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ON A NON-ABELIAN VARIETY OF GROUPS WHICH ARE SYMMETRIC ALGEBRAS

ERNEST PŁONKA

I.

It is known that symmetric operations have nice properties and there are many types of algebraic systems in which commutative systems play very special role. This is the case, for example, of groups, rings and modules. From the algebraic point of view two algebras $(A; \mathbf{F}_1)$ and $(A; \mathbf{F}_2)$ are equal if the sets $A(\mathbf{F}_1)$ and $A(\mathbf{F}_2)$ of all their algebraic operations (= all superpositions of fundamental operations and the projections) coincide (cf. [2]). It may happen that not all fundamental operations $\mathbf{F}$ of the algebra $(A; \mathbf{F})$ are symmetric, i.e. do admit of all permutations of their variables but $(A; \mathbf{F})$ is a symmetric algebra. This means that all non-symmetric operations from $\mathbf{F}$ can be presented as a superpositions of symmetric algebraic operations of the algebra $(A; \mathbf{F})$ and the projections $e_n^k(x_1, x_2, \ldots, x_n) = x_k$, $1 \leqq k \leqq n$, $n = 1, 2, 3, \ldots$.

It is clear that the Abelian group $(G; ., ^{-1}, 1)$ is symmetric algebra in the sense. In [4] E. Marczewski has asked whether there are non-Abelian groups which are symmetric algebras. Since 0-ary and 1-ary operations are symmetric the question is whether the group operation $\cdot$ can be expressed as a superposition of projections and of symmetric algebraic operations, which, in the case of groups, are symmetric words. It turned out [5] that such group exists.

In this note we find a non-Abelian variety (= equationally definable class) of groups which have the same property.

We prove the following

THEOREM. *If the group G satisfies the following identities*

(1) $$x^6 = 1$$

(2) $$[x, y]^3 = 1$$

(3) $$[x^2 y^2] = 1$$

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then G is symmetric algebra. Namely we have

$$xy = w(q(q(x, y), y^4), w(w^4(x, y), s(x, y, x))), \tag{4}$$

where

$$w(x, y) = xy[x, y] \tag{5}$$

$$q(x, y) = xy[x, y]x^2y^2 \tag{6}$$

$$s(x, y, z) = [x, y, z][z, y, x] \tag{7}$$

are symmetric operations in the group G.

II.

Let us begin with notations and auxiliary results which enable us to prove the theorem. As usual, $[x, y] = x^{-1}y^{-1}xy$, and

$$[x_1, x_2, \ldots, x_n] = [[x_1, x_2, \ldots, x_{n-1}], x_n] \quad \text{for} \quad n > 2.$$

The following relations

$$xy = yx[x, y] \tag{8}$$

$$[x, y]^{-1} = [y, x] \tag{9}$$

$$[xy, z] = [x, z][x, z, y][y, z] \tag{10}$$

$$[x, yz] = [x, z][x, y][x, y, z] \tag{11}$$

are identities in any group [cf. [1]).

LEMMA 1. *If* (1)–(3) *are identities in G, then the following equations*

$$[[x, y], z^2] = 1 \tag{12}$$

$$[[x, y], [z, u]] = 1 \tag{13}$$

$$[y, x, z] = [x, y, z]^2 \tag{14}$$

$$[y, x, y, x] = [x, y, y, x]^2 \tag{15}$$

$$[y, x, y, x] = [y, x, x, y] \tag{16}$$

$$[y, x, x, y, y] = [y, x, x, y] \tag{17}$$

$$[y, x, y, x, x] = [y, x, y, x] \tag{18}$$

$$[y, x, x, x] = [y, x, x] \tag{19}$$

$$[x, y^{2k}, y] = 1 \tag{20}$$

are identities in G for $k = 1, 2, 3, \ldots$

PROOF. If we take $[x,z]^2$ instead of x into (3) and apply (2) we get (12). This implies (13) by putting $[z,u]^2$ instead of z. Using (2), (10) and (13) we have

$$[y,x,z] = [[x,y]^{-1},z] = [[x,y]^2,z]$$
$$= [x,y,z][[x,y,z],[x,y]][x,y,z] = [x,y,z]^2,$$

which yields (14).

From (3), (10), (12) and (14) it follows

$$[y,x,y,x] = [[x,y,y]^2,x] = [x,y,y,x]^2$$

which gives (15).

It is known (cf. e.g. [1]) that in metabelian groups, i.e. in the groups with identity (12), the following Jacobi identity

$$[x,y,z][y,z,x][z,x,y] = 1 \tag{21}$$

holds. Thus we have

$$1 = [x,y,[x,y]][y,[x,y],x][x,y,x,y]$$

which together with (2), (3), (9), (10), (12) and (14) gives

$$[x,y,x,y] = [y,[x,y],x]^{-1} = [[x,y,y]^2,x]^2 = [x,y,y,x]^4 = [x,y,y,x]$$

i.e. equality (16) is fulfiled.

The equalities (17), (18) and (19) follow from (2), (11) and (12), because we have

$$1 = [y,x,x,y^2] = [y,x,x,y]^2[y,x,x,y,y]$$
$$1 = [y,x,y,x^2] = [y,x,y,x]^2[y,x,y,x,x]$$
$$1 = [y,x,x^2] = [y,x,x]^2[y,x,x,x].$$

Now the Jacoby identity yields

$$1 = [x,y^{2k},y][y^{2k},y,x][y,x,y^{2k}],$$

which together with (12) gives (20) and lemma 1 follows.

LEMMA 2. *If the equations* (1)–(3) *are identities in the group G, then the words w, q and s defined by the formulas* (5), (6) *and* (7), *respectively, are symmetric operations in G.*

PROOF. By (2), (8) and (9) we have

$$w(y,x) = yx[y,x] = xy[y,x]^2 = xy[x,y] = w(x,y)$$

Using this and (12) we get

$$q(y,x) = yx[y,x]y^2x^2 = w(y,x)y^2x^2 = w(x,y)x^2y^2 = q(x,y).$$

To prove s is ternary symmetric operation in G observe that the cycles $(2, 3, 1)$ and $(1, 3)$ generate the symmeetric group S_3 of all permutations on three leters x, y and z. Now from (13) we have

$$s(z, y, x) = [z, y, x][x, y, z] = s(x, y, z)$$

We have also by (2), (13), (14) and (21)

$$s(y, z, x) = [y, z, x][x, z, y] = [z, y, x][z, y, x][x, z, y]$$

$$= [z, y, x][y, x, z]^{-1} = [x, y, z][z, y, x] = [x, y, z],$$

which completes the proof.

III. Proof of the theorem.

First of all we calculate $q^2(x, y)$. Using (8) we get

$$q^2(x, y) = xy[x, y]x^2y^2xy[x, y]x^2y^2$$

$$= x^2y[x, y, x]x^2y^2[y^2, x]y[x, y]x^2y^2$$

Observe that $[y^2, x] = [x, y^2]^2$, because of (2) and (9). This together with (20) gives $[y^2, x]y = y[y^2, x]$. Hence, once more from (8), we get

$$q^2(x, y) = x^2y^2[x, y, x][x, y, x, y]x^2[x^2, y]y^2[y^2, x][x, y]x^2y^2$$

Now the equality (13), (9) and (10) yield

$$[x^2, y][y^2, x] = [x, y][x, y, x][x, y][y, x][y, x, y][y, x] = [x, y, x][y, x, y].$$

Therefore, in view of (1), (2), (3), (12) and (13) we get

$$q^2(x, y) = [x, y][x, y, x]^2[y, x, y][x, y, x, y]$$

Now we are going to calculate $a = q(q(x, y), y^4)$. Taking into account (1), (3), (6), (10), (12) and (13) we get

$$a = xy[x, y]x^2y^2y^4[xy[x, y]x^2y^2, y^4]q^2(x, y)y^8$$

$$= xyx^2y^2[x, y][xy, y^4]q^2(x, y)$$

It follows from (10) and (13) that

$$[x, y^2] = [x, y]^2[x, y, y] \tag{22}$$

This together with (2), (10) and (13) gives

$$[xy, y^4] = [x, y^4][x, y^4, y] = [x, y^4] = [x, y^2]^2 = [x, y][x, y, y]^2$$

Therefore we have

$$a = x^3y[y,x^2]y^2[x,y]^2[x,y,y]^2q^2(x,y) = x^3y^3[y,x,x][x,y,y]^2q^2(x,y),$$

as a consequence of (8), (9) and (13). Now using (13) and (14) we obtain

$$a = x^3y^3[x,y][x,y,y][x,y,x][x,y,x,y] = x^3y^3c_1$$

for a suitable product c_1 of commutators. It follows from (13) and the definitions of the operations s and w that

$$b = w(w^4(x,y), s(x,y,x)) = (xy[x,y])^4[x,y,x]^2$$

Thus, in viewof (2), (3), (6) and (9), we get

$$(xy[x,y])^4 = (xy[x,y]xy[x,y])^2 = (x^2y[y,x][x,y][x,y,x]y[x,y])^2$$
$$= (x^2y^2[x,y][x,y,x][x,y,x,y])^2 = x^4y^4[y,x][x,y,x]^2[x,y,x,y]^2$$

and consequently

$$b = x^4y^4[y,x][x,y,x][x,y,x,y]^2 = x^4y^4c_2$$

for a suitable product c_2 of commutators.

In order to calculate $[a,b]$ let us consider the commutator $[x^3y^3c_1,\alpha]$, α being x or y or else c_2. In view of (3), (6) and (10) we have

$$[x^3y^3c_1,\alpha] = [xyc_1,\alpha][xy,\alpha] = [x,\alpha][x,\alpha,y][y,\alpha]$$

This together with (11) and (20) yields

$$[a,b] = [x^3y^3c_1, x^4y^4c_2] = [x^3y^3c_1, x^4][x^3y^3c_1, y^4][x^3y^3c_1, c_2]$$
$$= [y,x^4][x,y^4][x,c_2][x,c_2,y][y,c_2]$$

Now by (3) and (22) we have

$$[y,x^4] = [y,x^2]^2 = ([y,x]^2[y,x,x])^2 = [y,x]^4[y,x,x]^2$$

and similarly

$$[x,y^4] = [x,y]^4[x,y,y]^2$$

which together with (13) and (14) gives

$$[a,b] = [x,y,x][y,x,y][x,c_2][x,c_2,y][y,c_2]$$

It follows from (2), (11) and (13) that

$$[x,c_2] = [y,x,x]^2 = [x,y,x,x]^2[[x,y,x,y]^2,x]^2,$$

which, in view of (14), (15), (16) and (18), can be rewrite as

$$[x,c_2] = [x,y,x][y,x,x][y,x,x,y,x]^2 = [x,y,y,x]$$

Thus we get

$$[x, c_2, y] = [x, y, y, x, y] = [x, y, x, y, y] = [x, y, x, y]$$

as it follows from (16) and (18).

Using the same arguments we have also

$$[y, c_2] = [y, x, y]^2[x, y, x, y]^2[[x, y, x, y]^2, y]^2$$
$$= [x, y, y][y, x, x, y][y, x, x, y]^2 = [x, y, y].$$

Thus

$$[a, b] = [x, y, x][y, x, y][x, y, y, x][x, y, x, y][x, y, y]$$
$$= [x, y, x][y, x, y, x]$$

as it follows from (2), (3), (14), (15) and (16).

Now we are able to calculate $w(a, b)$. The identities (1)–(3) and (8) give

$$w(a, b) = ab[a, b]$$
$$= x^3y^3[x, y][x, y, x][x, y, y][x, y, x, y]x^4y^4[y, x][x, y, x][x, y, x, y]^2ab$$
$$= x^3y^3[x, y, x]^2x^4y^4[x, y, x][x, y, y][y, x, y, x] = x^3yx^4[x, y, y][y, x, y, x]$$
$$= xy[x^2, y][x, y, y][y, x, y, x]$$

Since $[x^2, y] = [x, y]^2[x, y, x]$ we have, by (16),

$$w(a, b) = xy[y, x][x, y, x][x, y, y][y, x, x, y]$$

Thus, it is enough to prove that the last product of commutators equals 1. To do this we use (2), (3), (10) and (11). We have

$$1 = [y^2, x^2] = [y, x^2]^2[y, x^2, y] = ([y, x]^2[y, x, x])^2[[x, y][y, x, x], y]$$
$$= [y, x][x, y, x][x, y, y][y, x, x, y],$$

which completes the proof of the theorem.

COROLLARY. *In the normal product Z_3Z_2 of the cyclic group Z_3 by the group Z_2 of all its automorphisms* (i.e. *the group S_3*) *all equations* (1)–(3) *are fulfilled. This gives another proof of a result from* [5].

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INSYTITUTE OF MATHEMATICS
THE SILESIAN TECHNICAL UNIVERSITY OF GLIWICE
UL. KASZUBSKA 23
44–100 GLIWICE,
POLAND