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A THEOREM OF GROTHENDIECK USING PICARD GROUPS FOR THE ALGEBRAIST*

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Abstract.

This is an application of a new algebraic reformulation of the Picard group pic(G) of a quasi-compact subset $G \subseteq X = \operatorname{Spec} A$ for a commutative ring A. A torsion theoretic (algebraic) proof is given of A. Grothendieck's theorem that a complete intersection that is factorial in co-dimension 3 is factorial. Our proof is along the lines of Grothendieck's SGA2 proof, but eliminates the need for spectral sequences and (formal) sheaf theory.

All torsion theoretic details are given in a lengthy appendix, including the proof of the long standing conjecture that $Q_G = \varinjlim Q_U$ for G quasi-compact and U open. This yields the interesting result that restriction of an O_X-Module to a quasi-compact, generically closed subset does not require the sheafification process.

One of A. Grothendieck's theorems is that a complete intersection ring which is locally a unique factorization domain (UFD) in codimension 3, is a UFD. This conjecture of P. Samuel was proved about thirty years ago [9, Corollaire XI 3.14]. Grothendieck's proof uses the "most sophisticated techniques of algebraic geometry" [8, p. 3]. We provide in this paper a purely algebraic proof of this purely algebraic theorem.

Such a proof, accessible to the pure algebraist is sorely needed. Our technique is "merely" to translate Grothendieck's proof into commutative algebra, using torsion theory to finesse the material on formal schemes, and to avoid the occasional spectral sequence. We have used this technique in [5] to find a simple proof of the local Lichtenbaum-Hartshorne theorem. This illustrates again how torsion theory can handle difficult algebraic geometry with relative ease. Of course, some simplifications come by narrowing the focus of the highly complex and general theory that [9] provides.

Many of the ideas of our proof parallel those in [9], and for comparison, references to [9] are provided for those familiar with the algebraic geometry language. Algebraists should find most interesting the three methods (Steps 1, 2,

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and 3) of "lifting" (modulo nilpotents or a regular element) or "extending" a finitely generated module that is locally free on just a portion of spec A.

The difficult part of Grothendieck's proof is the theorem that a complete intersection (A, m) of dimension $\geqq 4$ is parafactorial, i.e., depth $A \geqq 2$ and the Picard group $\mathrm{Pic}(U)$ of the punctured spectrum $U = \mathrm{Spec}\, A - m$ is trivial. This leads to a study of a purely algebraic formulation, $\mathrm{pic}(G)$, where $G \subseteq X = \mathrm{Spec}\, A$. Our object is to define an abelian group $\mathrm{pic}(G)$ that agrees with the algebraic geometers' $\mathrm{Pic}(G, \mathcal{O}_G)$, the group of isomorphism classes of invertible $\mathcal{O}_G$-Modules where $\mathcal{O}_G = \mathcal{O}_X|_G$, yet is easy to manipulate and is analogous in definition to the ring theorists' $\mathrm{Pic}(A)$ ($\cong \mathrm{Pic}(X)$), the group of isomorphism classes of finitely generated, locally free A-modules of constant rank 1 (these are just the (finitely generated) rank 1 projectives).

Formulations of $\mathrm{pic}(G)$ are being developed in the literature, e.g. [20, 19, 18, 6], and we have provided a reworking in an extensive appendix (see Definition A-6 and the variations that follow) when G is quasi-compact. This includes the three cases (1) when Spec A is noeetherian and G is arbitrary, (2) when A is arbitrary and G is an intersection of quasi-compact opens, and (3) when A is a Krull domain and G is the set of height 1 primes. We only show that $\mathrm{pic}(G) \cong \mathrm{Pic}(G)$ for case (2), and though our application is for A noetherian and G open, our $\mathrm{pic}(G)$ is useful in the other cases as well. Part of our work shows that, in case (2), the definition of $\tilde{M}|_G$ does not require the sheafification process even though G may not be open.

The proofs of the results in the Appendix and of the Grothendieck-Samuel theorem will use the torsion functor $\mathcal{T}_G$ and the localization functor Q_G, as well as some techniques from torsion theory, so we have included this needed material in order to make the paper easier to read.

We hope this presentation will encourage other commutative ring theorists to use torsion theory to explore algebraic geometry from the algebraic point of view.

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We need to discuss when a noetherian local ring is a unique factorization domain (UFD). If a noetherian ring is already a domain, then it is a UFD if and only if each height 1 prime is principal [15, Theorem 13.1]. Any localization of a UFD is a UFD, and a noetherian UFD is a normal domain, i.e., noetherian and integrally closed [15, 13.2 and 13.3]. We refer the reader to the Appendix (A-6, A-7, A-11) for our module theoretic formulation of pic.

The following definition should be compared with [9, XI Proposition 3.5].

DEFINITION 1. A noetherian local ring (A, m) of dimension $\geqslant 2$ is called *parafactorial* if depth $A \geqq 2$ and $\operatorname{pic}(U) = 0$ where $U = \operatorname{spec} A - m$.

The next item is useful in an induction proof.

PROPOSITION 2 [9, XI Corollaire 3.10]. *A noetherian local ring (A, m) of dimension $\geqq 2$ is a* UFD *if and only if A is parafactorial and A_p is a* UFD *for all $p \neq m$.*

PROOF. ($\Leftarrow$) Since A_p is a normal domain for each $p \neq m$, Serre's criterion for normality $(R_1) + (S_2)$ is satisfied by A for the cases $p \neq m$ [13, Theorem 23.8]. $\operatorname{Dim} A \geqq 2$, and depth $A \geqq 2$ from the parafactorial property, covers the case $p = m$. The normal ring A is then a finite direct product of normal domains [13, Exercise 9.11] and, since it is local, A must be a domain. The result (A-13) is applicable as a normal domain is Krull [13, Theorem 12.4].

($\Rightarrow$) Use (A-13) again.

LEMMA 3. *Let $\varphi\colon A \to B$ be a flat homomorphism of local rings such that ${}_A mB$ is primary to the maximal ideal ${}_B m$ of B. Then* $\dim A = \dim B$ *and* depth $A =$ depth B. *If* $\dim B \geqq 2$ *and B is parafactorial, so is A.*

PROOF. For flat local homomorphisms we have dimension and depth formulas $\dim B = \dim A + \dim B/{}_A mB$ and depth $B = \operatorname{depth} A + \operatorname{depth} B/{}_A mB$ [13, Theorem 15.1 and Corollary to Theorem 23.3]. The first parts follow from $\dim B/{}_A mB = 0 = \operatorname{depth} B/{}_A mB$.

Let $U = \operatorname{spec} A - {}_A m$ and $U' = \operatorname{spec} B - {}_B m$. We use (A-11) for our description of $\operatorname{pic}(U)$: the group of isomorphism classes $[[M]]_U$ of finitely generated reflexive A-modules M, locally free rank 1 on U. Then by the flatness of $B, M \otimes_A B$ is reflexive, $[[M \otimes B]]_{U'} \in \operatorname{pic}(U') = 0$, so $M \otimes B \cong B$. Faithful flatness (M is finitely generated) and the Ext^1 condition will show M is projective, hence free over the local ring A, of rank 1 (cf., [9, XI Lemme 3.6]).

Recall that a regular local ring R is a noetherian local ring (R, m) such that the number of generators of a minimal generating set for m is equal to the dimension of R. Regular local rings are UFD's [13, Theorem 20.3]. If R is regular local, so is R_p for each $p \in \operatorname{spec} R$ [13, Theorem 19.3]. A noetherian local ring is regular if and only if its m-adic completion is regular [2, Proposition 11.24].

The next definition has been updated to include completions.

DEFINITION 4. A (noetherian) local ring (A, m) is a *complete intersection* (c.i.) if its m-adic completion $\hat{A} = R/(x_1, \ldots, x_s)$, where R is a (complete) regular local ring and $x_1, \ldots, x_s$ is an R-sequence.

A (c.i.) is Gorenstein, hence Cohen-Macaulay [13, Theorems 21.3 and 18.1], so depth $A = \dim A$.

THEOREM 5 [9, Théorème XI 3.13 (ii)]. *Any complete intersection of dimension $\geqq 4$ is parafactorial.*

PROOF. We induct on s. If $s = 0$, we mean $\hat{A}$ is regular. Then A is regular, hence parafactorial by Proposition 2.

For $s > 0$, it is sufficient to consider the complete case, Lemma 3. We now change notation. For the induction hypothesis, for a fixed $s \geqq 0$, we suppose every (c.i.) of dimension $\geqq 4$ whose completion can be written as a regular local ring mod a regular sequence of length s is parafactorial. Let A_1 be a complete (c.i.) of dimension $\geqq 4$, $A_1 = A/tA$ where $A = R/(x_1, \dots, x_s)$ is complete of dimension $\geqq 5$, parafactorial by the induction hypothesis, and t is a non-zero-divisor of A in the maximal ideal m of A. We need to show $\operatorname{pic}(U_1) = 0$ where we denote by U_i the punctured spectrum of $A_i = A/t^iA$ (A_i is also Cohen-Macaulay, depth $A_i = \dim A_i \geqq 4$, so the depth condition in the definition of parafactorial is automatically satisfied for every i). We accomplish this in three steps.

Let $U = \operatorname{spec} A - m$, $G = g(V(tA) - m) = \{p \in \operatorname{spec} A \mid p \subseteq \text{some } q \in \operatorname{spec} A, t \in q \neq m\} \subseteq U$. We have natural group homomorphisms

$$0 = \operatorname{pic}(U) \xrightarrow{\alpha_1} \operatorname{pic}(G) \xrightarrow{\alpha_2} \varprojlim_i \operatorname{pic}(U_i) \xrightarrow{\alpha_3} \operatorname{pic}(U_1)$$

which we will discuss each in turn.

Step (1). First note that $U \backslash G = \{p \mid \dim A/p = 1 \text{ and } t \notin p\}$. This is because if $p \in U \backslash G$ then the principal ideal generated by t in A/p is primary to the maximal ideal. By Krull's principal ideal theorem [13, Theorem 13.5], $\dim A/p \leqq 1$. Secondly, since A_p is Cohen-Macaulay [13, Theorem 17.3 (iii)], $U \supseteq G \supseteq \{p \in \operatorname{spec} A \mid \operatorname{depth} A_p \leqq 1\}$. Thus, we may use (A-11) for our description of pic, and define the group homomorphism α_1 by $\alpha_1([[M]]_U) = [[M_p]]_G \in \operatorname{pic}(G)$ where M is a finitely generated reflexive A-module, locally free of rank 1 on U.

To show α_1 is surjective, let $[[M]]_G \in \operatorname{pic}(G)$ where M is a finitely generated reflexive A-mdule, locally free rank 1 on G. M is actually free on all of U, since $[[M]]_{U_p} \in \operatorname{pic}(U_p)$ where U_p is the punctured spectrum of A_p, $p \in U \backslash G$. This is true since $U_p \subseteq G$ and M is locally free on G (of course, A_p and U_p satisfy the hypothesis of (A-11) since $\dim A_p \geqq 4 > 1$). Since A is a regular local ring modulo a regular sequence of length s, the same is true for A_p by earlier remarks. The dimension of $A_p \geqq 4$ and thus, by the induction hypothesis, A_p is parafactorial, and $\operatorname{pic}(U_p) = 0$. We conclude that $M_p \cong A_p$, $[[M]]_U \in \operatorname{pic}(U)$, and it follows that α_1 is surjective (compare with [9, XI Proposition 3.12 and X Exemple 2.1]).

Step (2). For the groups $\operatorname{pic}(G)$ and $\operatorname{pic}(U_i)$ we use (A-6). The natural group homomorphisms, for $j \geqq i$, π_{ij}: $\mathrm{pic}(U_j) \to \mathrm{pic}(U_i)$ are defined by $\pi_{ij}([\mathrm{M}_j]_{U_j}) = [M_j \otimes A_i]_{U_i}$. These maps from an inverse system and the natural maps $\mathrm{pic}(G) \to \mathrm{pic}(U_i)$ factor through the π_{ij}, hence we have a homomorphism α_2: $\mathrm{pic}(G) \to \varprojlim_i \mathrm{pic}(U_i)$ given by $\alpha_2([M]_G) = ([M \otimes A_i]_{U_i})_{i \geqq 1}$. To show α_2 is surjective, let $([M_i]_{U_i})_{i \geqq 1} \in \varprojlim_i \mathrm{pic}(U_i)$. By the remarks at the end of (A-11) we can assume the representative M_i is a finitely generated A_i-module and $M_i = Q_{U_i}(M_i)$ $(= M_i^{**})$. Furthermore, we fix isomorphisms, for $j = i + 1, i \geqq 1, Q_U(M_j \otimes A_i) \cong Q_{U_i}(M_j \otimes A_i) \cong M_i$ where the first equality comes from (A-1j) and the second from compatibility relations in the inverse limit. Set $Q = Q_U$ for ease of notation, and define, for $j = i + 1$, A-module maps φ_{ij}: $M_j \to M_j \otimes A_i \to Q(M_j \otimes A_i) \cong M_i$. The obvious compositions give us an inverse system of maps between any pair of the M_i's. $M = \varprojlim M_i$ is the module we want. To show this, we first claim that the φ's satisfy the Mittag-Leffler condition which we proceed to prove (cf., [9, IX Théorème 2.2]). These φ's fit into a large commutative diagram with many maps.

$$(*)\qquad \begin{array}{ccccccccc}
 & & & & & & & \leftrightarrows & \\
 & & & & & & M_4 & \cong & \cdots \\
 & & & & & \leftrightarrows & \downarrow\uparrow & & \\
 & & & & M_3 & \cong & Q(M_4 \otimes A_3) & \cong & \cdots \\
 & & & \varphi \leftrightarrows \psi & \downarrow\uparrow & & \mathrm{nat}\downarrow\uparrow t & & \\
 & & M_2 & \cong & Q(M_3 \otimes A_2) & \cong & Q(M_4 \otimes A_2) & \cong & \cdots \\
 & \leftrightarrows & \downarrow\uparrow & & \downarrow\uparrow & & \downarrow\uparrow & & \\
M_1 & \cong & Q(M_2 \otimes A_1) & \cong & Q(M_3 \otimes A_1) & \cong & Q(M_4 \otimes A_1) & \cong & \cdots
\end{array}$$

In each column of isomorphisms, the top one is the chosen fixed one mentioned earlier (thus the small triangles with the φ's commute by definition). The remaining ones in the column are induced from these by tensoring with A_i and applying Q. The downward maps are the natural ones induced from the surjections $M_k \otimes A_{i+1} \xrightarrow{\mathrm{nat}} M_k \otimes A_i$. The upward maps are induced from $M_k \otimes A_i \xrightarrow{t} M_k \otimes A_{i+1}$, and these are used to construct, for $j > i$, maps ψ_{ji}: $M_i \to M_j$, i.e., the small triangles commute by the definition of the ψ's. If we can show that the squares commute (both kinds) then we can work along any column to study the M_i's. But this is easy to establish; set $N = M_k$, $N^+ = M_{k+1}$, and let $j = i + 1$. We have a diagram

$$(**)\qquad \begin{array}{ccccc}
N \otimes A_j & \cong & Q(N^+ \otimes A_k) \otimes A_j & \leftarrow & N^+ \otimes A_j \\
\downarrow\uparrow\ 1 \otimes t & & \downarrow\uparrow\ 1 \otimes t & & \downarrow\uparrow\ 1 \otimes t \\
N \otimes A_i & \cong & Q(N^+ \otimes A_k) \otimes A_i & \leftarrow & N^+ \otimes A_i
\end{array}$$

The squares commute, the horizontal arrows are locally isomorphisms on U, so when Q is applied to (**) we obtain the small squares in (*), by (A-1c).

We are ready to extract information about the M_i's and φ's from (*) by working along a column. First, the φ's and ψ's yield, for any i and j, a long exact sequence $0 \to M_i \xrightarrow{\psi} M_{i+j} \xrightarrow{\varphi} M_j \to R^1Q(M_i) \to R^1Q(M_{i+j}) \to \dots$, where we have deleted the subscripts on φ and ψ for clarity. To obtain this sequence, let $k > i + j$ and set $N = M_k$. The *right* exact sequence $0 \to N \otimes A_i \xrightarrow{t^j} N \otimes A_{i+j} \to N \otimes A_j \to 0$ becomes exact locally at each $p \in U$ since $N = M_k$ is locally either zero or free, and t is not a zero-divisor on A. By (A-1c), the kernel of the map marked t^j is $\mathcal{T}_U$-torsion (see the introduction to the Appendix for terminology). Replace $N \otimes A_i$ by its image K_i and apply Q to the resulting short exact sequence to obtain a long exact sequence. Since the class of $\mathcal{T}_U$-torsion modules is closed under the formation of injective envelopes when A is noetherian [6, Proposition 6.3 (6)] and Q kills this torsion class by (A-1c), it follows that $R^nQ(N \otimes A_i) \xrightarrow{\sim} R^nQ(K_i), n \geqq 0$. Substituting the M_i's (and φ's and ψ's) yields the quoted long exact sequence.

Now for the claimed Mittag-Leffler condition. Fix $d \geqq 1$. We must show that the images of the M_i's ($i \geqq d$) in M_d stabilize for all i sufficiently large. Use the preceding long exact sequences to construct, for each $j \geqq 2$, the commutative diagrams

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & M_d \arrow[r, equal] \arrow[d, "\psi"'] & M_d \arrow[d] & & \\
0 \arrow[r] & M_{jd} \arrow[r, "\varphi"] \arrow[d, "\varphi"'] & M_{(j+1)d} \arrow[r, "\varphi"] \arrow[d] & M_d \arrow[d, equal] & \\
0 \arrow[r] & M_{(j-1)d} \arrow[r] \arrow[d] \arrow[ur, phantom, "t^d"] & M_{jd} \arrow[r] \arrow[d] & M_d & \\
 & R^1Q(M_d) \arrow[r, equal] & R^1Q(M_d) & &
\end{tikzcd} \qquad (j \geqq 2)$$

Denote by $c_j = c_j(d)$ the image of M_{jd} in M_d and by $l_j = l_j(d)$ the image of M_{jd} in $R^1Q(M_d)$. Conclude that $c_{j+1} \subseteq c_j \subseteq M_d$, $l_{j-1} \subseteq l_j \subseteq R^1Q(M_d)$, and $l_j/l_{j-1} \cong c_j/c_{j+1}$ by the snake lemma. It is clear that $R^1Q(M_d) = \varinjlim_j \operatorname{Ext}^1_A(m^j, M_d) = H^2_m(M_d)$, from (A-1i) and the fact that the powers of the maximal ideal m of A are cofinal in the torsion filter F_U (see the introduction to the Appendix). By a standard change of rings formula [6, Theorem 7.2 (9)], the second local cohomology $H^2_m(M_d) = H^2_{mA_d}(M_d)$. By duality for the Gorenstein ring A_d [6, Proposition 7.10] or [9, V Proposition 3.5], $H^2_{mA_d}(M_d) = \operatorname{Ext}^{n-3}_{A_d}(M_d, A_d)^*$ where $*$ is the Matlis dual $\operatorname{Hom}_{A_d}(\ , E(A_d/mA_d))$ and $n = \dim A = \dim A_d + 1 \geqq 5$. A quick check locally gives that the finitely generated A_d-module $\operatorname{Ext}^{n-3}_{A_d}(M_d, A_d)$ vanishes on U_d since

$n - 3 \geqq 1$, hence is of finite length. Thus $R^1Q(M_d)$ is also of finite length and the l_j's stabilize, hence so do the c_j's, say to $c(d)$, $d \geqq 1$. This establishes the Mittag-Leffler claim.

We have yet to prove that $M = \varprojlim_i M_i = \varprojlim_d c(d)$ has the required properties: M is finitely generated as an A-module, locally free rank one on G, and $Q(M \otimes A_i) \cong M_i$. For the last item, apply $\varprojlim_j$ to the family of short exact sequences $0 \to M_{(j-1)d} \to M_{jd} \to c(d) \to 0$ (for $j >> 0$) to obtain the exact sequence $0 \to M \to M \to c(d) \to 0$ (Mittag-Leffler is used here). The endomorphism of M is just multiplication by t^d, a fact deduced from the diagonal map in the diagrams ($j \geqq 2$) above. All the φ's are locally surjective at each $p \in U$ so we have $Q(M_d) = Q(c(d))$. Thus $M_d = Q(M_d) = Q(c(d)) \cong Q(M/t^dM) = Q_{U_d}(M/t^dM)$ for each d. We now use [13, Theorem 8.4] to prove M is finitely generated. It is clear from the definition of $M = \varprojlim c(d)$ that $0 = \cap \ker(M \to c(d)) = \cap t^dM$ so that M is t-adically separated. Also, $M/tM \cong c(1) \subseteq M_1$ is finitely generated over the (t-adically) complete ring A. Thus M is finitely generated. For local freeness on G, let $p \in V(tA) - m$. Then $M_p/t^dM_p \cong c(d)_p \cong (M_d)_p \cong A_p/t^dA_p$, $d \geqq 1$. Nakayama's lemma and Krull's intersection theorem $\cap t^dA_p = 0$ imply $M_p \cong A_p$. We have completed the proof that α_2 is surjective.

Step (3). Grothendieck quotes some (now) well-known theorems from algebraic geometry to prove the natural maps $\operatorname{Pic}(U_{i+1}) \to \operatorname{Pic}(U_i)$ are isomorphisms [9, XI Proposition 1.1], hence so is the projection α_3 from the inverse limit to the first coordinate. His argument is that the Picard group can be written as the first Čech cohomology of a sheaf of units and that this is the same as the first (Grothendieck) sheaf cohomology [10, Exercises III 4.4 and 4.5]. Then $\operatorname{Pic}(U_{i+1})$ and $\operatorname{Pic}(U_i)$ fit into part of a long exact sequence obtained by modding out a nilpotent ideal sheaf of order 2 [10, Exercise III 4.6]. Thus the kernel and cokernel of $\operatorname{Pic}(U_{i+1}) \to \operatorname{Pic}(U_i)$ lie in first (respectively, second) cohomology groups, which can be shifted to the local cohomology modules $H^2_m(t^iA/t^{i+1}A)$ and $H^3_m(t^iA/t^{i+1}A)$, respectively. Since $t^iA/t^{i+1}A \cong A/tA$ and $\operatorname{depth} A/tA = \dim A/tA \geqq 4$, these modules are zero, and $\operatorname{Pic}(U_{i+1}) \cong \operatorname{Pic}(U_i)$, for all i. This is Grothendieck's proof that α_3 is surjective. If the reader accepts these ideas, then he or she may continue to the main theorem.

However, for anyone who would like a "purely algebraic" proof of this theorem, we provide directions.

First we need that a U-invertible module M can be determined by local data, where U is the punctured spectrum of, say, any noetherian local ring (A, m). We assume depth $A \geqq 2$, $M = Q_U(M) = M^{**}$, and that M is U-invertible. From (A-3), choose a finite covering of $U = \cup D(f_i)$ such that $M_{f_i} = Q_{D(f_i)}(M) \cong Q_{D(f_i)}(A) = A_{f_i}$, and let $\sigma_i \colon M_{f_i} \xrightarrow{\sim} A_{f_i}$ be these isomorphisms. For each i,j set

$\sigma_{ij} = \sigma_i \otimes A_{f_j}\colon M_{f_i f_j} \xrightarrow{\simeq} A_{f_i f_j}$. We identify $(M_{f_i})_{f_j} = M_{f_i f_j} = M_{f_j f_i} = (M_{f_j})_{f_i}$. Then for each $x_i \in M_{f_i}$ we have $\sigma_i(x_i)' = \sigma_{ij}(x_i')$, where we use "$'$" to denote an image in a further localization. Also, for $i < j$, $r_{ij} := \sigma_{ij}\sigma_{ji}^{-1}$ is an automorphism of $A_{f_i f_j}$, so corresponds to a unit in $A_{f_i f_j}$, i.e., $r_{ij} \in A^{\times}_{f_i f_j}$. These r_{ij}, $i < j$, satisfy in $A_{f_i f_j f_k}$ the relations $r'_{jk} r'^{-1}_{ik} r'_{ij} = 1$. The point is that M can be reconstructed (up to isomorphism) from the local data $r_{ij} \in A^{\times}_{f_i f_j}$, $i < j$. We know $M = Q_U(M)$ can be written [6, Theorem 5.1] as a kernel of a map which is part of the commutative diagram

$$\begin{array}{ccccccc} 0 & \to & M & \to & \bigoplus_i M_{f_i} & \to & \bigoplus_{i<j} M_{f_i f_j} \\ & & & & \simeq \big\downarrow \oplus\sigma_i & & \simeq \big\downarrow \oplus\sigma_{ij} \\ 0 & \to & K & \to & \bigoplus A_{f_i} & \to & \bigoplus A_{f_i f_j} \end{array}$$

where M is (isomorphic to) the kernel of the (standard) homomorphism that sends $(x_i) \in \bigoplus M_{f_i}$ to the element of $\bigoplus M_{f_i f_j}$ whose (i,j)-coordinate is $x_j' - x_i'$. K is the kernel of the twist map that sends $(a_i) \in \bigoplus A_{f_i}$ to $(r_{ij} a_j' - a_i') \in \bigoplus A_{f_i f_j}$. However, if for this M we have another set of σ_i's, i.e., of the form $\sigma_i c_i$ where $c_i \in A^{\times}_{f_i}$ corresponds to an automorphism, then the old local data $(r_{ij}) \in \Pi A^{\times}_{f_i f_j}$ is related to the new local data (s_{ij}) by the formula $(c_j' c_i'^{-1})(s_{ij}) = (r_{ij})$. Conversely, given any element $(r_{ij}) \in \Pi_{i<j} A^{\times}_{f_i f_j}$ such that $r'_{ij} r'_{jk} = r'_{ik}$ in $A_{f_i f_j f_k}$, the kernel K of the twist map will be trivial on each $D(f_i)$ (we mean K_{f_i} is a free A_{f_i}-module; for example K_{f_1} is generated by $(1, r_{12}^{-1}, \ldots, r_{1n}^{-1})$). By (A-3), K is U-invertible. Moreover, $K = Q_U(K)$ since each A_{f_i} and $A_{f_i f_j}$ is stable under Q_U and Q_U is left exact, (A-1e) and (A-1a). Furthermore, K is finitely generated, by the observation in (A-11). If we also have any other element $(s_{ij}) \in \Pi_{i<j} A^{\times}_{f_i f_j}$ and $(c_i) \in \Pi_i A^{\times}_{f_i}$ such that $c_j' c_i'^{-1} s_{ij} = r_{ij}$ then $s'_{ij} s'_{jk} = s'_{ik}$ and the constructed kernel L of the twist map will be isomorphic to K, as the next commutative diagram shows

$$\begin{array}{ccccccc} 0 & \to & \mathrm{K} & \to & \bigoplus A_{f_i} & \to & \bigoplus A_{f_i f_j} \\ & & & & \simeq \big\downarrow \oplus c_i & & \simeq \big\downarrow \oplus_{i<j} c_i' \\ 0 & \to & L & \to & \bigoplus A_{f_i} & \to & \bigoplus A_{f_i f_j} \end{array}$$

All the above may be nicely summarized in terms of the first cohomology group of a Čech complex. Consider the complex of multiplicative abelian groups with (standard) coboundary homomorphisms [14, p. 72]:

$$1 \to \Pi_i A^{\times}_{f_i} \xrightarrow{\delta_1} \Pi_{i<j} A^{\times}_{f_i f_j} \xrightarrow{\delta_2} \Pi_{i<j<k} A^{\times}_{f_i f_j f_k} \to \cdots$$

where $\delta_1((a_i)) = (a_j' a_i'^{-1})$, $\delta_2((a_{ij})) = (a'_{jk} a'^{-1}_{ik} a'_{ij})$. Then for each (finite) cover of $U = \cup D(f_i)$ there is a one-to-one correspondence between the elements of the cohomology group $C := \operatorname{Ker}\delta_2/\operatorname{Im}\delta_1$ and those isomorphism classes $[[M]] \in \operatorname{pic}(U)$ such that M_{f_i} is a free A_{f_i}-module for all i.

Next we show how to lift this data (cohomology) mod nilpotents. Note the rather astonishing fact that if I is an ideal of a commutative ring A, and $I^2 = 0$

then there is an exact sequence $0 \to I \to A^\times \to (A/I)^\times \to 1$ of *abelian groups*, in which I has as its group structure its natural additive structure, and the other two groups are multiplicative. The first map sends $a \in I$ to the unit $1 + a$. For ease of notation, we consider lifting a module from U_1 to U_2 where U_i is the punctured spectrum of $A_i = A/I^i$, $i = 1, 2$. Let $[M] \in \mathrm{pic}(U_1)$ and pick suitable elements $f_i \in A$ such that σ_i: $M_{f_i} \cong (A_1)_{f_i}$ and $U_1 = \cup D(f_i A_1)$ (hence $U_2 = \cup D(f_i A_2)$). Then the σ's determine an element in the first cohomology group C_1 as described above. We have a commutative diagram of abelian groups (and standard maps) with columns that are exact and rows that are complexes.

$$\begin{array}{ccccccc}
0 & & 0 & & 0 & & \\
\downarrow & & \downarrow & & \downarrow & & \\
\oplus_i (I/I^2)_{f_i} & \to & \oplus_{i<j}(I/I^2)_{f_i f_j} & \to & \oplus_{i<j<k}(I/I^2)_{f_i f_j f_k} & \to & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
\Pi(A/I^2)^\times_{f_i} & \to & \Pi(A/I^2)^\times_{f_i f_j} & \to & \Pi(A/I^2)^\times_{f_i f_j f_k} & \to & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
\Pi(A/I)^\times_{f_i} & \to & \Pi(A/I)^\times_{f_i f_j} & \to & \Pi(A/I)^\times_{f_i f_j f_k} & \to & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
1 & & 1 & & 1 & &
\end{array}$$

A diagram chase shows that the cohomology C_2 of the second row at the second term maps onto our cohomology C_1 (of the third row at the second term), if the first row is exact (as abelian groups or as A-modules) at the third term. Exactness occurs precisely when the local cohomology module $H^3_m(I/I^2) = 0$.

An explanation of this last remark is given in [17, Proposition 2.3, p. 78], or we may see this in the following way. Let N be any module over a noetherian ring A and first suppose the f_i's form a regular sequence. Then the complex $D^{\cdot}_A(N) = D^{\cdot}_A(A) \otimes_A N$: $0 \to N \to \oplus_i N_{f_i} \to \oplus_{i<j} N_{f_i f_j} \to \cdots \to N_{f_1 f_2 \ldots f_n} \to 0$ has ith cohomology $H^i(D^{\cdot}(N)) = \varinjlim_j \mathrm{Ext}^i(A/(f_1^j, \ldots, f_n^j), N) = H^i_m(N)$, since (1) the Koszul complex $K.(f_1^j, \ldots, f_n^j)$ is a resolution of $A/(f_1^j, \ldots, f_n^j)$ by free modules of finite rank (the f_i^j's also form a regular sequence [13, Theorem 16.1]), (2) the complex $\mathrm{Hom}(K.(f_1^j, \ldots, f_n^j), N)$ is chain isomorphic to $K.(f_1^j, \ldots, f_n^j) \otimes N$, and (3) $\varinjlim_j K.(f_1^j, \ldots, f_n^j)$ is $D^{\cdot}(A)$ indexed in the reverse order. In case the f_i's do not form an A-sequence, map the polynomial ring $B = \mathbf{Z}[X_1, \ldots, X_n]$ to A by sending the indeterminate X_i to f_i. Then we have that $H^i_m(N) = H^i_{(X_1, \ldots, X_n)}(N) = H^i(D^{\cdot}_B(B) \otimes_B N) = H^i(D^{\cdot}_A(A) \otimes_A N) = H^i(D^{\cdot}_A(N))$, even as modules.

Return to our application. In our case, $N = I/I^2 = tA/t^2A \cong A/tA$ (in general $I^i/I^{i+1} \cong A/tA$) and we know $H^3_m(A/tA) = 0$, so we can lift. Thus, given $[[M]] = [[K]] \in \mathrm{pic}(U_1)$ with K the kernel of a twist map defined by data $(r_{ij}) \in \Pi(A/tA)^\times_{f_i f_j}$ induced from M, we can construct, by the above, a kernel K'

from some lifting $(s_{ij}) \in \Pi(A/t^2A)^{\times}_{f_i f_j}$ of (r_{ij}), with $[[K']] \in \mathrm{pic}(U_2)$, It remains to show $Q(K'/tK') \cong Q(K) = K$. This is easy: localizing the natural map $K'/tK' \to K$ at any of the f_i's will yield a surjection of rank one free modules, which must then be an isomorphism. The conclusion follows from (A1-c). So we have that π_{12} is surjective, and in general each of the maps $\mathrm{pic}(U_{i+1}) \to \mathrm{pic}(U_i)$ is surjective, hence so is α_3.

This completes the proof that $\mathrm{pic}(U_1) = 0$ and the induction step.

REMARKS 6. (i) Grothendieck uses $\varinjlim \mathrm{Pic}(U_\lambda)$, the direct limit taken over all open $U_\lambda \supseteq G$, instead of our $\mathrm{pic}(G)$.

(ii) His proof uses the Picard group of a formal scheme where we have used $\varprojlim \mathrm{pic}(U_i)$. These are frequently equal, in general [10, Exercise II 9.6]. The methods of Step 2 are based on those of [9, IX §2].

(iii) Grothendieck actually shows when α_1, α_2, and α_3 are injective or bijective using minimal hypothesis (see, for example, [8, Lemma 18.14]). This can be translated to some extent as well using Q and methods similar to the above. For example, in proving that $\alpha_2 \circ \alpha_1$ is injective, one uses the long exact sequence $0 \to Q(M) \to Q(M) \to Q(M/t^iM) \to R^1Q(M) \to \cdots$, where the first map is multiplication by t^i, to show $Q(M)/t^iQ(M) \cong Q(M/t^iM)$ for $i >> 0$. These ideas are left to the reader to explore, as more complicated techniques need be developed first (e.g., we found [1, §2] to be helpful).

Now for the main result [9, Corollaire XI 3.14].

THEOREM 7 (Grothendieck-Samuel). *If A is a complete intersection and A_p is a* UFD *for all primes $p \in \mathrm{spec}\, A$ of height $\leqq 3$, then A is a* UFD.

PROOF. We prove that each A_p is a UFD for all $p \in \mathrm{spec}\, A$ by inducting on the height of p. The cases $ht(p) \leqq 3$ being trivial by hypothesis, assume $ht(p) \geqq 4$, and for all $p' \subsetneq p$ that $A_{p'}$ is a UFD. Let $q \in \mathrm{spec}\, \hat{A}$ be minimal over $p\hat{A}$. Then $p = q \cap \hat{A}$, $A_p \to \hat{A}_q$ is flat, $p\hat{A}_q$ is primary to $q\hat{A}_q$. By Lemma 3, $\dim \hat{A}_q \geqq 4$, and it is clear that $\hat{A}_q$ is also a complete intersection. Theorem 5 gives that $\hat{A}_q$ is parafactorial, hence so is A_p by Lemma 3. Thus A_p is a UFD from Proposition 2.

EXAMPLES 8. (1) The height condition in Theorem 5 and Theorem 7 cannot be lowered. If $A = k[X, Y, U, V]_m/(XY - UV)$, where $m = (X, Y, U, V)$ and k is a field, then A is a complete intersection of dimension 3, and A_p is regular, hence a UFD, for $\mathrm{ht}(p) \leqq 2$ (at least one of the elements $x, y, u, v \in A$ becomes a unit in A_p). However, A is not a UFD since $xy = uv$ (or the ideal of A generated by x and u is a height one prime that is not principal), hence not parafactorial.

(2) In general, the complete intersection hypothesis cannot be replaced by Gorenstein in Theorem 5 and Theorem 7. Let $A = k[X_{ij}]_m/I$, $1 \leqq i \leqq 3$ and

$1 \leqq j \leqq 3$, where m is generated by all the indeterminates X_{ij} and I is generated by all the 2×2 minors of the (generic) 3×3 matrix $[X_{ij}]$. A is Gorenstein and $\operatorname{ht}(I) = \operatorname{gd}(I) = (3 - 2 + 1)^2 = 4$ [4, Corollary 8.9 and Theorem 2.5, respectively]. Thus, $\dim A = 9 - 4 = 5$. For $p \nsupseteq mA$, A_p is a regular local ring (same argument as in (1)), hence a UFD. Let J be the ideal of A generated by x_{11}, x_{21}, x_{31}. Then $A/J \cong k[Y_{ij}]_{m'}/I'$, $1 \leqq i \leqq 3$ and $1 \leqq j \leqq 2$, where m' is generated by all the (indeterminates) Y_{ij} and I' is generated by all the 2×2 minors of the 3×2 matrix $[Y_{ij}]$. Thus, $\operatorname{ht}(I') = \operatorname{gd}(I') = (3 - 2 + 1)(2 - 2 + 1) = 2$ and $\dim A/J = 6 - 2 = 4$, i.e., $\operatorname{ht}(J) = 5 - 4 = 1$. A/J is Cohen-Macaulay [11, Theorem 1] means depth $A/J = \dim A/J = 4 > 2$, and A/J is locally regular on the punctured spectrum, so by Serre's criterion for normality A/J is a normal domain and J is a height one prime of the normal domain A that is not principal. Hence A is not a UFD and not parafactorial.

(3) In general, $\mathcal{O}(G)$ and $Q_G(A)$ may not agree when G is a quasi-compact (cf. Theorem A-15). Let $A = S^{-1}\mathbf{Z}$ where $S = \mathbf{Z}\backslash(p) \cup (q)$, p and q distinct prime numbers. Set $m_1 = pA$ and $m_2 = qA$, $G = \{m_1, m_2\}$. Since G is discrete in the induced topology, $\tilde{M}(G) = M_{m_1} \times M_{m_2}$ for any A-module M. But $g(G) = X = \operatorname{spec} A$, so $Q_G(M) = Q_X(M) = M$. None-the-less, $\operatorname{Pic}(G) = \operatorname{pic}(G) = 0$.

Appendix.

It is in this appendix that we recall some facts from torsion theory and describe, for any commutative ring A, a module-theoretic formulation of the Picard group of a generically closed, quasi-compact subset of $X = \operatorname{Spec} A$ (the structure sheaf is that induced from the natural one $\mathcal{O}_X = \hat{A}$ on X by restriction). This formulation uses no sheaf theory and is reasonably easy to use in practice, so we consider a more general case as well.

For the torsion theory preliminaries, let A be any commutative ring, $G \subseteq X = \operatorname{spec} A$, with *generic closure* $g(G) = \{q \in X \mid q \subseteq p \text{ for some } p \in G\}$. Denote by $F_G = \{I \subseteq A \mid I \nsubseteq \text{any } p \in G\}$ the *torsion filter corresponding to* G (we may assume that G is generically closed, i.e., $G = g(G)$). G is quasi-compact if and only if F_G contains a cofinal subset of finitely generated ideals [6, Theorem 3.3].

F_G is a directed set under reverse inclusion so we can form, for any A-module M, the A-module $P_G(M) := \varinjlim_{I \in F_G} \operatorname{Hom}(I, M)$. Let $Q_G(M) := P_G(P_G(M))$ and $\mathcal{T}_G(M) := \{x \in M \mid \operatorname{Ann}_A x \in F_G\} \cong \varinjlim_{I \in F_G} \operatorname{Hom}(A/I, M)$, the *$\mathcal{T}_G$-torsion submodule of M*. M is *$\mathcal{T}_G$-torsion* if $\mathcal{T}_G(M) = M$ and *$\mathcal{T}_G$-torsionfree* if $\mathcal{T}_G(M) = 0$. Note that $\bar{M} := M/\mathcal{T}_G(M)$ is $\mathcal{T}_G$-torsionfree. We summarize some facts about $\mathcal{T}$ and Q in the following.

PROPOSITION A-1. (a) *$\mathcal{T}_G$ and Q_G are left exact idempotent functors* [6, *Theorem* 1.3 (1)].

(b) *There is a canonical map* $\varphi_M^G\colon M \to Q_G(M)$ *whose kernel is* $\mathcal{T}_G(M)$ *and cokernel is the first cohomology* $R^1\mathcal{T}_G(\bar{M})$. $Q_G(M) = \{x \in E(\bar{M}) \mid \exists I \in F_G, Ix \subseteq \bar{M}\}$ *where* $E(\bar{M})$ *is the injective envelope of* $\bar{M}$ [6, *Theorem* 1.3 (6′) *and* (3)].

(c) *If* $\alpha\colon M \to M'$ *is a homomorphism of* A*-modules, then* $\alpha \otimes_A A_p$ *is an isomorphism for each* $p \in G$ *if and only if* $\ker \alpha$ *and* $\operatorname{coker} \alpha$ *are* $\mathcal{T}_G$*-torsion, if and only if* $Q_G(\alpha)\colon Q_G(M) \xrightarrow{\cong} Q_G(M')$ *is an isomorphism. In particular,* $M_p \cong Q_G(M)_p$ *for any* $p \in G$ [6, *Theorem* 1.3 (4)].

(d) $Q(A)$ *is a commutative ring,* $Q(M)$ *is a* $Q(A)$*-module, and* $\operatorname{Hom}_A(M, Q(A)) = \operatorname{Hom}_A(Q(M), Q(A)) = \operatorname{Hom}_{Q(A)}(Q(M), Q(A))$ *for any* $Q = Q_G$ [6, p. 10–11, *Proposition* 4.1 (1) *and* (2)].

(e) *If* $G \supseteq H$ *then there is a unique map* $Q_G(M) \to Q_H(M)$ *compatible with* φ_M^G *and* φ_M^H. *Thus* $Q_H Q_G = Q_H = Q_G Q_H$ [6, *Proposition* 4.1 (2) *and* 4.5].

(f) *Given* $\alpha\colon M \to N$ *then* $N = Q_G(M)$ *if and only if* $\ker \alpha$ *and* $\operatorname{coker} \alpha$ *are* $\mathcal{T}_G$*-torsion,* N *is* $\mathcal{T}_G$*-torsionfree, and the natural map* $N = \operatorname{Hom}(A, N) \to \operatorname{Hom}(I, N)$ *is surjective for all* $I \in F_G$ [6, *Theorem* 1.3 (5), *Remarks* (i) *on* p. 7, *and Proposition* 1.5].

(g) *If* G *and* H *are generically closed, quasi-compact subsets of* $\operatorname{spec} A$ *then* $Q_G Q_H(M) = Q_{G \cap H}(M)$. *In particular,* $Q_G(M)_p = Q_{G \cap \operatorname{spec} A_p}(M_p)$ *for any* $p \in \operatorname{spec} A$ [6, *Theorem* A-6; *or* 19, *Corollary* 5.23].

(h) *If* A *is a domain then* $Q_G(A) = \bigcap_{p \in G} A_p$. *This follows easily from the description of* $Q_G(A)$ *in terms of* $E(A) = K$, *the quotient field of* A, *given in* (b) *above.*

(i) *If* A *is a noetherian ring then* $Q_G = P_G$ *for any subset* $G \subseteq \operatorname{spec} A$ [6, *Proposition* 6.3 (8)].

(j) *If* $\varphi\colon A \to B$ *is a homomorphism of noetherian rings,* $G \subseteq \operatorname{spec} A$, $G' = {}^a\varphi^{-1}(G) \subseteq \operatorname{spec} B$, *and* M *is a* B*-module, then* $Q_G(M) = Q_{G'}(M)$. *This is deduced by applying* [6, *Lemma* 7.3 *and Example* 10.4] *to part* (b) *above.*

We first investigate the notion of invertibility on a subset of $\operatorname{spec} A$ for an arbitrary commutative ring A.

DEFINITION A-2. Let $G \subseteq \operatorname{spec} A$. An A-module M is called *G-invertible* if there is a finitely generated submodule $N \subseteq M$ such that for all $p \in G$ we have $A_p \cong N_p = M_p$.

To show the needed equivalent conditions of G-invertible we use an idea of G. Picavet [16]. Let G be a quasi-compact subset of $X = \operatorname{spec} A$, M an A-module, t an indeterminate over A and M. Set $S_G = \{h \in A[t] \mid c(h) \in F_G\}$ where $c(h)$ is the ideal of A generated by the coefficients of the polynomial h. Let $\mathbf{G}(M) := S_G^{-1}M[t]$. Then the natural composition $A \to A[t] \to \mathbf{G}(A)$ is a flat ring homomorphism and $\mathbf{G}(M) = \mathbf{G}(A) \otimes_A M$ so that $\mathbf{G}$ is an exact functor. If $G = X$, write $M(t)$ in place of $\mathbf{X}(M)$ (cf. [15, p. 17–18]). Each maximal ideal of $\mathbf{G}(A)$ is of the form $p\mathbf{G}(A)$ where p is a (maximal) element of G [16, Lemma IV 11]. Then

$\mathbf{G}(M)_{p\mathbf{G}(A)} = M_p(t)$, and $\mathbf{G}$ vanishes precisely on the class of $\mathscr{T}_G$-torsion modules since each maximal element of the quasi-compact set G is the contraction of a maximal ideal of $\mathbf{G}(A)$ [16, Proposition IV 3].

We are ready for the key lemma, a generalization of [6, Theorem 8.2]. Recall that a topological space is *quasi-noetherian* [12] if it is quasi-compact and has a quasi-compact open basis (closed under finite intersections).

LEMMA A-3. *Let G be a quasi-compact subset of* spec A. *Then the following two conditions on the A-module M are equivalent.*

(1) *M is G-invertible.*

(2) a) *There is a complex $A^r \to A^s \to M \to 0$ which is exact at each $p \in G$, and*

b) *$M_p \cong A_p$ for all $p \in G$.*

These imply:

(3) *There is a finite covering of $G = \cup G_i$ by relatively open subsets G_i with $Q_{G_i}(M) \cong Q_{G_i}(A)$.*

Conversely, if the G_i can be chosen to be quasi-compact (e.g., *if G is quasi-noetherian*) *then* (3) *is equivalent to* (1) *and* (2).

PROOF. (1) $\Leftrightarrow$ (2) G-invertible means that 2 (b) holds and there is a complex $A^s \to M \to 0$ that is exact at each $p \in G$. So let $K = \ker(A^s \to M)$ and apply the exact functor $\mathbf{G}$ to obtain an *exact* sequence $0 \to \mathbf{G}(K) \to \mathbf{G}(A)^s \to \mathbf{G}(M) \to 0$. From the above remarks, $\mathbf{G}(M)$ is a locally free rank one $\mathbf{G}(A)$-module. It follows from the flatness of $\mathbf{G}(A)$ that there is a finitely generated A-submodule $L \subseteq K$ such that $\mathbf{G}(K/L) = \mathbf{G}(A) \otimes (K/L) = \mathbf{G}(A) \otimes K/\mathbf{G}(A) \otimes L = 0$, i.e., $K_p = L_p$ for all $p \in G$. Use this L to construct the desired complex 2 (a).

(1) $\Rightarrow$ (3) If $p \in G$, pick $x \in N$ that generates M at p, and map a rank one free onto $Ax \subseteq N \subseteq M$. Since N is finitely generated and M is locally free on G, there is a sufficiently small neighborhood $G_f = G \cap D(f)$ of p such that the composition $A \to N \subseteq M$ is locally an isomorphism at every $q \in G_f$. This composition induces the homomorphism $Q_{G_f}(A) \xrightarrow{\cong} Q_{G_f}(M)$ by (A-1c).

(3) $\Rightarrow$ (1) If $p \in G_i \subseteq G$ then $M_p = Q_i(M)_p \cong Q_i(A)_p = A_p$ by (A-1c) where we use the notation Q_i in place of Q_{G_i}. So we have that M is locally free on G. To construct a suitable $N \subseteq M$, let x be a generator of the cyclic $Q_i(A)$-module $Q_i(M)$. The hypothesis that G_i is quasi-compact implies that there is a finitely generated ideal $I \in F_{G_i}$ with $Ix \subseteq \bar{M} = \mathrm{Im}(M \to Q_i(M))$. Select a finitely generated A-submodule $N_i \subseteq M$ that maps onto Ix, (A-1b). This gives the diagram

$$\begin{array}{ccccc} N_i & \subseteq & M & & \\ \downarrow & & \downarrow & & \\ Ix & \subseteq & \bar{M} & \subseteq & Q_i(M). \end{array}$$

At each $p \in G_i$, all these maps become bijective so that N_i generates M locally at each such p. The finitely generated submodule $N = \Sigma N_i$, for the finite number of modules N_i so constructed, is the desired submodule.

COROLLARY A-4. *If G is quasi-compact and M is G-invertible then, for each $p \in G$, we have* $\mathrm{Hom}_A(M, Q_G(A))_p = \mathrm{Hom}_{A_p}(M_p, Q_G(A)_p) \cong A_p$.

PROOF. As in the end of the proof of [6, Theorem 8.2, p. 59–60] or [19, Lemma 5.28].

REMARK A-5. More generally, M can be locally of any (finite) constant rank on G in (A-3) and (A-4).

We now come to our definition of the Picard group of a quasi-compact subset pf the prime spectrum.

DEFINITION A-6. Let G be a quasi-compact subset of spec A. We shall say that two A-modules M and M' are *G-equivalent* ($M \sim_G M'$) if there is another module L, and maps $\varphi\colon M \to L$ and $\varphi'\colon M' \to L$ such that for all $p \in G$, $\varphi \otimes A_p\colon M_p \to L_p$ and $\varphi' \otimes A_p\colon M'_p \to L_p$ are isomorphisms. G-equivalence is indeed an equivalence relation since $M \sim_G M'$ if and only if $Q_G(M) \cong Q_G(M')$, by (A-1c) (or, to check transitivity, use the pushout $L_1 \oplus L_2/\{\varphi'_1(x) - \varphi'_2(x) \mid x \in M'\}$). We use $[M]_G$ or just $[M]$ to denote the equivalence class.

Let $\mathrm{pic}(G) = \{[M]_G \mid M$ is a G-invertible A-module$\}$. That $\mathrm{pic}(G)$ is a set follows from the mapping $M \to \Pi_{p \in G} M_p \cong \Pi A_p$ since M is G-equivalent to its image in the last module. The *picard group of* G is $\mathrm{pic}(G)$ with addition $[M_1]_G + [M_2]_G = [M_1 \otimes_A M_2]_G$, identity $[A]_G$, and inverse $-[M]_G = [\mathrm{Hom}(M, Q_G(A))]_G$. The verification of inverse requires special handling, but the other properties are straightforward and left to the reader.

First $\mathrm{Hom}(M, Q_G(A))$ is locally free rank one on G, by (A-4). We can replace G by its generic closure without loss of generality. Since M is assumed G-invertible, there is a (finite) cover of $G = \cup G_i$ by sets G_i such that $Q_i(M) \cong Q_i(A)$ (see (A-3)). We may assume that the G_i are quasi-compact since G is now quasi-noetherian. If $M \to Q_G(A)$ is any A-linear homomorphism then there is induced a homomorphism $Q_i(A) \cong Q_i(M) \to Q_i(Q_G(A)) = Q_i(A)$, by (A-1e). This mapping yields the commutative diagram, for each $p \in G_i$,

$$\begin{array}{ccc} \mathrm{Hom}(M, Q_G(A))_p & \to & \mathrm{Hom}(Q_i(M), Q_i(A))_p \\ \simeq\downarrow & & \simeq\downarrow \\ \mathrm{Hom}(M_p, Q_G(A)_p) & \xrightarrow{\simeq} & \mathrm{Hom}(Q_i(A)_p, Q_i(A)_p) \end{array}$$

The right hand vertical map is an isomorphism since $Q_i(M)$ is a free $Q_i(A)$-module and from (A–1d); the left hand from (A-4); the bottom from $p \in G_i \subseteq G$, $Q_i(M) \cong Q_i(A)$, and (A-1c and e). This proves $Q_i(\mathrm{Hom}(M, Q_G(A))$ $\xrightarrow{\simeq} Q_i(Q_i(A)) = Q_i(A)$ is an isomorphism. Since G is quasi-noetherian, it follows that $\mathrm{Hom}(M, Q_G(A))$ is G-invertible, (A-3). The natural homomorphism $M \otimes_A \mathrm{Hom}(M, Q_G(A)) \to Q_G(A) \sim_G A$ is locally an isomorphism at each $p \in G$, thus $[\mathrm{Hom}(M, Q_G(A))]_G$ is an inverse of $[M]_G$.

We should remark that an alternative formulation of $Q_G(A)$ is $\{g/h \in \mathbf{G}(A) \mid g = \Sigma a_i t^i,\ h = \Sigma b_i t^i,\ b_i(g/h) = a_i/1 \text{ for all } i\}$ for any quasi-compact G [6, Theorem A-7], but we do not exploit this representation here.

Observe that pic is natural in that given a ring homomorphism φ: $A \to B$ and quasi-compact subsets $G \subseteq \operatorname{spec} A$, $G' \subseteq \operatorname{spec} B$, $G \supseteq {}^a\varphi(G')$ then there exists a natural group homomorphism $\mathrm{pic}(G) \to \mathrm{pic}(G')$ given by sending $[M]_G$ to $[M \otimes_A B]_{G'}$.

Here are some variations of the above definition of pic that under certain hypotheses may be more suitable.

VARIATION A-7. The representative $Q_G(M) \in [M]_G$ is uniquely determined up to isomorphism. Thus it is clear we can use *isomorphism* classes $[[M]]$ where M is G-invertible and $M = Q_G(M)$, (A-1a). The identity is then $[[Q_G(A)]]$, $[[M_1]] + [[M_2]] = [[Q_G(M_1 \otimes M_2)]]$, but the inverse remains $[[\mathrm{Hom}(M, Q_G(A))]]$, by [6, Proposition 4.1 (3)]. In this case, the group homomorphism $\mathrm{pic}(G) \to \mathrm{pic}(G')$ mentioned above sends $[[M]]$ to $[[Q_{G'}(M \otimes_A B)]]$.

VARIATION A-8. In the inverse, we could have replaced $Q_G(A)$ by $\bar{A} = A/\mathscr{T}_G(A)$ by doing the following complicated maneuver: first choose a finitely generated $N \subseteq M$ that demonstrates M is G-invertible; then show $\mathrm{Hom}(N, \bar{A})$ is G-invertible and G-equivalent to $\mathrm{Hom}(M, Q(A))$, where we denote Q_G by Q. For the latter, set $T = Q(A)/\bar{A}$, a $\mathscr{T}_G$-torsion module, (A-1b). In the exact sequence $0 \to \mathrm{Hom}(N, \bar{A}) \to \mathrm{Hom}(N, Q(A)) \to \mathrm{Hom}(N, T)$, the last term is $\mathscr{T}_G$-torsion since N is finitely generated, so the first two terms are G-equivalent. But the middle term is isomorphic to $\mathrm{Hom}(Q(N), Q(A)) = \mathrm{Hom}(Q(M), Q(A)) = \mathrm{Hom}(M, Q(A))$, (A-1c and d). Demonstrating $\mathrm{Hom}(N, \bar{A})$ is G-invertible is easy now, for it amounts to proving the following: if φ_M: $M \to Q_G(M)$ is the canonical map and $Q_G(M)$ is G-invertible, so is M. But from the quasi-compactness of G, there is a finitely generated $I \in F_G$ with $IN \subseteq \bar{M}$, from which we can construct the necessary submodule of M.

VARIATION A-9. If $T = \mathscr{T}_G(A)$ is *bounded*, i.e., if there is an ideal $I \in F_G$ such that $IT = 0$, then we may use $[\mathrm{Hom}(N, A)]_G$ for the inverse. For then we have an exact sequence $0 \to \mathrm{Hom}(N, T) \to \mathrm{Hom}(N, A) \to \mathrm{Hom}(N, \bar{A}) \to \mathrm{Ext}^1(N, T)$, with $\mathrm{Ext}^i(N, T)$ being $\mathscr{T}_G$-torsion (it's killed by I) for $i = 0, 1$. Thus, $\mathrm{Hom}(N, A)$ and $\mathrm{Hom}(N, \bar{A})$ are G-equivalent.

While this variation includes the case when A is a domain, it also is true that we may use $[\mathrm{Hom}(M, Q_G(A))]_G$ where $Q_G(A) = \cap_{p \in G} A_p$, by (A-1h). A Krull domain, for example, is particularly nice if $G = \{p \in \operatorname{spec} A \mid \operatorname{ht}(p) = 1\}$, for then $Q_G(A) = A$.

VARIATION A-10. If A is noetherian, then we need only use the finitely generated M in our equivalence classes, with inverse $-[M]_G = [\operatorname{Hom}(M, A)]_G$. This is because the natural homomorphism $M \otimes \operatorname{Hom}_A(M, A) \to A$ is an isomorphism locally at each $p \in G$ (for *any* $G \subseteq \operatorname{spec} A$) when M is finitely generated and locally free rank one on G. Of course, $\operatorname{Hom}(M, A)$ is finitely generated locally free rank one, too. Our pic is then defined without any reference to Q, for any subset $G \subseteq \operatorname{spec} A$.

VARIATION A-11. If A is noetherian *and* G contains those primes p such that depth $A_p \leqq 1$ (but otherwise arbitrary) then we have the classical situation: isomorphism classes $[[M]]$ of finitely generated, reflexive A-modules which are locally free rank one on G. This happens because each representative of the isomorphism class in (A-7) is of this form, due to the relations $\operatorname{Hom}(N, Q_G(A)) = \operatorname{Hom}(M, Q_G(A))$, $Q_G(M) \cong \operatorname{Hom}(\operatorname{Hom}(M, Q_G(A)), Q_G(A))$ (checked locally on G and use [6, Proposition 4.1 (3)]), and $A = Q_G(A)$, which we prove in the next proposition. The addition is then $[[M_1]] + [[M_2]] = [[(M_1 \otimes M_2)^{**}]]$, the identity is $[[A]]$, and inverse is $-[[M]] = [[M^*]]$ where the dual is $\operatorname{Hom}_A(\ , A)$. Also, the natural group homomorphism $\operatorname{pic}(G) \to \operatorname{pic}(G')$ (with both G and G' satisfying the depth hypothesis) sends $[[M]]$ to $[[(M \otimes B)^{**}]]$ where the dual is $\operatorname{Hom}_B(\ , B)$.

If the double dual proves awkward in an application, we note that (A-6) could also be used, with the additional knowledge that every element in the equivalence class $[M]_G$ is finitely generated and that $Q_G(M) = M^{**}$ whenever M is G-invertible.

PROPOSITION A-12. *Let A be a noetherian ring, M an A-module. Let $H \subseteq G$ be two generically closed subsets of* $\operatorname{spec} A$ *such that for all $p \in G \backslash H$ either $M_p = 0$ or $0 \neq M_p$ is finitely generated and* depth $M_p \geqq 2$. *Then $Q_G(M) = Q_H(M)$.*

PROOF. If we do not have an isomorphism locally at each $p \in G$, choose a $p \in G$ of smallest height where $M_p = Q_G(M)_p \neq Q_H(M)_p = Q_{H'}(M_p)$, $H' = H \cap \operatorname{spec} A_p$, by (A-1g). By (A-1g and c), we can enlarge H' to be the punctured spectrum of A_p, by the choice of p, so we are reduced to the case of a local ring (A, m), $U = \operatorname{spec} A - m$, $0 \neq M$ finitely generated of depth $\geqq 2$, and $M \neq Q_U(M)$. But depth $M > 0$ implies $\operatorname{Hom}(A/m, M) = 0$, so $M = \bar{M}$. In regards to (A-1b), we see, for a non-zero-divisor $a \in m$ on M, that $\operatorname{Hom}(A/m, E(M)/M) \cong \operatorname{Ext}^1(A/m, M) \cong \operatorname{Hom}(A/m, M/aM) = 0$, since depth $M \geqq 2$. Thus $Q_U(M)/M = \mathscr{T}_U(E(M)/M) = 0$, a contradiction.

We try our formulation of pic on a known result for Krull domains.

Let $H_1 = \{p \in \operatorname{spec} A \mid \operatorname{ht}(p) = 1\}$. Recall that a domain A is *Krull* if (i) $A = \cap_{p \in H_1} A_p$, (ii) for each $p \in H_1$, A_p is a principal ideal domain (i.e., A_p is a DVR), and (iii) each $0 \neq a \in A$ is contained in only finitely many elements of H_1. It is clear that H_1 is a (quasi-)noetherian space.

PROPOSITION A-13. *Let A be a Krull domain, $X = \operatorname{spec} A$. Suppose $G \subseteq X$ is quasi-compact and contains H_1. Then*

(1) $\operatorname{pic}(X) \subseteq \operatorname{pic}(G) \subseteq \operatorname{pic}(H_1)$.

(2) *A is a* UFD *if and only if* $\operatorname{pic}(H_1) = 0$.

(3) *If each element in H_1 is finitely generated and if A_p is a* UFD *for each $p \in G$ then* $\operatorname{pic}(G) = \operatorname{pic}(H_1)$.

PROOF. (1) Set $H = H_1$. There are natural group homomorphisms $\operatorname{pic}(X) \to \operatorname{pic}(G) \to \operatorname{pic}(H)$ given by restricting the equivalence relations to smaller sets. So to prove injectivity, let $[M]_G \in \operatorname{pic}(G)$. If $[M]_H = [A]_H \in \operatorname{pic}(H)$ then $Q_H(M) \cong Q_H(A) = \cap_{p \in H} A_p = A$ since A is Krull, (A-1b). We check that the natural map $M \to Q_H(M)$ is locally an isomorphism at each $p \in G \supseteq H$, using $M_p \cong A_p$ in the commutative diagram

$$\begin{array}{ccccccc} & & M_p & \cong & A_p & & \\ & \swarrow & \downarrow & & \downarrow & \searrow^{\cong} & \\ Q_H(M)_p & = & Q_H(M_p) & \cong & Q_H(A_p) & = & Q_{H'}(A_p), \end{array}$$

where we have used (A-1g), and have put $H' := H \cap \operatorname{spec} A_p$, the set of height one primes of the Krull domain A_p. Then $M \sim_G Q_H(M) \cong A$ and our maps are injective.

(2) Now suppose $\operatorname{pic}(H) = 0$ and p is any height one prime. For A to be a UFD we need to show [8, Proposition 6.1] that p is principal (i.e., free rank one). It is easy to see from the above two defining properties (ii) and (iii) of a Krull domain that p is H-invertible, so that $\operatorname{pic}(H) = 0$ implies $Q_H(p) \cong Q_H(A) = A$. But [6, Lemma 8.9 (2)] says for any $I \subseteq A$ that $Q_H(I)$ is the intersection of all the symbolic powers (containing I) of height one primes. In particular $Q_H(p) = p$, hence p is principal for each $p \in H$ and A is a UFD.

Conversely, if A is a UFD, let $[[M]] \in \operatorname{pic}(H)$ (we use (A-7) here) with $M = Q_H(M)$. Then M embeds in $\Pi_{p \in H} M_p \cong \Pi A_p$, a torsionfree (usual) module over the domain A. So M is isomorphic to a non-zero A-submodule $I \subseteq M_{(0)} \cong K$, the quotient field of A. $I \neq 0$ implies $I^{-1} \cong I^* \neq 0$, (A-4). If $0 \neq x \in I^{-1}$ then $I \cong xI \subseteq A$ so we may assume M is isomorphic to an ideal I of A with $I = Q_H(I) = \cap p_i^{(n_i)}$. Now it is true in any domain that a symbolic power of a principal prime p_i is an ordinary power, and since there are no containment relations among the p_i's, their intersections will agree with their products, so I is principal, i.e., rank one free, and $\operatorname{pic}(H) = 0$.

(3) If A_q is a UFD for all $q \in G$, and $p \in H$ is finitely generated then p is

G-invertible and $[p]_G \in \text{pic}(G)$ maps to $[p]_H \in \text{pic}(H)$. On the other hand, we have seen earlier that each equivalence class in $\text{pic}(H)$ has a representative $I = Q_H(I) = \cap p_i^{(n_i)}$, $p_i \in H$. But $Q_H(\otimes_i p_i^{\otimes n_i}) = Q_H(\Pi p_i^{n_i}) = \cap p_i^{(n_i)} = I$ (check locally on H using A-1c)) so that $\text{pic}(H)$ is generated by the classes of height one primes and our map is surjective.

The reader may prove that indeed $\text{pic}(H_1)$ is just the *divisor class group of* A, the group of divisorial fractional ideals modulo the subgroup of principal fractional ideals. Also, $\text{pic}(X)$ ($= \text{Pic}(A)$) is just the *ideal class group of* A, the invertible fractional ideals modulo the principal's. For these results the following are useful: if $I^* \neq 0$ then $I^{**} \cong I^{-1-1}$; if I is not the quotient field K of A then $I \xrightarrow{\simeq} Q_H(I) \cong I^{**}$ if and only if $I = I^{-1-1}$; $I \cong J$ if and only if $I = xJ$ for some $0 \neq x \in K$; Q_X is the identity functor so $M \otimes M^* \xrightarrow{\simeq} A$ whenever M is X-invertible ($=$ rank one projective).

Our next goal is to show that, with sufficient hypothesis on $G \subseteq X = \text{spec}\, A$, $\text{pic}(G)$ is isomorphic to the algebraic geometers' $\text{Pic}(G)$, the group of isomorphism classes of invertible ($=$ locally free rank one) $\mathcal{O}_G$-Modules, where $\mathcal{O}_G$ is the natural structure sheaf $\mathcal{O}_X$ on X restricted down to G. There is a discussion of this in [19, Proposition 7.10], but a different approach (sheaf theoretic) to $\text{pic}(G)$ is used. Some of our ideas are from [19], but our method of proof is different in that we first prove an important conjectured result, a generalization of [19, 5.25].

LEMMA A-14. *If G is a quasi-compact subset of $X = \text{spec}\, A$ then for each A-module M we have $Q_G(M) = \varinjlim Q_U(M)$, where the direct limit* (*with the natural restriction maps of* (A-1e)) *is taken over all open subsets U of X containing G. Furthermore, these isomorphisms are compatible with the restriction maps $Q_G(M) \to Q_{G'}(M)$ for all quasi-compact G and G' with $G \supseteq G'$.*

PROOF. We need only use the quasi-compact open $U \supseteq G$ in the direct limit as they form a cofinal subset. The natural maps $\varphi_M^U\colon M \to Q_U(M)$ induce $\varphi\colon M \to \varinjlim Q_U(M)$. We need, by (A-1f), that $\ker \varphi$ and $\text{coker}\, \varphi$ are $\mathcal{T}_G$-torsion. This follows from (A-1c) since for each $p \in G \subseteq U$, $M_p \to Q_U(M)_p$ is an isomorphism, so that $\varphi \otimes A_p$ is an isomorphism, too.

We also need that this direct limit is $\mathcal{T}_G$-torsionfree. Let $[x] \in \varinjlim Q_U(M)$, and $I = (a_1, \ldots, a_n) \in F_G$ a finitely generated ideal such that $I[x] = 0$ (G is quasi-compact). Then each $a_i x$ maps to zero in the direct limit, hence to zero in some $Q_{U_i}(M)$, $U_i \supseteq G$. Thus $Ix = 0$ in $Q_V(M)$, $V = \cap U_i$. Since $I \in F_U \subseteq F_V$ and $Q_V(M)$ is always $\mathcal{T}_V$-torsionfree, $x = 0$ and we have proved this part.

It remains to prove that for any $I \in F_G$, a homomorphism $f\colon I \to \varinjlim Q_U(M)$ lifts to A. We can assume $I = (a_1, \ldots, a_n)$ is finitely generated. Map a free module A^n onto I with kernel $K \subseteq A^n$. Since I is U-invertible, $U = D(I)$, apply (A-3) to find a finitely generated $L \subseteq K$ such that K/L is $\mathscr{T}_U$-torsion (or prove this directly). Now lift the $f(a_i)$'s to x_i's in a common $Q_V(M)$, V open and $G \subseteq V \subseteq U$. Then the images of a finite generating set for L in $Q_V(M)$ will map to zero in the direct limit. Replacing V by a sufficiently smaller open set containing G, we can assume L maps to zero in $Q_V(M)$, hence the images of K and K/L are the same in $Q_V(M)$. However, K/L is $\mathscr{T}_U$-torsion, hence $\mathscr{T}_V$-torsion, while $Q_V(M)$ is $\mathscr{T}_V$-torsionfree. Thus K/L (hence K) maps to zero in $Q_V(M)$. Deduce from this that f factors through some $f'\colon I \to Q_V(M)$. But $I \in F_U \subseteq F_V$, so by the lifting property of $Q_V(M)$ there is a lifting of f' to A from which is obtained the desired lifting f'' of f.

$$\begin{tikzcd}
K \arrow[r, phantom, "\subseteq"] \arrow[d] & A^n \arrow[r, two heads] \arrow[d] & I \arrow[r, phantom, "\subseteq"] \arrow[dl, dashed, "f'"'] \arrow[d, "f"] & A \arrow[dl, dashed, "f''"] \\
K/L \arrow[r, "0"] & Q_V(M) \arrow[r] & \varinjlim Q_U(M) &
\end{tikzcd}$$

To show compatibility of our isomorphisms with the restriction maps, for $G \supseteq G'$ quasi-compact, look at the diagram

$$\begin{tikzcd}
Q_G(M) \arrow[rr, "\simeq"] \arrow[dd] & & \varinjlim\limits_{U \supseteq G} Q_U(M) \arrow[dd] \\
& M \arrow[ul] \arrow[ur] \arrow[dl] \arrow[dr] & \\
Q_{G'}(M) \arrow[rr, "\simeq"] & & \varinjlim\limits_{V \supseteq G'} Q_V(M)
\end{tikzcd}$$

which has commutative triangles. Since $\operatorname{Hom}(M, Q_{G'}(M)) = \operatorname{Hom}(Q_G(M), Q_{G'}(M))$ by (A-1d), the square commutes.

Recall that a quasi-compact, generically closed subset of spec A is always the image of the (reverse) spec map ${}^a\varphi$ of some flat ring homomorphism $\varphi\colon A \to B$, and conversely. Since these subsets form a basis of the closed sets of the flat topology [7, Theorem 2.2], we shall refer to them as the *flat subsets of spec A*. A flat subset G of spec A is quasi-noetherian since subsets of the form $G \cap U$ where U is a quasi-compact open subset of spec A is again quasi-compact.

For an arbitrary commutative ring A, let $\mathcal{O} = \mathcal{O}_X = \tilde{A}$ denote the natural structure sheaf of rings on $X = \operatorname{Spec} A$, and, for an A-module M, let $\tilde{M}$ be the $\mathcal{O}$-Module canonically associated to M. Let $G \subseteq \operatorname{Spec} A$. Recall that the definition of the restriction sheaf $\mathcal{O}_G := \mathcal{O}|_G$ and the $\mathcal{O}_G$-Module $\tilde{M}|_G$ involve direct limits of sections over open $U \supseteq G$ to define a presheaf, followed by the sheafification process [10, p. 65 and Proposition-Definition II 1.2].

THEOREM A-15. *Let G be a flat subset of $X = \operatorname{Spec} A$, M an A-module. Then, in the definition of $\mathcal{O}_G$ and $\tilde{M}|_G$, the sheafification process is not needed. More* *precisely, the presheaf assignment* $G_0 \mapsto \varinjlim \{\tilde{M}(U) \mid G_0 \subseteq U \text{ open } \subseteq X\}$ *defined on the basis of all flat relatively open subsets* G_0 *of* G *is, in fact, a sheaf on this basis, with* $\tilde{M}|_G(G \cap U) \cong \varprojlim \{Q_{G \cap U_0}(M) \mid G \cap U_0 \subseteq G \cap U, U_0 \text{ quasi-compact open } \subseteq X\}$ *for any open* $U \subseteq X$. *In particular,* $\tilde{M}(G) := \tilde{M}|_G(G) \cong Q_G(M)$ *and* $\mathcal{O}(G) := \mathcal{O}_G(G) \cong Q_G(A)$, *and these isomorphisms commute with restrictions.*

PROOF. For a discussion of (pre)sheaves on a basis, see [14, p. 32–34]. If U_0 is a quasi-compact open subset of X then, if G is flat, $G_0 = G \cap U_0$ is a quasi-compact relatively open subset of G. By [6, Theorem 5.1; or 19, Proposition 5.16] and (A-14), the assignment $G_0 \mapsto Q_{G_0}(M)$ defines our above stated presheaf (on the basis of quasi-compact relatively open subsets G_0 of G) for the sheaf $\tilde{M}|_G$. But this presheaf is actually a sheaf on this basis, as we now show.

(1) If $G_0 = \cup G_i$, $G_0 \subseteq G$, and $x \in Q_{G_0}(M)$ is such that x is zero in $Q_{G_i}(M)$ for each i, then $x \in \cap_i \mathcal{T}_{G_i}(Q_{G_0}(M)) = \mathcal{T}(Q_{G_0}(M)) = 0$, (A-1b and e). This establishes uniqueness.

(2) Let $G_0 = \cup G_i$, where G_0 and the G_i are flat relatively open subsets of G, and let $\{x_i\}$, $x_i \in Q_{G_i}(M)$, be a family of elements that agree on overlaps $G_i \cap G_j$. We want to find an $x \in Q_{G_0}(M)$ that will agree with the x_i's (x is unique by part (1)). We can assume the G_i are of the form $G \cap D(f_i)$, $f_i \in A$, and argue as in [6, Theorem A-2, p. 105] using this refined cover of G_0 (just as in that proof, one needs to use part (1) again to show that the chosen x agrees with the x_i's on the original cover $\{G_i\}$).

The compatibility follows from the facts (a) the presheaf defines the sheaf, (b) it is true for all quasi-compact open $U \subseteq X$, and (c) the last item in (A-14).

REMARKS A-16. It is not known if $\tilde{M}|_G \cong Q_G(M)$ when G is *not* quasi-compact, even if G is open in X. Nor has the case where G is not generically closed been studied in the literature (see Example 8.3 before the Appendix). Also, there still remains the question of whether $\tilde{M}|_G(G \cap U) = Q_{G \cap U}(M)$ when G is flat and U is *any* open subset of X.

PROPOSITION A-17. *Let* G *be a flat subset of* Spec A, *and* $\mathcal{L}$ *an invertible* $\mathcal{O}_G$*-module on* G. *Then* $\mathcal{L} \cong \widetilde{\mathcal{L}(G)}|_G$, *and for each* $x = p \in G$ *we have the stalk* $\mathcal{L}_x \cong \mathcal{L}(G)_p$.

PROOF. Our aim is to define the local maps of sections over a typical flat relatively open subset $G_0 \subseteq G$. First choose a cover of G of flat relatively open subsets G_i such that $\mathcal{L}|_{G_i} \cong \mathcal{O}_{G_i}$ (from the definition of an invertible sheaf). We have a commutative diagram, since $\mathcal{L}$ is a sheaf,

$$(*) \qquad \begin{array}{ccccccc} 0 \to & \mathcal{L}(G) & \to & \oplus \mathcal{L}(G_i) & \to & \oplus \mathcal{L}(G_i \cap G_j) \\ & \downarrow & & \downarrow & & \downarrow \\ 0 \to & \mathcal{L}(G_0) & \to & \oplus \mathcal{L}(G_0 \cap G_i) & \to & \oplus \mathcal{L}(G_0 \cap G_i \cap G_j) \end{array}$$

Now the commutative diagram (use (A-15))

$$\begin{array}{ccccc} \mathscr{L}(G_i) & \cong & \mathcal{O}(G_i) & \cong & Q_{G_i}(A) \\ \downarrow & & \downarrow & & \downarrow \\ \mathscr{L}(G_0 \cap G_i) & \cong & \mathcal{O}(G_0 \cap G_i) & \cong & Q_{G_0 \cap G_i}(A) \end{array}$$

yields upon application of Q_{G_0} that $Q_{G_0}(\mathscr{L}(G_i)) \cong \mathscr{L}(G_0 \cap G_i)$, by (A-1g). Similarly, $Q_{G_0}(\mathscr{L}(G_i \cap G_j)) \cong \mathscr{L}(G_0 \cap G_i \cap G_j)$. From (*), (A-15), and these isomorphisms, we conclude that $\widetilde{\mathscr{L}(G)}(G_0) \cong Q_{G_0}(\mathscr{L}(G)) \xrightarrow{\cong} \mathscr{L}(G_0)$. These maps will define an isomorphism $\widetilde{\mathscr{L}(G)} \xrightarrow{\cong} \mathscr{L}$ of sheaves (on a basis) if we show they are compatible with restrictions. To see this, let $G_0 \supseteq G_0'$. Then the commutative diagram

$$\begin{array}{ccc} \mathscr{L}(G) & \to & \mathscr{L}(G_0) \\ \| & & \downarrow \\ \mathscr{L}(G) & \to & \mathscr{L}(G_0') \end{array}$$

induces

$$\begin{tikzcd} \mathscr{L}(G) \arrow[rr] \arrow[dr] \arrow[ddd, equal] & & \mathscr{L}(G_0) \arrow[ddd] \\ & Q_{G_0}(\mathscr{L}(G)) \arrow[ur] \arrow[d] & \\ & Q_{G_0'}(\mathscr{L}(G)) \arrow[dr] & \\ \mathscr{L}(G) \arrow[rr] \arrow[ur] & & \mathscr{L}(G_0') \end{tikzcd}$$

We have just shown the upper and lower triangles are commutative, and the left side is (A-1e), so the right hand trapezoid is commutative, by (A-1d), whence the sheaf isomorphism is established.

Now take stalks at $x = p \in G$ to show $\mathscr{L}(G)_p \cong \mathscr{L}_x$.

THEOREM A-18. *Let G be a flat subset of $X = \operatorname{Spec} A$. Then* $\operatorname{pic}(G) \cong \operatorname{Pic}(G, \mathcal{O}_G)$.

PROOF. If $\mathscr{L}$ is an invertible $\mathcal{O}_G$-Module then (A-15) and (A-17) tell us that $Q_G(\mathscr{L}(G)) = \mathscr{L}(G)$. It is also clear that if $\mathscr{L} \cong \mathscr{L}'$ then $\mathscr{L}(G) \cong \mathscr{L}'(G)$. Thus, if we use (A-7), the map $\operatorname{Pic}(G) \to \operatorname{pic}(G) = \{[[M]] \mid M = Q_G(M), M \text{ is } G\text{-invertible}\}$ given by $[[\mathscr{L}]]$ goes to $[[\mathscr{L}(G)]]$ is well defined provided we show that $\mathscr{L}(G)$ is G-invertible. Cover $G = \cup G_i$, $G_i = G \cap D(f_i)$, $f_i \in A$, so that $\mathscr{L}|_{G_i} \cong \mathcal{O}_{G_i}$. Then $Q_{G_i}(\mathscr{L}(G)) \cong \mathscr{L}(G_i) \cong \mathcal{O}(G_i) \cong Q_{G_i}(A)$, by (A-15) and (A-17). From (A-3), we have that $\mathscr{L}(G)$ is G-invertible.

To see that the map is bijective, define a map in the reverse direction by sending the isomorphism class $[[M]]$ to $[[\tilde{M}|_G]] \in \operatorname{Pic}(G)$; of course, M is G-invertible and $M = Q_G(M)$. $\tilde{M}|_G$ is, indeed, an invertible sheaf since, for a suitable covering of $G = \cup G_i$, we have $Q_{G_i}(M) \cong Q_{G_i}(A)$, by (A-3). Hence, the canonical homomorphism $M \to Q_{G_i}(M)$ induces the isomorphism $\tilde{M}|_{G_i} \xrightarrow{\cong} \widetilde{Q_{G_i}(M)}|_{G_i} \cong \widetilde{Q_{G_i}(A)}|_{G_i} \xleftarrow{\cong} \tilde{A}|_{G_i} \cong \mathcal{O}_{G_i}$, and $\tilde{M}|_G$ is an invertible sheaf. These maps are inverses of each other since $\mathscr{L} \cong \widetilde{\mathscr{L}(G)}|_G$ by (A-17), and $\tilde{M}|_G(G) \cong Q_G(M) = M$, by (A-15).

Our mappings are group homomorphisms. To see this, let $[[\mathscr{L}]]$, $[[\mathscr{L}']] \in \operatorname{Pic}(G)$ and consider the morphism of preschemes to the sheafification $\mathscr{L} \otimes_{\mathcal{O}_G} \mathscr{L}'$. The homomorphism over the set G is then $\mathscr{L}(G) \otimes_{\mathcal{O}G)} \mathscr{L}'(G) \to (\mathscr{L} \otimes \mathscr{L}')(G)$. We claim this homomorphism of A-modulus is locally an isomorphism at each $p = x \in G$. It induces the commutative diagram

$$\begin{array}{ccc}
\mathscr{L}(G)_p \otimes_{\mathcal{O}(G)_p} \mathscr{L}'(G)_p \xrightarrow{\cong} (\mathscr{L}(G) \otimes_{\mathcal{O}(G)} \mathscr{L}'(G))_p & \to & (\mathscr{L} \otimes \mathscr{L}')(G)_p \\
\downarrow{\cong} & & \cong\downarrow \\
\mathscr{L}_x \otimes_{\mathcal{O}_x} \mathscr{L}'_x \qquad \xrightarrow{\cong} & & (\mathscr{L} \otimes \mathscr{L}')_x
\end{array}$$

The vertical maps are isomorphisms from (A-17), the lower one an isomorphism since a presheaf and its sheafification have the same stalks [10, p. 64]. From these isomorphisms, for each $p \in G$, we conclude $Q_G(\mathscr{L}(G) \otimes_A \mathscr{L}'(G)) \cong Q_G((\mathscr{L} \otimes \mathscr{L}')(G)) = (\mathscr{L} \otimes \mathscr{L}')(G)$ since $\mathscr{L} \otimes \mathscr{L}'$ is an invertible sheaf on G.

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