\{z, e, z\} = z^2$ for all $z \in Z$. Polarizing, we get $\{z, e, w\} = (zw + wz)/2, z, w \in Z$. Let $a \in H$, then as in the proof of Proposition 2.3, $\exp(itL_a) = L_{\exp(ita)}$, $t \in \mathrm{R}$, belong to the group $K = \mathrm{Aut}(D) \cap \mathrm{GL}(Z)$ of linear isometries of Z. By Proposition 1.2 the members of K are automorphisms of the partial triple product. Hence, for any $z \in Z$ and $t \in \mathrm{R}$,

$$\begin{aligned}\exp(ita)z^2 &= \exp(itL_a)\{z, e, z\} \\ &= \{\exp(itL_a)(z), exp(itL_a)(e), \exp(itL_a)(z)\} \\ &= \{\exp(ita)z, \exp(ita), \exp(ita)z\}.\end{aligned}$$

Differentiating with respect to t at 0, we get

$$az^2 = 2\{az, e, z\} - \{z, a, z\} = (az)z + z(az) - \{z, a, z\} = az^2 + zaz - \{z, a, z\}.$$

Thus, $\{z, a, z\} = zaz$. It follows that for $a = a_1 + ia_2 \in Z_s$ with $a_1, a_2 \in H$ and $z \in Z$,

$$\{z, a, z\} = \{z, a_1, z\} - i\{z, a_2, z\} = za_1z - iza_2z = za^\# z.$$

Polarizing, we get $\{z, a, w\} = (za^\# w + wa^\# z)/2$ for every $z, w \in Z$ and $a \in Z_s$.

Next, we show that $(ab)^\# = b^\# a^\#$ for all $a, b \in Z_s$. Since $Z_s = H + iH$, it is certainly enough to show that $(ab)^\# = ba$ for $a, b \in H$. By the above arguments we have for all $t \in \mathrm{R}$

$$\exp(ita)b = \{\exp(ita), \exp(ita)b, \exp(ita)\}.$$

Differentiating with respect to t at 0, we get $iab = i(ab + ba) - i\{e, ab, e\}$. However, by Proposition 2.1, $\{e, z, e\} = z^{\#}$ for all $z \in Z_s$. Applying this with $z = ab$ (and using the fact that Z_s is an algebra), we get $(ab)^{\#} = ba$ as desired.

The C^*-axiom $\|a^{\#} a\| = \|a\|^2$ in Z_s, follows from the C^*-axiom for JB*-triples, see Section 1 above. Indeed, in our case $D(a, a)z = (aa^{\#} z + za^{\#} a)/2, z \in Z$. Hence,

$$\|a\|^2 = \|D(a, a)\|_{B(Z_s)} \leqq (\|a^{\#} a\| + \|aa^{\#}\|)/2 \leqq \|a\| \, \|a^{\#}\|,$$

and thus, $\|a\| \leqq \|a^{\#}\|$ for all $a \in Z_s$. Replacing a by $a^{\#}$ we get $\|a\| = \|a^{\#}\|, a \in Z_s$. It follows that equality holds in the above inequality, and so $\|a^{\#} a\| = \|aa^{\#}\| = \|a\|^2, a \in Z_s$. This completes the proof.

2.7. Remark. We have a direct argument yielding $\|z^{\#}\| = \|z\|$ for every $z \in Z_s$. Indeed, let $\alpha_n \in \mathrm{R}$ be so that $\tanh(t) = \sum_{n=0}^{\infty} \alpha_n t^{2n+1}$, $t \in \mathrm{R}$, and let $a \in Z_s$. It is easy to check (from the initial value problem defining $\exp(th_a)$) that $\exp(th_a)(0) = \tanh(ta) := \sum_{n=0}^{\infty} \alpha_n t^{2n+1} a(a^{\#} a)^n$, $t \in \mathrm{R}$. It follows from the anti multiplicativity of $\#$ that $(\exp(th_a)(0))^{\#} = \exp(th_{a^{\#}}) = \tanh(ta^{\#}) \in D_s$. Since $D_s = \mathrm{Aut}(D)(0)$ and $K(0) = \{0\}$, we get $D_s = \exp(p)(0)$, where $\exp(p)$ is the subgroup of $\mathrm{Aut}(D)$ generated by $\{\exp(h_a); a \in Z_s\}$. It follows that $(D_s)^{\#} = D_s$, and this is equivalent to $\|z^{\#}\| = \|z\|$ for all $z \in Z_s$.

2.8. Problem. Does $e \in Z_s$ imply the von Neumann inequality in Z? In particular, does $e \in Z_s$ imply that $h_e(z) = e - z^2$, i.e. that $\{z, e, z\} = z^2$ for all $z \in Z$? Notice that, by the proof of Theorem 2.6, the last identity is equivalent to the identity $\{z, a, z\} = za^{\#} z$ for all $z \in Z$ and $a \in Z_s$.

3. The isometries and the Hermitean operators.

3.1. Theorem. *Let Z, W be unital Banach algebras which satisfy the von-Neumann inequality, and let $\varphi: Z \to W$ be a surjective isometry. Then,*

(i) $u := \varphi(e)$ is a unitary element of W_s;

(ii) $\varphi(z) = u\psi(z), z \in Z$, where ψ is an isometric Jordan isomorphism of Z onto W. Namely, ψ is an isometry satisfying $\psi(e) = e$ and

$$\psi(ab + ba) = \psi(a)\psi(b) + \psi(b)\psi(a), \text{ for } a, b \in Z;$$

$$\psi(Z_s) = W_s \text{ and } \psi(z)^{\#} = \psi(z^{\#}) \text{ for } z \in Z_s.$$

3.2. Remarks. (i) Theorem 3.1 reduces the study of a geometrical problem (the description of the isometries of Banach algebras satisfying the von Neumann inequality) to that of an algebraic one (namely, the description of the Jordan isomorphisms of these algebras). It is our generalization of Kadison's theorem discussed in the introduction.

(ii) The partial Jordan triple product is expressed in terms of the (binary) Jordan product $x \circ y := (xy + yx)/2$ and the involution $\#$ on Z_s:

$$\{x, y, z\} = x \circ (y^{\#} \circ z) + z \circ (y^{\#} \circ x) - (x \circ z) \circ y^{\#}, \quad x, z \in Z, \quad y \in Z_s.$$

It follows that a linear map which preserves the Jordan product and the partial involution preserves also the partial Jordan triple product.

(iii) We do not know whether the converse of Theorem 3.1 is true, namely whether a Jordan isomorphism of Banach algebras satisfying the von Neumann inequality must be an isometry. This is true in C^*-algebras. More generally, an automorphism of a JB*-triple must be an isometry.

We need the characterization of tripotents as partial isometries.

3.3. PROPOSITION. *Let Z be as Theorem* 3.1 *and let $u \in Z_s$.*

(i) *u is a tripotent (namely, $\{u, u, u\} = u$) if and only if it is a partial isometry (namely, $u^{\#}u$ is an idempotent);*

(ii) *u is a unitary tripotent (namely, $\{u, u, z\} = z$ for every $z \in Z$) if and only if $u^{\#}u = e = uu^{\#}$*, i.e. *u is a unitary element of the C^*-algebra Z_s.*

PROOF. Part (i) is well known (see, for instance, [H]), If u is a unitary element of Z_s, then for every $z \in Z$, $2\{u, u, z\} = uu^{\#}z + zu^{\#}u = 2z$. On the other hand, if $\{u, u, z\} = z$ for every $z \in Z$, then $e = \{u, u, e\} = (u^{\#}u + uu^{\#})/2$ and u is a partial isometry. Since e is an extreme point of the unit ball of Z_s (this holds in any unital Banach algebra, see [BD1, Chap. 1, Th. 5, p. 38]), we get $u^{\#}u = uu^{\#} = e$.

PROOF OF THEOREM 3.1. By Proposition 1.2, $\varphi(Z_s) = W_s$ and φ preserves the partial triple product. Using Theorem 2.6 we get

$$\varphi(xy^{\#}z + zy^{\#}x) = \varphi(x)\varphi(y)^{\#}\varphi(z) + \varphi(z)\varphi(y)^{\#}\varphi(x)$$

for all $x, z \in Z$ and $y \in Z_s$. Set $u = \varphi(e)$ and notice that for every $z \in Z$, $\varphi(z) = \varphi(\{e, e, z\}) = \{\varphi(e), \varphi(e), \varphi(z)\} = \{u, u, \varphi(z)\}$. It follows from Proposition 3.3 that u is a unitary element of W_s. Let $\psi(z) := u^{\#}\varphi(z)$. Then ψ is an isometry of Z onto W and $\psi(e) = e$. By proposition 1.2, ψ preserves the partial triple product. Hence, for every $z \in Z_s$

$$\psi(z^{\#}) = \psi(\{e, z, e\}) = \{\psi(e), \psi(z), \psi(e)\} = \{e, \psi(z), e\} = \psi(z)^{\#}.$$

Thus ψ is self adjoint. Finally, for every $z \in Z$,

$$\psi(z^2) = \psi(\{z, e, z\}) = \{\psi(z), \psi(e), \psi(z)\} = \psi(z)^2$$

Polarizing this identity we see that ψ preserves the Jordan product. This completes the proof since $\varphi(z) = u\psi(z)$.

The next result deals with Hermitean operators on Z and it is the Lie-algebraic analog of Theorem 3.1. By Proposition 1.2 we know that every Hermitean operator $T: Z \to Z$ satisfies $T(Z_s) \subseteq Z_s$. Moreover, iT is a derivation of the partial triple product, namely the identity in Proposition 1.2 (ii) holds with

$h = iT$. An operator $A\colon Z \to Z$ is a *derivation of the (binary) Jordan product* $x \circ y := (xy + yx)/2$ if $A(x \circ y) = Ax \circ y + x \circ Ay$ for all $x, y \in Z$. It is obvious that in this case $A(e) = 0$. An operator $T\colon Z \to Z$ is *self-adjoint* if $T(H) \subseteq H$. This is equivalent to $T(H + iH) \subseteq H + iH$ and $(T(z))^{\#} = (T(z^{\#})$ for every $z \in H + iH$. T is *skew-adjoint* if iT is self-adjoint, and this is equivalent to $T(H + iH) \subseteq H + iH$ and $(T(z))^{\#} = -T(z^{\#})$ for every $z \in Z_s$.

3.4. PROPOSITION. *Let Z be a unital Banach algebra satisfying the von Neumann inequality. Then every Hermitean operator $T\colon Z \to Z$ has a unique decomposition $T = A + L_a$, where $a := T(e) \in H$ and A is a skew-adjoint derivation of the Jordan product.*

PROOF. Set $a := T(e)$. Then $a \in Z_s = H + iH$ and

$$a = T(e) = T\{e, e, e\} = 2\{a, e, e\} - \{e, a, e\} = 2a - a^{\#}.$$

Thus $a = a^{\#}$, and so $a \in H$. Set $A := T - L_a$. Then A is Hermitean and $A(e) = 0$. Since $Z_s = H + iH$, we get $A(H + iH) \subseteq H + iH$ and for $z \in Z_s$,

$$A(z^{\#}) = A\{e, z, e\} = 2\{A(e), z, e\} - \{e, Az, e\} = -(Az)^{\#}.$$

Thus A is skew adjoint. Next, for any $z \in Z$

$$A(z^2) = A\{z, e, z\} = 2\{Az, e, z\} - \{z, A(e), z\} = (Az)z + z(Az).$$

Polarizing, we get $A(z \circ w) = (Az) \circ w + z \circ (Aw)$ for every $z, w \in Z$. Thus A is also a derivation of the Jordan product. The uniqueness of the decomposition $T = A + L_a$ is obvious.

3.5. REMARKS. (i) Proposition 3.4 reduces a geometrical problem (the description of the Hermitean operators) to an algebraic one (the description of the skew-adjoint derivations of the Jordan product).

(ii) In the situation described in Proposition 3.3, T admits also the decomposition $T = B + R_a$, where $a = T(e) \in H$ and B is a skew-adjoint derivation of the Jordan product. Of course, $B = A + L_a - R_a$.

(iii) Let $T\colon Z \to Z$ be of one of the forms $T = A + L_a$ or $T = B + R_a$, where $a \in H$ and A, B are skew-adjoint derivations of the (binary) Jordan product. Then iT is a derivation of the partial Jordan triple product. This follows from Remark 3.2 (ii).

(iv) We do not know whether a bounded operator $T\colon Z \to Z$ for which iT is a derivation of the partial Jordan triple product must be Hermitean. It is easy to see that in this case $\exp(i\lambda T)$ is an automorphism of the triple product for all $\lambda \in \mathbb{R}$. But we do not know whether $\exp(i\lambda T)$ is in fact an isometry. See Remark 3.2 (iii).

4. Banach algebras which satisfy the von Neumann inequality.

The results obtained above lead naturally to the problem of the description of the class of Banach algebras satisfying the von Neumann inequality. This problem is interesting from many point of views, but it is difficult and far from being solved. We collect bellow some known facts relevant to this problem, and raise some questions.

The first point we would like to make is that *the von Neumann inequality is an inequality in the category of Jordan Banach algebras.* A *Jordan algebra* is a commutative algebra with a product $x \circ y$ (called a Jordan product), in which the associative law is replaced by the weaker law (the *Jordan algebra identity*): $x^2 \circ (x \circ y) = x \circ (x^2 \circ y)$, where $x^2 := x \circ x$.

It is easy to see that for every element x the powers x^n (which are defined inductively via $x^n = x \circ x^{n-1}$) satisfy $x^n \circ x^m = x^k \circ x^l$ whenever $n + m = k + l$. Thus the subalgebra generated by x is associative. A *Jordan Banach algebra* is a Jordan algebra which is also a Banach space, so that $\|x \circ y\| \leqq \|x\| \, \|y\|$ for all x and y. We will assume that the field of scalars is C. Clearly, it is possible to consider the von Neumann inequality in the category of Jordan Banach algebras, and it is interesting to characterize those Jordan Banach algebras in which this inequality holds. Notice that the von Neumann inequality has a local nature, namely it is a statement on all the singly generated subalgebras of Z.

For any Banach algebra Z, let $Z^J := \langle Z, +, \circ, \|\cdot\| \rangle$ denote the associated Jordan Banach algebra with the Jordan product $x \circ y := (xy + yx)/2$ and the same addition and norm. Notice that for every $z \in Z$, the powers z^n are the same in Z and in Z^J, thus the singly generated subalgebras are the same for Z and Z^J. Therefore we get,

4.1. COROLLARY. *The von Neumann inequality holds in Z if and only if it holds in Z^J.*

We remark that not all Jordan Banch algebras hae the form Z^J for some Banach algebra Z.

A JB*-*algebra* is a unital, complex Jordan Banach algebra Z with an involution $z \mapsto z^*$ (i.e. an anti-linear, multiplicative map of period 2), so that $\|\{z, z^*, z\}\| = \|z\|^3$ for every $z \in Z$, where the triple product is defined via the binary product and the involution by

$$\{x, y, z\} := x \circ (y^* \circ z) + z \circ (y^* \circ x) - y^* \circ (x \circ y).$$

In this case, $X := \{x \in Z;\, x^* = x\}$ is a real JB-*algebra* (i.e. a real Banach Jordan algebra satisfying $\|x\|^2 \leqq \|x^2 + y^2\|$ for all x and y) and $Z = X \otimes \mathbf{C}$. Conversely, the complexification of every real, unital JB-algebra has a unique norm with respect to which it is a JB*-algebra. An equivalent description is that a JB*-algebra is a JB*-triple having a unitary tripotent e, with the binary product $z \circ w := \{z, e, w\}$ and the involution $z^* := \{e, z, e\}$. Clearly, every C^*-algebra is a JB*-algebra.

4.2. PROPOSITION. *The von Neumann inequality holds in any* JB*-*algebra Z.*

Indeed, the vector field $h_e(z) := e - \{z, e, z\} = e - z^2$ is completely integrable since Z is a JB*-triple. The rest follows from Proposition 2.2 and the fact that the singly generated subalgebras of Z are associative.

It is obvious that if the von Neumann inequality holds in the Jordan Banach algebra Z then it holds in any subalgebra. The same is true for quotient algebras, but this require some explanation. A *quotient map* in the category of Banach spaces is a linear operator $T: X \to Y$ which maps the open unit ball of X onto the open unit bal of Y. A *multiplicative quotient map* in the category of Banach algebras (or, Jordan Banach algebras) is a quotient map which preserves the product (respectively, the Jordan product). Notice that a multiplicative quotient map must preserve the unit element. Also, a multiplicative quotient map $Q: Z \to W$ in the category of Banach algebras is also a multiplicative quotient map $Q: Z^J \to W^J$ in the category of Jordan Banach algebras. The converse is false.

4.3. PROPOSITION. *Assume that the von Neumann inequality holds in the Jordan Banach algebra Z and let Q be a multiplicative quotient map of Z onto a* JB-*algebra W. Then the von Neumann inequality holds in W. The same is true in the category of Banach algebras.*

PROOF. The first statement implies the second via Proposition 4.1. To prove the first, let f be a polynomial and let $w \in W$, $\|w\| < 1$. Let $z \in Z$ be so that $Qz = w$ and $\|z\| < 1$. Since $Q(z^n) = (Q(z))^n$ for all $n \geqq 0$, we get $Q(f(z)) = f(Q(z))$. Thus,

$$\|f(w)\| = \|f(Qz)\| = \|Q(f(z))\| \leqq \|f(z)\| \leqq \|f\|_\infty.$$

4.4. COROLLARY. *The von Neumann inequality holds in every subalgebra and in very quotient algebra of a* JB*-*algebra.*

There is an extensive literature on the generalizations of von Neumann inequality in the context of *commutative* Banach algebras. These generalizations deal mainly with the multi-variable (isometric and isomorphic) analogs of the von Neumann inequality. See, for instance, [Da], [MT], [DD] and the references therein. Nevertheless, to our knowledge, there is no complete characterization of the commutative Banach algebras which satisfy the von Neumann inequality. However, there is a characterization, due to I. G. Craw (see [Da, Lemma 3.1]), of the commutative Banach algebras satisfying a multi variable generalized version of the von Neumann inequality.

4.5. THEOREM. *Let Z be a commutative Banach algebra. Then*

$$\|f(z_1, z_2, \ldots, z_n)\| \leqq \|f\|_\infty := \sup\{|f(\zeta_1, \zeta_2, \ldots, \zeta_n)|; \zeta_j \in \mathbf{C}, |\zeta_j| = 1\}$$

for every finite sequence $z_1, z_2, \ldots, z_n \in Z$ with $\|z_j\| \leqq 1$ and every polynomial $f(\zeta_1, \zeta_2, \ldots, \zeta_n)$, if and only if Z is isometrically isomorphic to a quotient of a uniform algebra.

We would like to mention also the result of T. Ando, saying that if T_1, T_2 are commuting contractions on Hilbert space and $f(\zeta_1, \zeta_2)$ is a polynomial, then $\|f(T_1, T_2)\| \leqq \|f\|_\infty$. N. Th. Varopoulos [Va1], [Va2] showed that this result cannot be extended to three or more commuting contractions.

Let $\mathscr{P}$ be the algebra of all polynomials in the non-commuting variables $x_1, x_2, \ldots, x_n, \ldots$. For every Banach algebra Z consider the norm

$$\|f\|_{\mathscr{P}Z} := \sup\{\|f(a_1, a_2, \ldots, a_n, \ldots)\|; a_j \in Z, \|a_j\| \leqq 1, j = 1, 2, \ldots\}$$

on $\mathscr{P}$. Notice that $\|\cdot\|_{\mathscr{P}Z}$ is submultiplicative.

4.6. DEFINITION. ([Di1]). A class $\mathscr{C}$ of Banach algebras is called a *variety* if there exists a submultiplicative norm $\|\cdot\|_\mathscr{C}$ on $\mathscr{P}$ so that $\mathscr{C}$ is the class of all Banach algebras Z for which $\|f\|_{\mathscr{P}Z} \leqq \|f\|_\mathscr{C}$ for every $f \in \mathscr{P}$, namely the inclusion map $\langle \mathscr{P}, \|\cdot\|_\mathscr{C} \rangle \to \langle \mathscr{P}, \|\cdot\|_{\mathscr{P}Z} \rangle$ is contractive for all $Z \in \mathscr{C}$.

4.7. THEOREM ([Di1]). *A class $\mathscr{C}$ of Banach algebras is a variety if and only if it is closed under taking closed subalgebras, quotient algebras, l_∞-direct products and isometric isomorphisms.*

4.8. COROLLARY. *The class $\mathscr{VN}$ of all Banach algebras satisfying the von Neumann inequality is a variety.*

Indeed, it is easy to see that $\mathscr{VN}$ is closed under l_∞-direct products. Thus, Corollary 4.8 follows from Proposition 4.3 and Theorem 4.7.

4.9. PROBLEM. Compute the norm $\|f\|_{\mathscr{VN}}$ on $\mathscr{P}$.

It is plain that $\|f\|_\infty := \|f\|_{\mathscr{P}\mathbf{C}} \leqq \|f\|_{\mathscr{VN}}$ for every $f \in \mathscr{P}$, but the two norms are inequivalent. Notice that it f depends on one variable then the above inequality becomes an equality.

The following characterization of subalgebras of $B(H)$ is due to Bernard (see [Be], [Di2]).

4.10. THEOREM. *A Banach algebra Z is isometrically isomorphic to a subalgebra of $B(H)$ for some Hilbert space H if and only if $\|f\|_{\mathscr{P}Z} \leqq \|f\|_{\mathscr{P}B(l_2)}$ for every $f \in \mathscr{P}$, where l_2 is the separable, infinite dimensional Hilbert space.*

Thus, the Banach subalgebras of C^*-algebras form a variety, which is clearly contained in $\mathscr{VN}$, and so $\|f\|_{\mathscr{P}B(l_2)} \geqq \|f\|_{\mathscr{VN}}$ for every $f \in P$.

Algebras of the form $B(X)$, the bounded operators on a Banach space X, do not satisfy the von Neumann inequality in general:

4.11. THEOREM. ([Fo]). *Let X be a complex Banach space. Then $B(X)$ satisfies the von Neumann inequality if and only if X is isometric to a Hilbert space.*

Let D be the open unit ball of the Banach algebra Z. We denote by $A(D, Z)$ the Banach algebra of all bounded continuous functions on the closure of D which are analytic in D, with the pointwise multiplication and norm $\|f\|_{A(D,Z)} := \sup\{\|f(z)\|;\ z \in D\}$. Let A be the disk algebra, namely $A = A(\mathsf{D}, \mathsf{C})$. For every $f \in A$ let $Jf \in \mathscr{H}(D, Z)$ (= the space of all holomorphic functions from D into Z) be defined by the Cauchy integral

$$Jf(z) = f(z) = \frac{1}{2\pi i} \int_{|\zeta|=1} f(\zeta)(\zeta - z)^{-1} d\zeta.$$

In general, Jf need not be bounded on D.

4.12. PROPOSITION. *The following conditions are equivalent:*

(i) *Z satisfies the von Neumann inequality;*

(ii) *J maps A isometrically into $A(D, Z)$.*

(iii) *J maps A into $A(D, Z)$, and J: $A \to A(D, Z)$ is a contraction.*

PROOF. By definition, (i) $\leftrightarrow$ (iii). Let R: $A(D, Z) \to A$ be the restriction map, defined by $(Rf)(\zeta)e := f(\zeta e)$, $\zeta \in \mathsf{D}$. Then $\|Rf\|_\infty \leqq \|f\|_{A(D,Z)}$ and $RJf = f$ for every $f \in A$. It follows that for every $f \in A$, $\|f\|_\infty = \|RJf\|_\infty \leqq \|Jf\|_{A(D,Z)}$. Thus (ii) $\leftrightarrow$ (iii).

4.13. PROPOSITION. (i) *The map C: $Z \to A(D, Z)$ defined by $C_z(w) = z$ is an isometric homomorphism;*

(ii) *The map Q_0: $A(D, Z) \to Z$ defined by $Q_0 f = f(0)$ is a multiplicative quotient map.*

(iii) *CQ_0 is a projection of norm* 1;

(iv) *The von Neumann inequality holds in Z if and only if it holds in $A(D, Z)$.*

PROOF. Clearly, (iv) follows from (i) and (ii) via Proposition 4.3. Parts (i), (ii) and (iii) are easily checked.

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