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THE NEVANLINNA MATRIX OF ENTIRE FUNCTIONS ASSOCIATED WITH A SHIFTED INDETERMINATE HAMBURGER MOMENT PROBLEM

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Abstract.

The shifted moment problem is introduced and its associated polynomials of the first and second kind are calculated. The Nevanlinna matrix associated with the shifted problem is found and a one-to-one correspondence between the solutions to the shifted problem and those to the original is given. It is proved that any of the four functions in the Nevanlinna matrix associated with an indeterminate moment problem belongs to a certain class of functions, introduced by Hamburger. A two-variable analogue of this result is established and is applied to the r times shifted moment problem.

0. Introduction.

We consider a normalized Hamburger moment sequence $(s_n)_{n \geqq 0}$ and the associated polynomials of the first and second kind, $(P_k)_{k \geqq 0}$ and $(Q_k)_{k \geqq 0}$, following the notation of Akhiezer, [1]. The sequence $(P_k)_{k \geqq 0}$ forms an orthonormal system with respect to the inner product given by $\langle x^n, x^m \rangle = \int_{\mathbf{R}} x^{n+m} d\mu(x)$, where μ is any measure from the set $V = \{\mu \geqq 0 \mid s_n = \int x^n d\mu(x) \, \forall n \geqq 0\}$ of solutions to the moment problem. The P_k's are uniquely determined by the additional condition that their leading coefficients are positive. The sequence $(Q_k)_{k \geqq 0}$ is given by

$$Q_k(x) = \int_{\mathbf{R}} \frac{P_k(x) - P_k(t)}{x - t} d\mu(t),$$

where μ is any measure from V.

The sequences $(a_k)_{k \geqq 0}$ and $(b_k)_{k \geqq 0}$ given by the formulas $a_k = \int x P_k(x)^2 d\mu(x)$, $b_k = \int x P_k(x) P_{k+1}(x) d\mu(x)$ define the Jacobi-matrix

$$J = \begin{pmatrix} a_0 & b_0 & 0 & 0 & \dots \\ b_0 & a_1 & b_1 & 0 & \dots \\ 0 & b_1 & a_2 & b_2 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

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associated with the moment problem. The matrix obtained from J by deleting the first row and column is in fact still a Jacobi-matrix associated with a certain normalized moment sequence $(\tilde{s}_n)_{n \geqq 0}$. This is because of Favard's theorem, cf. [1] p. 5. The sequence $(\tilde{s}_n)_{n \geqq 0}$ is called the shifted moment sequence.

The investigation of the shifted moment problem goes back to Sherman, [7], using continued fractions.

In case of an indeterminate moment sequence, the series

$$p(z) \equiv \left(\sum_{k=0}^{\infty} |P_k(z)|^2 \right)^{1/2} \tag{1}$$

$$q(z) \equiv \left(\sum_{k=0}^{\infty} |Q_k(z)|^2 \right)^{1/2} \tag{2}$$

converge uniformly on compact subsets of the complex plane.

The so-called Nevanlinna matrix of entire functions, cf. [1] p. 55,

$$\begin{pmatrix} A & C \\ B & D \end{pmatrix}$$

plays an important role in the parametrization of all solutions to the moment problem. The functions are defined as follows:

$$\begin{aligned} A(z) &= z \sum_{k=0}^{\infty} Q_k(0) Q_k(z) \\ B(z) &= -1 + z \sum_{k=0}^{\infty} Q_k(0) P_k(z) \\ C(z) &= 1 + z \sum_{k=0}^{\infty} P_k(0) Q_k(z) \\ D(z) &= z \sum_{k=0}^{\infty} P_k(0) P_k(z). \end{aligned} \tag{3}$$

The objective of this paper is to express the Nevanlinna matrix associated with the shifted moment problem in terms of the functions (3) in the Nevanlinna matrix associated with the original moment problem. The functions A and C of the original problem are given in terms of the functions B and D of the shifted problem. This is applied to the interrelation between Nevanlinna matrices of indeterminate moment problems and a certain class $\mathscr{A}$ of entire functions, introduced by Hamburger [5]. A two-variable analogue of this interrelation is established and applied to the r times shifted moment problem. A one-to-one correspondence between the set of solutions to the shifted problem and V is derived.

The above description of A and C in terms of B and D of the shifted problem can be used to obtain results on the growth of the functions (1), (2) and (3) associated with an indeterminate moment problem, see Berg & Pedersen [3].

1. The shifted moment problem and its associated polynomials.

Suppose that $(s_n)_{n\geqq 0}$ is a normalized Hamburger moment sequence with associated polynomials $(P_k)_{k\geqq 0}$ and $(Q_k)_{k\geqq 0}$. Put

$$\tilde{a}_k \equiv a_{k+1}$$
$$\tilde{b}_k \equiv b_{k+1} \tag{4}$$
$$\tilde{P}_k(x) \equiv b_0 Q_{k+1}(x), \quad k \geqq 0$$

and consider the three-term recurrence relation

$$xY_k = \tilde{b}_k Y_{k+1} + \tilde{a}_k Y_k + \tilde{b}_{k-1} Y_{k-1}, \quad k \geqq 1. \tag{5}$$

The sequence $(\tilde{P}_k)_{k\geqq 0}$ satisfies this recurrence relation with initial values $\tilde{P}_0(x) = 1, \tilde{P}_1(x) = (x - \tilde{a}_0)/\tilde{b}_0$. Since the b_k's are all strictly positive so are the $\tilde{b}_k$'s. Favard's theorem then ensures the existence of a uniquely determined normalized moment sequence $(\tilde{s}_n)_{n\geqq 0}$ having the sequence $(\tilde{P}_k)_{k\geqq 0}$ as associated polynomials of the first kind. We denote by $(\tilde{Q}_k)_{k\geqq 0}$ the associated polynomials of the second kind. These can actually be determined in terms of the P_k's and Q_k's and we have

LEMMA 1.1. *The associated polynomials of the shifted moment problem* $(\tilde{P}_k)_{k\geqq 0}$ *and* $(Q_k)_{k\geqq 0}$ *are given by*

$$\tilde{P}_k(x) = b_0 Q_{k+1}(x)$$
$$\tilde{Q}_k(x) = P_1(x)Q_{k+1}(x) - \frac{1}{b_0} P_{k+1}(x). \tag{6}$$

PROOF. The polynomials $(\tilde{Q}_k)_{k\geqq 0}$ also satisfy the recurrence relation (5) with initial values $\tilde{Q}_0(x) = 0, \tilde{Q}_1(x) = 1/b_0$. The two sequences $(\tilde{P}_k)_{k\geqq 0}$ and $(Q_k)_{k\geqq 0}$ are clearly linearly independent and so they form a basis of the space of solutions to (5).

Now, the sequence $(P_{k+1})_{k\geqq 0}$ is in fact a solution to (5) and therefore there exist real numbers $\alpha(x)$, $\beta(x)$ such that $P_{k+1}(x) = \alpha(x)\tilde{P}_k(x) + \beta(x)\tilde{Q}_k(x)$, $k = 0, 1, 2, \ldots$. In particular $P_1(x) = \alpha(x)$ and $P_2(x) = P_1(x)b_0Q_2(x) + \beta(x)/b_1$, giving us $\beta(x) = -b_0$ so that

$$\tilde{Q}_k(x) = P_1(x)Q_{k+1}(x) - \frac{1}{b_0} P_{k+1}(x).$$

REMARK. The above lemma is a special case of [2], Lemma 2.5, where the associated polynomials of the "r times" shifted moment problem are considered.

2. The Nevanlinna matrix of the shifted problem.

Using the lemma above we are now able to compute the entire functions of the shifted problem.

PROPOSITION 2.1. *The shifted moment sequence $(\tilde{s}_n)_{n\geq 0}$ is indeterminate exactly when $(s_n)_{n\geq 0}$ is indeterminate, and the entire functions in the Nevanlinna matrix associated with the shifted problem, as well as the function in* (1) *can be expressed as follows:*

$$\tag{7}\begin{aligned}\tilde{p}(z) &= b_0 q(z)\\ \tilde{A}(z) &= b_0^{-2}(D(z) - a_0(z-a_0)A(z) - (z-a_0)C(z) + a_0 B(z))\\ \tilde{B}(z) &= -C(z) - a_0 A(z)\\ \tilde{C}(z) &= (z-a_0)A(z) - B(z)\\ \tilde{D}(z) &= b_0^2 A(z).\end{aligned}$$

PROOF. Since $\tilde{P}_k(x) = b_0 Q_{k+1}(x)$, the two moment problems are indeterminate simultaneously, cf. [1] p. 16 and p. 19. Using Lemma 1.1 and the formulas (1) and (3) we get

$$\tilde{p}(z) = \left(\sum_{k=0}^{\infty} |\tilde{P}_k(z)|^2\right)^{1/2} = b_0 q(z),$$

$$\tilde{D}(z) = b_0^2 \sum_{k=0}^{\infty} Q_{k+1}(0) Q_{k+1}(z) = b_0^2 A(z),$$

$$\begin{aligned}\tilde{B}(z) &= -1 + z \sum_{k=0}^{\infty} \tilde{Q}_k(0)\tilde{P}_k(z)\\ &= -1 + z \sum_{k=0}^{\infty} (-a_0 Q_{k+1}(0) - P_{k+1}(0))Q_{k+1}(z)\\ &= -C(z) - a_0 A(z).\end{aligned}$$

The functions $\tilde{C}$ and $\tilde{A}$ may be calculated similarly.

REMARK. The function $\tilde{q}(z)$ can also be calculated. By definition $\tilde{q}(z) = (\sum_{k=0}^{\infty} |\tilde{Q}_k(z)|^2)^{1/2}$ so that

$$b_0^2\tilde{q}(z)^2 = \sum_{k=0}^{\infty} |(z-a_0)Q_{k+1}(z) - P_{k+1}(z)|^2$$

$$= |z-a_0|^2 q(z)^2 + p(z)^2 - 1 - (z-a_0)\sum_{k=0}^{\infty} Q_k(z)P_k(\bar{z})$$

$$-(\bar{z}-a_0)\sum_{k=0}^{\infty} Q_k(\bar{z})P_k(z).$$

Following the notation of Buchwalter & Cassier [4], p. 175 we obtain ($z \notin \mathrm{R}$)

$$b_0^2\tilde{q}(z)^2 = |z-a_0|^2 q(z)^2 + p(z)^2 - 1 + \frac{(z-a_0)(B(z,\bar{z})+1)}{z-\bar{z}}$$

$$-\frac{(\bar{z}-a_0)(B(\bar{z},z)+1)}{\bar{z}-z}$$

$$= |z-a_0|^2 q(z)^2 + p(z)^2 + \frac{(z-a_0)B(z,\bar{z}) - (\bar{z}-a_0)B(\bar{z},z)}{z-\bar{z}}.$$

Using the equality $B(z,w) = B(w)C(z) - A(z)D(w)$, see [4], p. 177 we get

$$b_0^2\tilde{q}(z)^2 = |z-a_0|^2 q(z)^2 + p(z)^2 + \frac{\operatorname{Im}\{(z-a_0)(B(\bar{z})C(z) - A(z)D(\bar{z}))\}}{\operatorname{Im} z}.$$

For $z \neq w$ we have

$$\sum_{k=0}^{\infty} Q_k(z)P_k(w) = \frac{B(z,w)+1}{w-z} = \frac{B(w)C(z) - A(z)D(w) + 1}{w-z},$$

and making $w \to z$ we get

$$\sum_{k=0}^{\infty} Q_k(z)P_k(z) = B'(z)C(z) - A(z)D'(z).$$

This implies the equality

$$b_0^2\tilde{q}(x)^2 = (x-a_0)^2 q(x)^2 + p(x)^2 - 1 - 2(x-a_0)(B'(x)C(x) - A(x)D'(x))$$

for $x \in \mathrm{R}$.

3. The Nevanlinna parametrization of the shifted problem.

When the sequence $(s_n)_{n\geq 0}$ is indeterminate, we have the so-called Nevanlinna parametrization of the set V of representing measures by means of the class $\mathscr{P}$ of Pick-functions in the upper half-plane H^+:

$$\mathscr{P} \equiv \{f \in \mathscr{H}(H^+) \mid \operatorname{Im} f \geqq 0\}.$$

There is a one-to-one correspondence $\varphi \leftrightarrow \mu_\varphi$ between $\mathscr{P} \cup \{\infty\}$ and the set V, given by

$$(8) \qquad \int_{\mathbb{R}} \frac{d\mu_\varphi(t)}{z-t} = \frac{A(z)\varphi(z) - C(z)}{B(z)\varphi(z) - D(z)}, \quad z \in \mathbb{C}\setminus\mathbb{R}.$$

The solutions to the shifted problem can be described in terms of those to the original problem; we have:

PROPOSITION 3.1. *The transformation $\varphi \mapsto -b_0^2/\varphi - a_0 = \varphi^*$ is a homeomorphism of $\mathbb{P} \cup \{\infty\}$ onto itself, and we have the following one-to-one correspondence between the sets $\tilde{V}$ and V:*

$$(9) \qquad b_0^2 \int \frac{d\tilde{\mu}_\varphi(t)}{z-t} = z - a_0 - \left(\int \frac{d\mu_{\varphi^*}(t)}{z-t}\right)^{-1}.$$

PROOF. Since $-1/\varphi \in \mathscr{P}$ whenever $\varphi \in \mathscr{P}\setminus\{0\}$, $\varphi \mapsto \varphi^*$ is easily seen to be a homeomorphism. Using (8) and Proposition 2.1 we get

$$b_0^2 \int \frac{d\tilde{\mu}_\varphi(t)}{z-t} = b_0^2 \frac{\tilde{A}(z)\varphi(z) - \tilde{C}(z)}{\tilde{B}(z)\varphi(z) - \tilde{D}(z)}$$

$$= \frac{(z-a_0)(A(z)(a_0\varphi(z) + b_0^2) + C(z)\varphi(z)) - D(z)\varphi(z) - B(z)(a_0\varphi(z) + b_0^2)}{A(z)(a_0\varphi(z) + b_0^2) + C(z)\varphi(z)}$$

$$= z - a_0 - \left(\frac{A(z)(b_0^2\varphi(z)^{-1} + a_0) + C(z)}{B(z)(b_0^2\varphi(z)^{-1} + a_0) + D(z)}\right)^{-1}$$

$$= z - a_0 - \left(\int \frac{d\mu_{\varphi^*}(t)}{z-t}\right)^{-1}.$$

4. On Hamburger's class $\mathscr{A}$ of entire functions.

The Nevanlinna matrix and the so-called Nevanlinna-extremal solutions (i.e. the subset $\{\mu_t \mid t \in \mathbb{R} \cup \{\infty\}\}$ of V) are closely interrelated. There has been made attempts to characterize these solutions in terms of entire functions, see [1], p. 161 ff. and [5] p. 515 (the result is partly wrong, see [6]).

Hamburger introduced a certain class $\mathscr{A}$ of entire functions. An entire function f belongs to this class if it is real, if it has infinitely many zeros $(\lambda_n)_{n \geq 1}$ all of which are real and simple, and if the following relations hold

$$\sum_{n=1}^{\infty} \left|\frac{\lambda_n^k}{f'(\lambda_n)}\right| < \infty \quad \text{for each} \quad k \in \mathbb{N},$$

$$\frac{1}{f(z)} = \sum_{n=1}^{\infty} \frac{1}{f'(\lambda_n)(z-\lambda_n)} \quad \text{for} \quad z \in \mathbb{C}\setminus\{\lambda_n \mid n \geqq 1\}.$$

It is proved in [1] p. 165 that any combination $tB - D$, $t \in \mathsf{R} \cup \{\infty\}$ ($\equiv B$ if $t = \infty$) belongs to the class $\mathscr{A}$, but nothing is stated about the natural question whether the "numerator" functions $tA - C$ occuring in the Nevanlinna parametrization (8) also belong to this class $\mathscr{A}$.

PROPOSITION 4.1. *For any $t \in \mathsf{R} \cup \{\infty\}$ the function $tA - C$ belongs to $\mathscr{A}$. In particular A, B, C, D belong to $\mathscr{A}$.*

PROOF. Using the formulas for $\tilde{B}$ and $\tilde{D}$ in Proposition 2.1 we get that

$$tA(z) - C(t) = \frac{t + a_0}{b_0^2}\tilde{D}(z) + \tilde{B}(z). \tag{10}$$

Applying the theorem in [1] p. 165 to the shifted moment problem we get that the righthand side of (10) belongs to the class $\mathscr{A}$, and so does the "numerator" function $tA - C$.

5. A class of entire functions of two variables.

We denote by $\mathscr{A}_2$ the class of entire functions F of two complex variables such that

$$\begin{aligned} &\forall u \in \mathsf{R}\colon (v \mapsto F(u,v)) \in \mathscr{A} \\ &\forall v \in \mathsf{R}\colon (u \mapsto F(u,v)) \in \mathscr{A} \end{aligned} \tag{11}$$

It is possible to define four entire functions of two complex variables associated with an indeterminate moment problem, see [4], p. 175. These functions generalize the functions in the Nevanlinna matrix and in terms of those they may be written as follows, see [4], p. 177.

$$\begin{aligned} A(u,v) &= A(v)C(u) - A(u)C(v) \\ B(u,v) &= B(v)C(u) - A(u)D(v) \\ C(u,v) &= A(v)D(u) - B(u)C(v) \\ D(u,v) &= B(v)D(u) - B(u)D(v) \end{aligned} \tag{12}$$

PROPOSITION 5.1. *The two-variable entire functions A, B, C and D belong to the class $\mathscr{A}_2$.*

PROOF. The above equations (12), the fact that $A(z)D(z) - B(z)C(z) \equiv 1$ and Proposition 4.1 yield the result.

We shall now compute the two-variable functions associated with the r times shifted moment problem. To do this we need some notation. We define $A_1 \equiv \tilde{A}$,

$A_2 \equiv (A_1)\tilde{}, \ldots, A_r \equiv (A_{r-1})\tilde{}$ and similarly for the other functions B, C and D. We also put, for $k \geqq 0$,

$$M_k(u,v) = \begin{pmatrix} b_k^{-2}(u-a_k)(v-a_k) & -b_k^{-2}(u-a_k) & -b_k^{-2}(v-a_k) & b_k^{-2} \\ u-a_k & 0 & -1 & 0 \\ v-a_k & -1 & 0 & 0 \\ b_k^2 & 0 & 0 & 0 \end{pmatrix}.$$

With this notation we can formulate:

PROPOSITION 5.2. *Let $r \geqq 1$. Then,*

$$\begin{pmatrix} A_r \\ B_r \\ C_r \\ D_r \end{pmatrix} = M_{r-1}M_{r-2}\cdot\ldots\cdot M_1M_0\begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} \tag{13}$$

PROOF. A computation as in the proof of Proposition 2.1 (or simply (12) and Proposition 2.1) gives

$$\tilde{A}(u,v) = b_0^{-2}((u-a_0)(v-a_0)A(u,v) - (u-a_0)B(u,v) - (v-a_0)C(u,v) + D(u,v))$$

$$\tilde{B}(u,v) = (u-a_0)A(u,v) - C(u,v)$$

$$\tilde{C}(u,v) = (v-a_0)A(u,v) - B(u,v)$$

$$\tilde{D}(u,v) = b_0^2A(u,v).$$

This can be written more compactly as

$$\begin{pmatrix} A_1 \\ B_1 \\ C_1 \\ D_1 \end{pmatrix} = M_0\begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix}.$$

Then (13) follows by induction.

By Proposition 5.1 we see:

COROLLARY 5.3. *The four coordinates of the vector*

$$M_{r-1}M_{r-2}\cdot\ldots\cdot M_1M_0\begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix}$$

belong to the class $\mathscr{A}_2$.

REMARK. Proposition 5.2 generalizes Proposition 2.1 and Corollary 5.3 generalizes Proposition 4.1. I wish to thank the referee for suggesting the class $\mathscr{A}_2$ and these generalizations.

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