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ON EMBEDDINGS OF PROPER SMOOTH G-MANIFOLDS

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By a linear Lie group we mean a Lie group isomorphic to a closed subgroup of a general linear group. A euclidean space R^n equipped with the linear action of G via some representation $\varrho\colon G \to \mathrm{Gl}(n, \mathsf{R})$ is denoted by $\mathsf{R}^n(\varrho)$ and called a linear G-space. If M is a smooth, i.e., a C^∞-differentiable manifold and the action $G \times M \to M$ is smooth, we call M a smooth G-manifold. In case the mapping $G \times M \to M \times M$, $(g, x) \mapsto (gx, x)$, is proper, i.e., if the inverse image of every compact set is compact, we call M a proper smooth G-manifold. This definition of properness is equivalent to the definition used in [Pa3]. The purpose of this paper is to prove the following result:

THEOREM. *Let G be a linear Lie group and M a proper smooth G-manifold having only finitely many orbit types. Then there exists a G-equivariant, closed smooth embedding of M into some linear G-space.*

Suppose for a moment G is an arbitrary Lie group. Let G act on itself via multiplication on the left. This action makes G a proper smooth G-manifold having only one orbit type. Assume there exists a G-equivariant topological embedding f of the G-manifold G into a linear G-space $\mathsf{R}^n(\varrho)$. Then $f(g) = \varrho(g)f(e)$ for every $g \in G$ where e is the identity element of G. Since f is injective it follows that also $\varrho\colon G \to \mathrm{Gl}(n, \mathsf{R})$ is injective. It now follows from Proposition 5.1.2 in [Pr] that ϱ is a smooth immersion. Since the mappings $\varrho(G) \to \mathsf{R}^n(\varrho)$, $\varrho(g) \mapsto \varrho(g)f(e)$, and $f(G) \to G$, $f(g) \mapsto g$, are continuous, it follows that their composition $\varrho(G) \to G$, $\varrho(g) \mapsto g$, is continuous. Theorem II 2.10 in [He] finally implies that $\varrho(G)$ is closed in $\mathrm{Gl}(n, \mathsf{R})$, i.e., that G is a linear Lie group. Thus we see that in the previous theorem it is necessary to assume G is a linear Lie group.

The smooth embedding is constructed essentially in the same way as the topological embedding in [Pa3] and the subanalytic embedding in [Ka]. It will

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be used in a paper to appear where we will prove the real analytic version of the theorem by using G-equivariant real analytic approximations.

If $x \in M$ we denote its isotropy subgroup by G_x. We note that if G acts properly on M, then G_x is compact for every $x \in M$. For a subgroup H of G we denote $(H) = \{gHg^{-1} \mid g \in G\}$ and $M_{(H)} = \{x \in M \mid (G_x) = (H)\}$. The set (G_x) is called the orbit type of x. Let S be a G_x-invariant smooth submanifold of M containing x. If GS is an open subset of M and there exists a G-equivariant smooth mapping $f\colon GS \to G/G_x$ such that $S = f^{-1}(eG_x)$, then we call S a slice at x and GS a tube at x. It has been proven in [Pa3] that if M is a proper smooth G-manifold, then there exists a slice at every $x \in M$.

Suppose S is a slice at x. Let $g_0 \in G$, U be an open neighbourhood of g_0G_x in G/G_x and $\sigma\colon U \to G$ be a local cross section. It follows from Proposition 2.1.2 in [Pa3] that the mapping $F\colon U \times S \to V, (u, s) \mapsto \sigma(u)s$, is a homeomorphism onto an open neighbourhood V of g_0S. In fact, F is a diffeomorphism with the inverse mapping $F^{-1}\colon V \to U \times S, gs \mapsto (gG_x, \sigma(gG_x)^{-1}gs)$.

1. LEMMA. *Assume G is a Lie group, M a smooth G-manifold, $x \in M$ such that G_x is compact and S a slice at x. Let A and B be disjoint, closed G-invariant subsets of GS. Then there exists a G-invariant smooth mapping $f\colon GS \to [0,1]$ such that $f \mid A = 0$ and $f \mid B = 1$.*

PROOF. We first remark that $A \cap S$ and $B \cap S$ are disjoint, closed G_x-invariant subsets of S. Thus there is a smooth mapping $f_1\colon S \to \mathsf{R}$ such that $f_1(y) = 0$ when $y \in A \cap S$ and $f_1(y) = 1$ when $y \in B \cap S$. By Theorem 0.3.3 in [Br] the mapping $f_2\colon S \to \mathsf{R}, y \mapsto \int_{G_x} f_1(gy)dg$, is smooth. Obviously, $f_2(y) = 0$ when $y \in A \cap S$ and $f_2(y) = 1$ when $y \in B \cap S$. Let $f\colon GS \to \mathsf{R}, gs \mapsto f_2(s)$. Let $g_0 \in G$ and U, V, σ and F be as in the previous paragraph. Let $p\colon U \times S \to S$ be the projection. Then $f \mid V = f_2 \circ p \circ F^{-1}$ is smooth as composite of smooth mappings. Since g_0 was arbitrary, it follows that f is smooth. Clearly, f is G-invariant.

2. PROPOSITION. *Let G be a Lie group, M a proper smooth G-manifold and A and B disjoint, closed G-invariant subsets of M. Then there exists a G-invariant smooth mapping $f\colon M \to [0,1]$ such that $f \mid A = 0$ and $f \mid B = 1$.*

PROOF. Let $\{GS_i\}_{i=1}^{\infty}$ be a cover of M by tubes. Since M/G is paracompact by Theorem 4.3.4 in [Pa3], $\{GS_i\}_{i=1}^{\infty}$ has locally finite refinements $\{W_i\}_{i=1}^{\infty}$ and $\{V_i\}_{i=1}^{\infty}$ by open G-invariant sets W_i and V_i, respectively, such that $\overline{V}_i \subset W_i$ and $\overline{W}_i \subset GS_i$ for every i. Then $B \cap \overline{V}_i$ and $W_i \backslash A$ are G-invariant subsets of GS_i, $W_i \backslash A$ is open, $B \cap \overline{V}_i \subset W_i \backslash A$ and the closure of $W_i \backslash A$ is a subset of GS_i for every i. Thus, by Lemma 1 there exists for every i a G-invariant smooth mapping $f_i'|\colon GS_i \to [0,1]$ such that $f_i'|(GS_i \backslash (W_i \backslash A)) = 0$ and $f_i'|(B \cap \overline{V}_i) = 1$. We extend f_i' to $f_i\colon M \to [0,1]$ by setting $f_i(y) = 0$ when $y \in M \backslash GS_i$ and $f_i(y) = f_i'(y)$ when

$y \in GS_i$. Then f_i is G-invariant, and since $\overline{W}_i \subset GS_i$, it follows that f_i is smooth. Since $\{\operatorname{supp} f_i\}_{i=1}^{\infty}$ is locally finite, it follows that $f_B\colon M \to \mathsf{R}$, $x \mapsto \sum_{i=1}^{\infty} f_i(x)$, is smooth. Clearly, f_B is G-invariant and non-negative, $f_B \mid A = 0$ and $f_B(x) > 0$ for every $x \in B$.

Let A' and B' be closed G-invariant neighbourhoods of A and B, respectively, such that $A' \cup B' = M$, $B \cap A' = \emptyset$ and $A \cap B' = \emptyset$. Then there exist non-negative G-invariant smooth mappings $f_{B'}$, $f_{A'}\colon M \to \mathsf{R}$ such that $f_{B'} \mid A = 0$, $f_{B'}(x) > 0$ for every $x \in B'$, $f_{A'} \mid B = 0$ and $f_{A'}(x) > 0$ for every $x \in A'$. Since $f_{A'}(x) + f_{B'}(x) \neq 0$ for every $x \in M$, the mapping

$$f\colon M \to [0,1], \qquad x \mapsto \frac{f_{B'}(x)}{f_{A'}(x) + f_{B'}(x)},$$

is well-defined. Since f is smooth and G-invariant, $f \mid A = 0$ and $f \mid B = 1$, the proposition follows.

3. PROPOSITION. *Assume G is a linear Lie group, M a proper smooth G-manifold and $x \in M$. Then there exists a slice S at x such that the tube GS admits a G-equivariant smooth embedding in a linear G-space.*

PROOF. Let S_0 be a relatively compact slice at x. Then, by Proposition IV 1.2 in [Br], S_0 only has finitely many orbit types when regarded as a G_x-space by restriction. It has been proven in [Mo] and in [Pa1] that there exists a representation $\varrho_0\colon G_x \to \mathrm{Gl}(n, \mathsf{R})$ for some $n \in \mathsf{N}$ and a G_x-equivariant smooth embedding $j_0\colon S_0 \to \mathsf{R}^n(\varrho_0)$. According to Theorem 3.1 in [Pa3], there exists a representation $\varrho\colon G \to \mathrm{Gl}(p, \mathsf{R})$ for some $p \geqq n$ and a linear G-space $\mathsf{R}^p(\varrho)$ which, considered as a linear G_x-space by restriction, contains $\mathsf{R}^n(\varrho_0)$ as an invariant linear subspace. Therefore we can regard j_0 as an embedding in $\mathsf{R}^p(\varrho)$.

Since G is a linear Lie group, Theorem 3.2 in [Ka] implies that there exists a representation $\psi\colon G \to \mathrm{Gl}(q, \mathsf{R})$ for some $q \in \mathsf{N}$ and a point $v \in \mathsf{R}^q(\psi)$ such that $G_v = G_x$ and the mapping $G/G_x \to \mathsf{R}^q(\psi)$, $gG_x \mapsto \psi(g)v$, is a closed smooth, in fact a real analytic, embedding. We define

$$j\colon GS_0 \to \mathsf{R}^{p+q}(\varrho \oplus \psi), \qquad gs \mapsto (\varrho(g)j_0(s), \psi(g)v).$$

Since j_0 is G_x-equivariant and injective, it immediately follows that j is G-equivariant and injective.

Let $g_0 \in G$ and $\sigma\colon U \to G$ be a local cross section at $g_0 G_x$. The mapping $F_0\colon U \times S_0 \to V_0$, $(u,s) \mapsto \sigma(u)s$, is a diffeomorphism onto an open neighbourhood V_0 of $g_0 S_0$. Also $h\colon U \times S_0 \to U \times j(S_0)$, $(u,s) \mapsto (u, j(s))$, is a diffeomorphism. Since easily $j(S_0)$ is a topological slice at $j(x)$ in the G-space $j(GS_0)$ the mapping $F\colon U \times j(S_0) \to V$, $(u, j(s)) \mapsto \sigma(u)j(s)$, is a homeomorphism onto an open neighbourhood V of $j(g_0 S_0)$ in $j(GS_0)$. Clearly F is smooth. Then $j \mid V_0 =$

$F \circ h \circ F_0^{-1}$ is a smooth homeomorphism onto V. Since g_0 was chosen arbitrarily it follows that j is smooth and $j^{-1}: j(GS_0) \to GS_0, j(gs) \mapsto gs$, is continuous.

The restriction $j \mid S_0$ is a smooth embedding. Since the mapping $Gx \to G/G_x$, $gx \mapsto gG_x$, is a smooth diffeomorphism (see Proposition 1.1.5 in [Pa3] and Theorem VI 1.2 in [Br]) and the mapping $G/G_x \to \mathrm{R}^{p+q}(\varrho \oplus \psi), gG_x \mapsto (\varrho(g)j_0(x), \psi(g)v)$, is a smooth embedding it follows that the restriction $j \mid Gx$ is a smooth embedding. Let $y = (y_1, y_2) \in T_x GS_0 = T_x S_0 \oplus T_x Gx$ and let $dj_x(y) = 0$. Let $j^1: GS_0 \to \mathrm{R}^p(\varrho)$, $gs \mapsto \varrho(g)j_0(s)$, and $j^2: GS_0 \to \mathrm{R}^q(\psi)$, $gs \mapsto \psi(g)v$. Then $dj_x^1(y_1) + dj_x^1(y_2) = 0$ and $dj_x^2(y_1) + dj_x^2(y_2) = 0$. Since $j^2 \mid S_0$ is a constant mapping, it follows that $dj_x^2(y_1) = 0$. Thus also $dj_x^2(y_2) = 0$. Since $dj_x^2 \mid T_x S_0 \oplus T_x Gx$ is injective, it follows that $y_2 = 0$. Since $dj_x^1 \mid T_x S_0$ is injective, it follows that also $y_1 = 0$. Thus $y = 0$, which implies that dj_x is injective. Therefore x has an open neighbourhood W in GS_0 such that $j \mid W$ is an immersion. Now, $S = GW \cap S_0$ is a slice at x and $GS = GW$. Obviously, $j \mid GS$: $GS \to \mathrm{R}^{p+q}(\varrho \oplus \psi)$ is a G-equivariant smooth embedding.

We next show that for each orbit type (H_i), $i = 1, \ldots, m$, in M there exists a representation $\varrho_i: G \to \mathrm{Gl}(q_i, \mathrm{R})$ such that every $x \in M_{(H_i)}$ has a tube which admits a G-equivariant smooth embedding in $\mathrm{R}^{q_i}(\varrho_i)$. The representations ϱ_i are constructed in Lemma 4. In Lemma 7 they are used in showing that there exists a representation $\varrho: G \to \mathrm{Gl}(q, \mathrm{R})$ for some $q \in \mathrm{N}$, such that M can be covered with finitely many open sets each of which admits a G-equivariant smooth embedding in $\mathrm{R}^q(\varrho)$. Finally, the embedding of M is constructed by using Lemma 7 and Proposition 2. Lemma 5 and Corollary 6 are needed to make the embedding of M closed.

4. LEMMA. *Suppose G is a linear Lie group and M a proper smooth G-manifold with only finitely many orbit types. Suppose H is a compact subgroup of G. Then there exists a representation $\nu: G \to \mathrm{Gl}(n, \mathrm{R})$ of G for some $n \in \mathrm{N}$ with the following property: If $x \in M_{(H)}$, there is a slice S_x at x such that the tube GS_x has a G-equivariant smooth embedding in $\mathrm{R}^n(\nu)$.*

PROOF. Proposition 4.4.2 in [Pa3] yields that M only has finitely many orbit types when regarded as an H-space by restriction. Let $\varphi: H \to O(m)$ be a representation for some $m \in \mathrm{N}$ such that there exists an H-equivariant smooth embedding $f: M \to \mathrm{R}^m(\varphi)$. The existence of f follows from [Mo] and [Pa1]. As in Proposition 3 we can consider f as an embedding in some linear G-space $\mathrm{R}^p(\varrho)$.

Let $x \in M$ be such that $G_x = H$ and let S'_x be a relatively compact slice at x. Let $\psi: G \to \mathrm{Gl}(q, \mathrm{R})$ and $v \in \mathrm{R}^q(\psi)$, where $q \in \mathrm{N}$, be such that the mapping $G/H \to \mathrm{R}^q(\psi)$, $gH \mapsto \psi(g)v$, is a closed smooth embedding. Proposition 3 implies that there exists a slice $S_x \subset S'_x$ at x such that $j_x: GS_x \to \mathrm{R}^{p+q}(\varrho \oplus \psi)$, $gs \mapsto (\varrho(g)f(s), \psi(g)v)$, is a G-equivariant smooth embedding. For every $g \in G$, gS_x

is a slice at gx and $G(gS_x) = GS_x$. Thus j_x embeds also $G(gS_x)$ and the lemma follows.

5. LEMMA. *Let G be a linear Lie group, H a compact subgroup of G and M a proper smooth G-manifold. Then there exists a representation ψ: $G \to \mathrm{Gl}(k, \mathsf{R})$ for some $k \in \mathsf{N}$ with the following property: If $x \in M_{(H)}$, S_x is a slice at x and K_x is a compact subset of S_x, then there exists a G-equivariant smooth mapping h_x: $GS_x \to \mathsf{R}^k(\psi)$ whose restriction to GK_x is proper.*

PROOF. Let $x \in M$ be such that $G_x = H$. The mapping f_x: $GS_x \to G/H$, $gs \mapsto gH$, is smooth. Let $f_x|$ be the restriction of f_x to GK_x and ϕ_x the restriction of the group action mapping to $G \times K_x$. Since the projection p_x: $G \times K_x \to G$ and the natural projection π: $G \to G/H$ are proper mappings, it follows that $f_x| \circ \phi_x = \pi \circ p_x$ is proper. Since $\phi_x(G \times K_x) = GK_x$ it follows that $f_x|$ is proper. Let f: $G/H \to \mathsf{R}^k(\psi)$ be a G-equivariant, closed smooth embedding in some linear G-space $\mathsf{R}^k(\psi)$. Then $h_x = f \circ f_x$: $GS_x \to \mathsf{R}^k(\psi)$ is a G-equivariant smooth mapping whose restriction to GK_x is proper.

Let $g \in G$, S_{gx} be a slice at gx and K_{gx} be a compact subset of S_{gx}. Then $g^{-1}S_{gx}$ is a slice at x and $g^{-1}K_{gx}$ is a compact subset of $g^{-1}S_{gx}$. Since $GS_{gx} = G(g^{-1}S_{gx})$ and $GK_{gx} = G(g^{-1}K_{gx})$ we can choose $h_{gx} = h_x$.

6. COROLLARY. *Assume G is a linear Lie group and M a proper smooth G-manifold having only finitely many orbit types. Let H be a compact subgroup of G. Then there exists a representation ϱ: $G \to \mathrm{Gl}(m, \mathsf{R})$ for some $m \in \mathsf{N}$ with the following property: If $x \in M_{(H)}$, then there is a slice S_x at x such that if K_x is a compact subset of S_x, the tube GS_x has a G-equivariant smooth embedding f_x in $\mathsf{R}^m(\varrho)$ where the restriction $f_x \mid GK_x$ is proper.*

PROOF. Let ν: $G \to \mathrm{Gl}(n, \mathsf{R})$ and ψ: $G \to \mathrm{Gl}(k, \mathsf{R})$ be as in Lemmas 4 and 5, respectively. Let $x \in M_{(H)}$, S_x be a slice at x as in Lemma 4 and K_x be a compact subset of S_x. Then, obviously, (h_x, j_x): $GS_x \to \mathsf{R}^{k+n}(\psi \oplus \nu)$ is the desired mapping.

7. LEMMA. *Let G be a linear Lie group and M a proper smooth G-manifold having only finitely many orbit types. Then M has covers $\{O'_k\}_{k=1}^n$ and $\{O_k\}_{k=1}^n$ for some $n \in \mathsf{N}$, satisfying the following three conditions:*

1) *Every O'_k and O_k is open and G-invariant.*
2) *$\bar{O}_k \subset O'_k$ for every k.*
3) *There exists a representation ϱ: $G \to \mathrm{Gl}(q, \mathsf{R})$ for some $q \in \mathsf{N}$ such that for every k there is a G-equivariant smooth embedding j_k: $O'_k \to \mathsf{R}^q(\varrho)$ whose restriction to $\bar{O}_k$ is proper.*

PROOF. Let $(H_1), \ldots, (H_m)$ be the orbit types of M. Let $\{GS_{x_i}\}_{i=1}^{\infty}$ be a cover of M by such tubes that every S_{x_i} has the same properties as the slice in Corollary 6. The orbit space M/G is a paracompact space with finite covering dimension.

Thus, by Theorem 1.8.2 in [Pa2], there is an open cover $\{O'_{k\beta} \mid \beta \in B_k, k = 1, \ldots, n\}$ refining $\{GS_{x_i}\}_{i=1}^{\infty}$ such that each $O'_{k\beta}$ is G-invariant and $O'_{k\beta} \cap O'_{k\beta'} = \emptyset$ if $\beta \neq \beta'$. Here we can assume that each $B_k \subset \mathsf{N}$ and that $\{O'_{k\beta} \mid \beta \in B_k, k = 1, \ldots, n\}$ is locally finite and has an open G-invariant refinement $\{O_{k\beta} \mid \beta \in B_k, k = 1, \ldots, n\}$, where $\bar{O}_{k\beta} \subset O'_{k\beta}$ for every k and β.

We next choose for every k and β a tube GS_i such that $O'_{k\beta} \subset GS_i$ and denote this tube by $GS_{k\beta}$. We divide the family $\{GS_{k\beta} \mid \beta \in B_k, k = 1, \ldots, n\}$ into m subfamilies $\{GS^1_{k\beta}\}, \ldots, \{GS^m_{k\beta}\}$ in such a way that exactly those tubes $GS_{k\beta}$ for which $(G_{x_{k\beta}}) = (H_l)$ belong to the family $\{GS^l_{k\beta}\}$. By Corollary 6, there exists for each $l \in \{1, \ldots, m\}$ a representation $\varrho_l \colon G \to \mathrm{Gl}(n_l, \mathsf{R})$ for some $n_l \in \mathsf{N}$, such that every tube $GS^l_{k\beta}$ admits a G-equivariant smooth embedding $j^l_{k\beta}$ in $\mathsf{R}^{n_l}(\varrho_l)$. Since $\bar{O}_{k\beta} \cap S_{k\beta}$ is compact and $\bar{O}_{k\beta} = G(\bar{O}_{k\beta} \cap S_{k\beta})$ we can assume that the restriction $j^l_{k\beta} \mid \bar{O}_{k\beta}$ is proper.

The representation $\tilde{\varrho} = \varrho_1 \oplus \cdots \oplus \varrho_m$ makes $\mathsf{R}^p(\tilde{\varrho}) = \mathsf{R}^{n_1 + \cdots + n_m}(\tilde{\varrho})$ a linear G-space. Then $j_{k\beta} \colon GS^l_{k\beta} \to \mathsf{R}^p(\tilde{\varrho})$, $y \mapsto (0, \ldots, 0, j^l_{k\beta}(y), 0, \ldots, 0)$, is a G-equivariant smooth embedding whose restriction to $\bar{O}_{k\beta}$ is proper. Finally, let

$$\varrho \colon G \to \mathrm{Gl}(p+1, \mathsf{R}), \qquad g \mapsto \begin{pmatrix} \tilde{\varrho}(g) & 0 \\ 0 & 1 \end{pmatrix}.$$

Since $O'_{k\beta} \cap O'_{k\beta'} = \emptyset$ when $\beta \neq \beta'$, it follows that $j_k \colon \bigcup_{\beta \in B_k} O'_{k\beta} \to \mathsf{R}^{p+1}(\varrho)$, $y \mapsto (j_{k\beta}(y), \beta)$ when $y \in O'_{k\beta}$, is a G-equivariant smooth embedding. Since only finitely many values of β can occur in any compact subset of R it follows that the restriction $j_k \mid \bigcup_{\beta \in B_k} \bar{O}_{k\beta}$ is proper. Thus we can choose $O'_k = \bigcup_{\beta \in B_k} O'_{k\beta}$ and $O_k = \bigcup_{\beta \in B_k} O_{k\beta}$.

Proof of the theorem. Let $\{O'_k\}_{k=1}^n$ and $\{O_k\}_{k=1}^n$ be the covers of M as in Lemma 7. Let $\{W_k\}_{k=1}^n$ be a refinement of $\{O_k\}_{k=1}^n$ by open G-invariant sets W_k, where $\overline{W}_k \subset O_k$ for every k. According to Proposition 2 there exists for every k a G-invariant smooth mapping $h_k \colon M \to [0, 1]$, which is identically one on $\overline{W}_k$ and zero outside O_k. Let $\varrho \colon G \to \mathrm{Gl}(q, \mathsf{R})$ be a representation such that for every k there is a G-equivariant smooth embedding $j_k \colon O'_k \to \mathsf{R}^q(\varrho)$ whose restriction to $\bar{O}_k$ is proper. Next, for every k let $j^*_k \colon M \to \mathsf{R}^q(\varrho)$ be a mapping defined by $j^*_k(x) = h_k(x) j_k(x)$ if $x \in O_k$ and $j^*_k(x) = 0$ if $x \in M \backslash O_k$. Then each j^*_k is smooth and G-equivariant. Let R^n be a euclidean space where G acts trivially. Then the mapping

$$j \colon M \to \mathsf{R}^n \oplus \mathsf{R}^q(\varrho) \oplus \cdots \oplus \mathsf{R}^q(\varrho), \qquad x \mapsto (h_1(x), \ldots, h_n(x), j^*_1(x), \ldots, j^*_n(x)),$$

is G-equivariant and smooth. It is an immersion since each j^*_k is immersive in W_k.

Let $x \in M$ and let $(x_d)_{d=1}^{\infty}$ be a sequence in M such that $j(x_d) \to j(x)$. We know that $x \in W_k$ for some k. Thus $h_k(x) = 1$. Since $h_k(x_d) \to h_k(x)$, it follows that $h_k(x_d) > 0$ for sufficiently large d. Thus $x_d \in O_k$ for sufficiently large d. Since

$h_k(x_d)j_k(x_d) \to h_k(x)j_k(x)$, it now follows that $j_k(x_d) \to j_k(x)$. Since the restriction $j_k \mid O_k$ is an embedding, it follows that $x_d \to x$. Therefore j is injective and j^{-1} is continuous.

Since all the restrictions $j_k^* \mid \overline{W}_k$ are proper also the restrictions $j \mid \overline{W}_k$ are proper for every k. Since $\{\overline{W}_k\}_{k=1}^n$ is a closed cover of M it follows that j is proper. This completes the proof.

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