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HOLOMORPHIC FUNCTIONS AND THE (BB)-PROPERTY

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§ 1. Introduction.

A holomorphic function on a balanced domain in a locally convex space may be regarded as a sequence of polynomials which satisfies certain growth conditions. Locally convex topologies on the space of all holomorphic functions are a quantification of these conditions on aggregates of functions. This quantification is frequently obtained by combining, in an appropriate fashion, estimates on spaces of homogeneous polynomials. In this introduction we give an intuitive view of the process and in doing so, reformulate a number of known results.

To be more specific we require some definitions. E will denote a locally convex space over the complex numbers $\mathbf{C}$, $\mathscr{P}(^nE)$ is the space of continuous n-homogeneous polynomials on E and $\mathscr{H}(U)$ will denote the space of $\mathbf{C}$-valued holomorphic functions on the open subset U of E. The three most frequently studied topologies in infinite dimensional holomorphy are τ_0, the compact open topology, τ_ω and τ_δ. A seminorm p on $\mathscr{H}(U)$ is said to be τ_ω continuous if there exists a compact subset K of U such that for every V open, $K \subset V \subset U$, there exists $c(V) > 0$ such that

$$p(f) \leqq c(V) \|f\|_V \quad \text{for all } f \in \mathscr{H}(U). \tag{1.1}$$

The τ_ω topology on $\mathscr{H}(U)$ is the locally convex topology generated by the τ_ω continuous semi-norms. A semi-norm p on $\mathscr{H}(U)$ is said to be τ_δ continuous if for every increasing countable open cover of $U, \mathscr{V} = (V_n)_n$, there exists $c > 0$ and a positive integer n_0 such that

$$p(f) \leqq c \|f\|_{V_{n_0}} \quad \text{for all } f \text{ in } \mathscr{H}(U).$$

We always have $\tau_0 \leqq \tau_\omega \leqq \tau_\delta$ and conditions for equality have been investigated by various authors [1, 2, 3, 12, 13, 16]. An important special case is obtained by taking U balanced since this leads, via the Taylor series expansion, to a Schauder decomposition of $\mathscr{H}(U)$. As $\mathscr{P}(^nE)$, n arbitrary, is a complemented subspace of

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$\mathscr{H}(U)$ for any of the above topologies we see that any equality of topologies on $\mathscr{H}(U)$ will lead to the same equality on each space of homogeneous polynomials. When we restrict to $\mathscr{P}({}^nE)$ simplifications and refinements are possible. On the one hand $\tau_\omega = \tau_\delta$ on $\mathscr{P}({}^nE)$ for all n. On the other hand $\mathscr{P}({}^nE)$ may be identified with the dual of $\left(\widehat{\bigotimes}_{n,\pi,s} E\right)$ – the completion of the space of symmetric n-tensors on E endowed with the projective π topology. This duality can be used to describe τ_0 and τ_ω as well as two other natural topologies in $\mathscr{P}({}^nE)$ (we use the notation $\bigotimes_n x_i$ for $x_i \otimes \ldots \otimes x_i$ (n times) and $\bigotimes_{i,n} x_i$ to denote $x_1 \otimes x_2 \ldots \otimes x_n$);

τ_b – the topology of uniform convergence on the bounded subsets of E,
β – the strong topology inherited from $\widehat{\bigotimes}_{n,\pi,s} E$.

The duality between $\widehat{\bigotimes}_{n,\pi,s} E$ and $\mathscr{P}({}^nE)$ is given by $\left\langle P, \bigotimes_n x \right\rangle = P(x)$. The τ_0, τ_b and β topologies on $\mathscr{P}({}^nE)$, E metrizable, are then identified as uniform convergence on the following subsets of $\widehat{\bigotimes}_{n,\pi,s} E$;

$$\bar{\Gamma}\left(\bigotimes_{n,s} K\right), \bar{\Gamma}\left(\bigotimes_{n,s} B\right), \bigcap_m \bar{\Gamma}\left(\bigotimes_{n,s} r_m U_m\right),$$

respectively, where K is compact in E, B is bounded in E, $(U_m)_m$ denotes a fundamental sequence of convex balanced neighbourhoods of zero in E and $(r_m)_m$ is a sequence of positive numbers, Γ is the convex balanced hull and $\bar{\Gamma}$ the closed convex balanced hull.

For each positive integer m let $\mathscr{P}({}^nE)_m$ denote the set of all $P \in \mathscr{P}({}^nE)$ such that $\|P\|_{U_m} < \infty$ and we endow this space with the topology of uniform convergence on U_m. $\mathscr{P}({}^nE)_m$ is a Banach space and $\mathscr{P}({}^nE) = \bigcup_m \mathscr{P}({}^nE)_m$. With the above duality $\mathscr{P}({}^nE)_m$ has the topology of uniform convergence on $\bar{\Gamma}\left(\bigotimes_{n,s} U_m\right)$ and

$$(\mathscr{P}({}^nE), \tau_\omega) = \lim_{\vec{m}} \mathscr{P}({}^nE)_m.$$

From the above description it is clear that $\tau_0 \leqq \tau_b \leqq \beta \leqq \tau_\omega$ on $\mathscr{P}({}^nE)$ for all n.

By the Hahn-Banach theorem we have $\tau_0 = \tau_b$ on $\mathscr{P}({}^nE)$ for some (and hence for all n) if and only if E is semi-Montel (i.e. the closed bounded subsets of E are compact).

We have $\tau_b = \beta$ on $\mathscr{P}({}^nE)$ if and only if the sets $\bar{\Gamma}\left(\bigotimes_{n,s} B\right)$, B bounded in E, form a fundamental system of bounded subsets of $\widehat{\bigotimes}_{n,\pi,s} E$. This is the n-fold symmetric version of Grothendieck's "Problème des topologies" which we now state.

For locally convex spaces E and F is every bounded subset of $E \widehat{\bigotimes_{\pi}} F$ *contained in the closed absolutely convex hull of a set of the form* $B_1 \otimes B_2$ *where* B_1 *is a bounded subset of E and* B_2 *is a bounded subset of F?*

If this is the case then the pair $\{E, F\}$ is said to have the (BB)-property. We shall say that the locally convex space E has the $(\mathrm{BB})_n$-property if each bounded subset of $\widehat{\bigotimes_{n,\pi,s}} E$ is contained in $\bar{\Gamma}\left(\bigotimes_{n,s} B\right)$ for some bounded subset B of E. If E has $(\mathrm{BB})_n$ for all n we say that E has $(\mathrm{BB})_\infty$. With our new notation we have that $\tau_b = \beta$ on $\mathscr{P}(^nE)$ if and only if E has the $(\mathrm{BB})_n$-property. If $\{E, E\}$ has the (BB)-property then E has the $(\mathrm{BB})_2$-property and $\tau_b = \beta$ on $\mathscr{P}(^2E)$. The history of Grothendieck's problem can be divided into two distinct phases: the positive solutions of Grothendieck ([20]), circa 1955, and the recent developments, all of which follows from Taskinen's fundamental 1986 paper [20]. It is no coincidence that the Fréchet spaces for which the most interesting results have been obtained in infinite dimensional holomorphy – Banach spaces, nuclear and Schwartz spaces – are included in the classes for which Grothendieck obtained positive solutions to the "problème des topologies" and that the class of spaces which often appeared as the critical case in infinite dimensional holomorphy – the Fréchet-Montel spaces – should yield mixed results (i.e. both positive results and counterexamples) to the "problème des topologies". It is our belief that the $(\mathrm{BB})_\infty$-property will frequently appear as an essential hypothesis in topological problems of infinite dimensional holomorphy. Taskinen and his followers have shown by example and counterexample, that the collection of pairs of spaces with the (BB)-property will probably not coincide with any of the usual linear collections but will contain large subcollections of interesting spaces. We shall consider the (BB)-property more closely in the next section and confine ourselves here to a general presentation.

If E is a Fréchet-Montel space and E has the $(\mathrm{BB})_n$-property then each bounded subset of $\widehat{\bigotimes_{n,\pi,s}} E$ is contained in a subset of the form $\bar{\Gamma}\left(\bigotimes_{n,s} K\right)$. By our previous remarks this implies that $\tau_0 = \beta$ on $\mathscr{P}(^nE)$ and, moreover, since $\bar{\Gamma}\left(\bigotimes_{n,s} K\right)$ is a compact subset of $\widehat{\bigotimes_{n,\pi,s}} E$ it follows that $\widehat{\bigotimes_{n,\pi,s}} E$ is itself Fréchet-Montel. Hence $(\mathscr{P}(^nE), \tau_0)$ is reflexive and $\tau_0 = \tau_\omega$ on $\mathscr{P}(^nE)$. To summarize we have the following result.

Proposition 1. *If E is a Fréchet-Montel space then E has the* $(\mathrm{BB})_n$*-property if and only if* $\tau_0 = \tau_\omega$ *on* $\mathscr{P}({}^nE)$.

Taskinen [22] constructed a Fréchet-Montel space which did not have the (BB)-property and a suitable modification by Ansemil-Taskinen [3] yielded the first example of a Fréchet-Montel space for which $\tau_0 \neq \tau_\omega$ on $\mathscr{P}({}^2E)$. For positive results arising from proposition 1 we refer to proposition 4 and § 2.

Since $\widehat{\bigotimes}_\pi$ is an associative functor the following is immediate.

Proposition 2. [8, *proposition* 8]. *If $\mathscr{E}$ is a collection of Fréchet spaces which is stable under the formation of completed projective tensor products and $\{E, F\}$ has the* (BB)*-property for any E, $F \in \mathscr{E}$ then each $E \in \mathscr{E}$ has the* $(\mathrm{BB})_\infty$*-property.*

The collection of tensor-(FG) spaces introduced in [8] satisfies the hypothesis of proposition 2. Remarks regarding the definition of this collection and examples are given in the next section.

The collection of separable Banach spaces is stable under completed projective tensor products. Using this fact, associativity of $\bigotimes_\pi$ and [21, proposition 2.13] modified for separable Banach spaces we can easily show the following.

Proposition 3. *If E is a separable Fréchet space, $\{E, F\}$ has the* (BB)*-property for every separable Banach space F, and E contains a fundamental system of absolutely convex bounded sets $\mathscr{B}$ such that E_B has the approximation property for all $B \in \mathscr{B}$ then E has the* $(\mathrm{BB})_\infty$*-property.*

Separable Hilbertizable Fréchet spaces satisfy the hypothesis of proposition 3. The hypothesis of proposition 3 may be weakened by using recent results from [6].

Finally we consider the equality $\beta = \tau_\omega$ on $\mathscr{P}({}^nE)$. Since τ_ω is the barrelled topology associated with τ_0 on $\mathscr{P}({}^nE)$ and $\tau_\omega \geqq \beta \geqq \tau_0$ it follows that τ_ω is also the barrelled topology associated with β. A locally convex space is said to be distinguished if its strong dual is barrelled. Hence for arbitrary Fréchet spaces we have $\beta = \tau_\omega$ on $\mathscr{P}({}^nE)$ if and only if $\widehat{\bigotimes}_{n,\pi,s} E$ is distinguished. Since we require this property for all n it is more convenient to consider the density condition [5]. A Fréchet space E is said to have the density condition if the bounded subsets of $E'_\beta\,(= (\mathscr{P}(({}^1E), \beta))$ are metrizable. If E has $(\mathrm{BB})_n$ and the density condition then $\widehat{\bigotimes}_{n,\pi,s} E$ also has the density condition ([5]) and hence is distinguished. Quasinormable spaces and Fréchet-Montel spaces have the density condition.

This concludes our first examination of locally convex topologies on spaces of homogeneous polynomials. We now consider the problem of how these locally convex spaces of homogeneous polynomials are combined to provide locally convex structures on $\mathscr{H}(U)$, U a balanced domain in a locally convex space E. In later sections we shall see how to lift estimates to obtain more precise information about $\mathscr{H}(U)$.

The τ_0 topology on $\mathscr{H}(U)$, U a balanced domain in a Fréchet space, is generated by all seminorms of the form

$$p_K(f) = \sum_{n=0}^{\infty} \left\| \frac{\hat{d}^n f(0)}{n!} \right\|_K \tag{1.2}$$

where $f = \sum_{n=0}^{\infty} \frac{\hat{d}^n f(0)}{n!} \in \mathscr{H}(U)$ and K is a compact subset of U.

The τ_ω topology on $\mathscr{H}(U)$, U a balanced domain in a Fréchet space (or indeed in any locally convex space) is generated by all seminorms which have the following three properties

$$p(f) = \sum_{n=0}^{\infty} p\left(\frac{\hat{d}^n f(0)}{n!}\right) \tag{1.3}$$

for all $f = \sum_{n=0}^{\infty} \frac{\hat{d}^n f(0)}{n!} \in \mathscr{H}(U)$,

$$p \,|\, \mathscr{P}(^nE) \text{ is } \tau_\omega \text{ continuous for all } n \tag{1.4}$$

there exists a compact subset K of U such that for every V open, $K \subset V \subset U$, there exists $c(V) > 0$ such that

$$p(P) \leqq c(V)\, \|P\|_V \tag{1.5}$$

for all $P \in \mathscr{P}(^nE)$ and all n.

The τ_δ topology on $\mathscr{H}(U)$ is generated by all seminorms which satisfy (1.3) and (1.4).

Ansemil-Ponte [2] used (1.2), holomorphic germs and a result of Mujica to show that if $\tau_0 = \tau_\omega$ on $\mathscr{P}(^nE)$ for all n where E is a Fréchet-Montel space then $\tau_0 = \tau_\omega$ on $\mathscr{H}(U)$, U an arbitrary balanced open subset of U. In our terminology and using proposition 1 we may rephrase this as follows.

Proposition 4. *If E is a Fréchet-Montel space then $\tau_0 = \tau_\omega$ on $\mathscr{H}(U)$ for any balanced open subset U of E if and only if E has the* (BB)$_\infty$*-property.*

If $\tau_0 = \tau_\omega$ on $\mathscr{P}(^nE)$ then condition (1.4) is equivalent to

(1.4)′ *there exists a compact subset K_n of E such that*

$$p(P) \leqq \|P\|_{K_n} \textit{ for all } P \in \mathscr{P}(^nE)$$

We may thus regard (1.4)′ and (1.5) as our estimates on spaces of homogeneous polynomials and (1.3) plays a role in putting these estimates together to obtain $\tau_0 = \tau_\omega$ on $\mathscr{H}(U)$. Proposition 4 solves the $\tau_0 = \tau_\omega$ problem as a holomorphic problem on balanced domains and what remains is now a polynomial problem. The combined conditions (1.3), (1.4) and (1.5) may be regarded as a refinement of (1.1).

More precise descriptions are available for certain classes of Fréchet spaces. By proposition 4, if E is a Fréchet-Schwartz space then the τ_ω topology on $\mathscr{H}(U)$, U balanced on E, is generated by all seminorms which satisfy (1.2).

If E is a Banach space with unit ball B and U is a balanced open subset of E then the τ_ω topology on $\mathscr{H}(U)$ is generated by all semi-norms of the form

$$p(f) = \sum_{n=0}^{\infty} \left\| \frac{\hat{d}^n f(0)}{n!} \right\|_{K+\alpha_n B} \tag{1.6}$$

where $f = \sum_{n=0}^{\infty} \frac{\hat{d}^n f(0)}{n!} \in \mathscr{H}(U)$, K is a compact subset of U and $(\alpha_n)_n \in c_0$ ([4, 12]).

We now consider the τ_b topology on $\mathscr{H}(U)$. A subset B of U is called a bounded subset of U if it is a bounded subset of E and there exists a neighbourhood V of 0 in E such that $B + V \subset U$. The natural analogue of τ_b on $\mathscr{H}(U)$ is the topology of uniform convergence on the bounded subsets of U. However, it is rarely the case that each holomorphic function on U is bounded on all the bounded subsets of U. In this situation attention is often restricted to those holomorphic functions which are bounded on the bounded subsets of U and one obtains a subspace of $\mathscr{H}(U)$ which is denoted by $\mathscr{H}_b(U)$ (see example 10). We are interested in $\mathscr{H}(U)$ and so motivated by (1.2) for Fréchet-Schwartz spaces and (1.6) for Banach spaces we define a new topology on $\mathscr{H}(U)$. We shall say that a sequence of subsets $(B_n)_n$, of a locally convex space E, converges to a subset B if for every neighbourhood V of 0 in E there exists a positive integer n_0 such that $B_n \subset B + V$ for all $n \geqq n_0$.

DEFINITION 5. If U is a balanced open subset of a locally convex space E then the *τ_b topology on $\mathscr{H}(U)$* is generated by all seminorms on $\mathscr{H}(U)$ which have the form

$$p(f) = \sum_{n=0}^{\infty} \left\| \frac{\hat{d}^n f(0)}{n!} \right\|_{B_n} \tag{1.7}$$

for all $f = \sum_{n=0}^{\infty} \frac{\hat{d}^n f(0)}{n!} \in \mathscr{H}(U)$, where $(B_n)_n$ is a sequence of bounded subsets of E which converges to some compact subset of U.

If U is an arbitrary open subset of U we let

$$(\mathscr{H}(U), \tau_b) = \varprojlim_{\xi, V} (\mathscr{H}(\xi + V), \tau_b)$$

where the projective limit is taken over all pairs (ξ, V) with $\xi \in U$, V balanced in E and $\xi + V \subset U$ (we use the canonical identification between $\mathscr{H}(V)$ and $\mathscr{H}(\xi + V)$ to define τ_b on $\mathscr{H}(\xi + V)$). Many of the properties of $\mathscr{H}(U)$, U balanced, shared by τ_0, τ_ω and τ_δ remain true for τ_b and are proved in the same way. For instance $\{(\{\mathscr{P}(^nE), \tau_b)\}_{n=0}^{\infty}$ is an $\mathscr{S}$-absolute decomposition for $(\mathscr{H}(U), \tau_b)$ ([13]).

Clearly we have $\tau_0 \leqq \tau_b \leqq \tau_\omega$ on $\mathscr{H}(U)$, and $\tau_0 = \tau_b$, for U balanced, if and only if E is a semi-Montel space. We have already noted that $\tau_b = \tau_\omega$ for balanced domains in Banach spaces and Fréchet-Schwartz spaces.

Our aim in this paper is to obtain a similar result for a large class of Fréchet spaces.

For unexplained terminology, general definitions and results mentioned without proof or reference we refer to [13].

§ 2. *T*-Schauder decompositions.

In § 1 we saw that a pair of locally convex spaces has the (BB)-property if every bounded subset of their completed π-projective tensor products splits (modulo taking the absolute convex hull). For this reason it is only to be expected that good splittings of the component spaces should lead to examples of pairs with the (BB)-property. A splitting of the whole space is a projection. The mere existence of projections is not sufficient, however, as bounded sets in Fréchet spaces are defined by estimates involving a sequence of semi-norms and it is necessary to have further interactions between the projections and the semi-norms. The first step in this direction was taken by Taskinen [20] who defined a class of Fréchet spaces with an unconditional basis and a property allowing the extension of norm estimates on subsets of the basis to their linear span in a uniform fashion. The projections, given by the basis, led to a partition of the basis into disjoint subsets on each of which tensor norms could be estimated and the results from the different sections combined to obtain the (BB)-property. The technique of using good estimates between different norms on sufficiently many projections appears to be fundamental as in the same paper (see also [22]) the author obtained counterexamples to the general problem by using projections. Taskinen's method has been developed to more general situations but in all cases it is possible to see implicitly or explicitly the presence of projections.

Bonet-Diaz [7] introduced *T*-spaces by replacing the basis by projections and Bonet-Diaz-Taskinen [8] went a step further and replaced projections with a partition of the identity. Diaz-Metafune [10] (see also [9]) characterized the standard quojections of Moscatelli type E such that $\{E, F\}$ has the (BB)-property for every Banach space F as those spaces for which E'' is a product of Banach spaces (in other words the spaces with a single twist or spaces whose second dual contains sufficiently many projections onto spaces with the (BB)-property (i.e. Banach spaces)). The other major class for which there is a positive solution are the Hilbertizable Fréchet space and here again projections are implicitly available as every closed subspace of a Hilbert space is complemented.

In infinite dimensional holomorphy the methods of Taskinen [20] and Bonet-Diaz [7] was extended to the n-fold symmetric case in [14] and [16] to obtain further examples of Fréchet-Montel spaces E for which $\tau_0 = \tau_\omega$ on $\mathscr{H}(U)$, U balanced in E.

In this article we define a collection of spaces which are very similar to the T-spaces of Bonet-Diaz [7] and adjust the method of [14] and [16] to obtain more precise estimates for the $(BB)_n$-property. These estimates are then combined to obtain examples of spaces of holomorphic functions with $\tau_b = \tau_\omega$.

Let $\{E_n\}_n$ denote an unconditional Schauder decomposition of the locally convex space E. For each n let P_n denote the canonical projection, defined by the decomposition, from E onto E_n and for any subset J of N let $P_J = \sum_{j \in J} P_j$.

DEFINITION 6. An *unconditional Schauder decomposition*, $\{E_n\}_n$ of a Fréchet space E is a T-Schauder decomposition if there exists a fundamental system of semi-norms for E, $(\|\cdot\|_k)_{k \in N}$ such that

(2.1) $\qquad \|P_J(x)\|_k \leqq \|x\|_k$ for all $J \subset N$, $k \in N$ and $x \in E$

(2.2) $\qquad$ for every sequence $\alpha = (\alpha_k)_k$, $0 < \alpha_k \leqq 1$, there exists a partition

$$J_\alpha = (J_{\alpha,k})_k \text{ of } N \text{ such that if } P_{\alpha,k} := P_{J_{\alpha,k}} \text{ then}$$

$$\|P_{\alpha,k}(x)\|_{k-1} \leqq \alpha_k \|P_{\alpha,k}(x)\|_k \text{ for all } x \in E \text{ and all } k \geqq 2.$$

(2.3) $\|\cdot\|_k$ defines the topology induced by E on $P_{\alpha,k}(E)$ for all α and all k.

A basic sequence of semi-norms satisfying (2.1), (2.2) and (2.3) is called T-adaptable and a subset A of E is said to be T-invariant if $P_J(A) \subset A$ for all $J \subset N$.

Definition 6 is very similar to the definition of T-decomposable space given in [7]. The extra condition, required in order to prove lemma 7, is that each $P_{\alpha,k}$ is obtained by adding together projections from the original Schauder decomposition and condition (2.3) shows that it may be difficult to construct an example of a T-decomposable space which does not have a T-Schauder decomposition. The FG-spaces of [8] are obtained by replacing the decomposition and (2.1) by a partition of the identity $\sum_{j=1}^{\infty} P_j$ and tensor (FG) spaces are FG-spaces for which $\left\|\sum_{j=1}^{l} P_j\right\|_k \leqq 1$ for all $l \leqq k$ (see the remarks following proposition 2).

LEMMA 7. *If the Fréchet space E has a T-Schauder decomposition and U is a T-invariant convex balanced open subset of E then each compact subset of U is contained in an absolutely convex T-invariant compact subset of U.*

PROOF. Let K denote a convex balanced compact subset of U. Let $\tilde{K} = \bigcup_{J \subset N} P_J(K)$. Clearly $\tilde{K} \subset U$ and we show that $\tilde{K}$ is relatively compact. Let $(y_n)_n$ denote a sequence in $\tilde{K}$ and for each n choose x_n in K and $J_n \subset N$ such that $y_n = P_{J_n}(x_n)$. Since K is compact, $(x_n)_n$ contains a convergent subsequence (which we may suppose is the original sequence) which converges to $x \in K$. We identify subsets of N with points in 2^N in the usual way. If the set consisting of 2 elements is given its discrete topology and 2^N is given the product topology then 2^N is a compact metric space. It follows that $\{J_n\}_n$ contains a subsequence, which we again assume to be the original sequence, which converges to an element J of N.

If k is any positive integer then

$$\begin{aligned}\|P_{J_n}(x_n) - P_J(x)\|_k &\leqq \|P_{J_n}(x_n) - P_{J_n}(x)\|_k + \|P_{J_n}(x) - P_J(x)\|_k \\ &\leqq \|x_n - x\|_k + \|P_{J_n}(x) - P_J(x)\|_k \\ &\to 0 \text{ as } n \to \infty.\end{aligned}$$

Hence $\bar{\Gamma}(\tilde{K})$ is a compact subset of E.

For any $J_1 \subset N$ we have

$$P_{J_1}(\tilde{K}) = \bigcup_{J \subset N} \mathrm{P}_{J_1} P_J(K) = \bigcup_{j \subset N} P_{J_1 \cap J}(K) \subset \tilde{K}$$

Hence

$$P_{J_1}(\bar{\Gamma}(\tilde{K})) = \bar{\Gamma}(P_{J_1}(\tilde{K})) \subset \bar{\Gamma}(\tilde{K})$$

and $\bar{\Gamma}(\tilde{K})$ is a convex balanced compact T-invariant subset of E which contains K. This completes the proof.

PROPOSITION 8. *Let $(\|\cdot\|_k)_k$ denote a T-adaptable set of seminorms for the T-Schauder decomposition $\{E_n\}_n$ of the Fréchet space E. Let K denote an absolutely convex compact T-invariant subset of E and let $U_1 = \{x \in E;\ \|x\|_1 \leqq 1\}$. Let q_1 denote the seminorm on E with closed unit ball $U_1 + K$ and let $q_k = \|\cdot\|_k$ for $k \geqq 2$. Then $(q_k)_k$ is a T-adaptable set of seminorms for the decomposition $\{E_n\}_n$.*

PROOF. We retain the same set of projections $(P_{\alpha,k})_k$ for the new set of semi-norms. Since K is compact the norms $\|\cdot\|_1$ and q_1 are equivalent and hence (2.3) is satisfied by $(q_k)_{k \geqq 1}$. To complete the proof we must check (2.1) for q_1 and (2.2) for $k = 2$. For (2.1) it suffices to show that $q_1(P_J x) \leqq q_1(x)$ for all $J \subset N$ and all $x \in E$. This is equivalent to showing $P_J(U_1 + K) \subset U_1 + K$. By (2.1) for $\|\cdot\|_1$, we have $P_J(U_1) \subset U_1$ and since K is T-invariant we have $P_J(K) \subset K$.

Hence $P_J(U_1 + K) = P_J(U_1) + P_J(K) \subset U_1 + K$.

For (2.2) we are required to show that

$$q_1(P_{\alpha,1}(x)) \leqq \alpha_2 q_2(P_{\alpha,1}(x)) \quad \text{for all } x \in E.$$

Since $U_1 \subset U_1 + K$ it follows that $q_1 \leqq \|\cdot\|_1$.

Hence

$$q_1(P_{\alpha,1}(x)) \leqq \|P_{\alpha,1}(x)\|_1 \leqq \alpha_2 \|P_{\alpha,1}(x)\|_2 = \alpha_2 q_2(P_{\alpha,1}(x))$$

and this completes the proof.

EXAMPLE 9 (of Fréchet spaces with T-Schauder decompositions).

(a) Fréchet spaces with unconditional basis of type (T), l^p and X valued Köthe sequence spaces ([7, 8, 22]). In [11] the authors show that a Köthe echelon space, $\lambda_p(I, A)$, $1 \leqq p \leqq \infty$ or $p = 0$ and I of arbitrary cardinality, has a T-Schauder decomposition if and only if it has a total bounded set or equivalently if and only if $\lambda_p(I, A)'_\beta$ admits a continuous norm.

(b) Banach spaces [7].

(c) Fréchet-Schwartz spaces which admit a continuous norm and a finite dimensional decomposition ([7]).

(d) Fréchet-Montel spaces with unconditional basis $(e_n)_n$ which satisfy

$$(c\Omega) \text{ for all } k, t \in N,\ k > t, \text{ there exists } M_{k,t} > 0 \text{ such that if}$$
$$\|e_i\|_k \leqq C\,\|e_i\|_t,\ i \in J, \text{ for some } C > 0 \text{ and } J \subset N, \text{ then}$$
$$\|x\|_k \leqq CM_{k,t}\,\|x\|_t \text{ for every } x \in \overline{sp(e_i, i \in J)} \text{ ([7, 9])}.$$

The above are all examples of T-decomposable spaces and the proofs given in [7] show that they all have T-Schauder decompositions. Furthermore, the proof of Observation 2 in [7] shows that the countable product of Fréchet spaces with T-Schauder decompositions also has a T-Schauder decomposition.

Condition (2.3) implies that all except a finite number of semi-norms agree on $P_{\alpha,k}(E)$. For this reason we find (see for instance the proof of proposition 3 in [7]) that in many examples the spaces $P_{\alpha,k}(E)$ are finite dimensional. Finite dimensionality also plays a key role in the final part of Taskinen's proof [20, theorem 3.1 and 3.3] and something similar is often necessary in the general case. Our next example gives a natural situation in which all of the spaces $P_{\alpha,k}(E)$ are infinite dimensional Banach spaces.

EXAMPLE 10. Let E denote an infinite dimensional Banach space. The Taylor series decomposition of $\mathscr{H}_b(E)$ (the entire holomorphic functions of bounded type endowed with the topology of uniform convergence on the bounded subsets of E) is a T-Schauder decomposition of the Fréchet space $\mathscr{H}_b(E)$. We have $E_n = \mathscr{P}(^nE)$

and $P_n(f) = \dfrac{\hat{d}^n f(0)}{n!}$ for all n and all $f \in \mathscr{H}_b(E)$. The topology on $\mathscr{H}_b(E)$ is generated by the seminorms

$$p_k(f) = \sum_{n=0}^{\infty} \left\| \frac{\hat{d}^n f(0)}{n!} \right\|_{kB}, \quad k = 1, 2 \dots$$

where B is the unit ball of E.

Clearly (2.1) is satisfied. Let $\alpha = (\alpha_n)_n$ be given. Since

$$p_k\left(\frac{\hat{d}^n f(0)}{n!}\right) = \left(\frac{k}{l}\right)^n p_l\left(\frac{\hat{d}^n f(0)}{n!}\right)$$

for all k, l and $n \in N$ we can choose an increasing sequence of positive integers $(n_k)_{k \geqq 2}$ such that $p_{k-1}(P) \leqq \alpha_k p_k(P)$ for all $P \in \mathscr{P}(^n E)$ and all $n \geqq n_k$. Let $J_1 = \{0, 1, \dots, n_2 - 1\}$ and $J_k = \{n_k, \dots, n_{k+1} - 1\}$ for $k \geqq 2$. Let $P_{\alpha,k}(f) = \dfrac{\hat{d}^j f(0)}{j!}$. Then

$$\begin{aligned} \|P_{\alpha,k}(f)\|_{k-1} &= \sum_{j \in J_k} \left\| \frac{\hat{d}^j f(0)}{j!} \right\|_{(k-1)B} \\ &\leqq \alpha_k \sum_{j \in J_k} \left\| \frac{\hat{d}^j f(0)}{j!} \right\|_{kB} \\ &= \alpha_k \, \|P_{\alpha,k}(f)\|_k \end{aligned}$$

and (2.2) is satisfied. Since $\|\cdot\|_k \sim \|\cdot\|_l$ on $\mathscr{P}(^n E)$ for all n, k and l and the projections $P_{\alpha,k}$ only involve a finite number of derivatives it follows that (2.3) is also satisfied. Hence $\mathscr{H}_b(E)$ has a T-Schauder decomposition. We note in passing that the norms given above for $\mathscr{H}_b(E)$ satisfy condition $(c\Omega)$ of example 9 (d) without any restriction on k and l. A similar proof works for U balanced.

The proof of the following proposition was motivated by the proofs of [20, theorems 3.1 and 3.3], [14, theorem 1] and [16, proposition 3]. The crucial point in proposition 11 is to obtain *symmetric tensor representations* and not just *tensor representations* which have previously been given in [14] and [16].

We let S denote the symmetrization (projection) from $\widehat{\bigotimes}_{d,\pi} E$ onto $\widehat{\bigotimes}_{d,\pi,s} E$ obtained by extending

$$S(x_1 \otimes x_2 \dots \otimes x_d) := \frac{1}{d!} \sum_{\pi \in S_d} x_{\pi(1)} \otimes \dots \otimes x_{\pi(d)}$$

where S_d is the set of all permutations of $\{1, \dots, d\}$. If B is a convex balanced subset of E then the polarization formula [13, p. 4] and duality theory show that for $\theta \in \widehat{\bigotimes}_{d,\pi,s} E$ we have

$$(2.4)\quad \|\theta\|_{B^d} := \inf\left\{\sum_{j=1}^{\infty} \|x_{j,1}\|_B \|x_{j,2}\|_B \dots \|x_{j,d}\|_B ; \theta = \sum_{j=1}^{\infty} x_{j,1} \otimes x_{j,2} \dots \otimes x_{j,d}\right\}$$

$$\leqq \|\theta\|_B := \inf\left\{\sum_{j=1}^{\infty} \|x_j\|_B^d ; \theta = \sum_{j=1}^{\infty} \bigotimes_d x_j\right\}$$

$$\leqq \frac{d^d}{d!} \|\theta\|_{B^d}$$

Proposition 11. *Let E denote a Fréchet space with T-Schauder decomposition $\{E_n\}$ and T-adaptable set of semi-norms $(\|\cdot\|_k)_{k \geqq 1}$. Let d be a positive integer, $U_n = \{x \in E; \|x_n\| \leqq 1\}$, B be a bounded subset of $\widehat{\bigotimes}_{d,\pi,s} E$ and $\varepsilon > 0$ be arbitrary. If $B \subset \bar{\Gamma}\left(\bigotimes_{d,s} U_1\right)$ then there exists a bounded subset A in $(1+\varepsilon)U_1$ such that $B \subset \bar{\Gamma}\left(\bigotimes_{d,s} A\right)$.*

Proof. We may suppose without loss of generality that $d > 1$ and $0 < \varepsilon < 1$. Since B is bounded there exists an increasing sequence of positive numbers, $(r_n)_{n=1}^{\infty}$, such that $B \subset \bigcap_n \bar{\Gamma}\left(\bigotimes_{d,s} r_n U_n\right)$. By our hypothesis we may take $r_1 = 1$ and, without loss generality, we may suppose $r_2 > \max\left((2d)^{d^2}; \dfrac{d^{2d}}{\varepsilon}\right)$.

For each positive integer k and each z in B we have a representation

$$z = \sum_{i=1}^{\infty} \lambda_{i,k}(z) \bigotimes_d x_{i,k}(z)$$

where

$$\sum_{i=1}^{\infty} |\lambda_{i,k}(z)| \leqq 1 + \varepsilon \text{ for all } k \in N \text{ and } z \in B$$

and $x_{i,k}(z) \in r_k U_k$ for all $i, k \in N$ and $z \in B$.

Let $\alpha_k = 2^{-k} r_k^{-(d+2)}$ for $k \in N$ and let $(P_{\alpha,k})_k$ be the decomposition for $\alpha := (\alpha_k)_k$ given by (2.2). For the d-tuple $k = (k_1, \dots, k_d)$ of positive integers let

$$A_k = \left\{i, 1 \leqq i \leqq d, \hat{k} := \sup_{1 \leqq j \leqq d} k_j = k_i\right\} \quad \text{and let } B_k = \{i; 1 \leqq i \leqq d, k_i = 1\}$$

Let $|A_k| = e$ and $|B_k| = f$. For $1 \leqq j \leqq d$ let

$$\Phi_j(k) = \begin{cases} r_{\hat{k}}^{(d+1-e)/e} & \text{if } j \in A_k \text{ and } f > 0 \\ r_{\hat{k}}^{(d-e)/e} & \text{if } j \in A_k \text{ and } f = 0 \\ r_{\hat{k}}^{-1} & \text{if } j \notin A_k \text{ and } j \notin B_k \\ r_{\hat{k}}^{-1-\frac{1}{f}} & \text{if } j \in B_k. \end{cases}$$

If $|B_k| = 0$ then

$$\prod_{j=1}^{d} \Phi_j(k) = (r_{\hat{k}}^{(d-e)/e})^e (r_{\hat{k}}^{-1})^{d-e} = r_{\hat{k}}^{d-e-d+e} = 1.$$

If $|B_k| = f > 0$ then

$$\prod_{j=1}^{d} \Phi_j(k) = (r_{\hat{k}}^{(d+1-e)/e})^e (r_{\hat{k}}^{-1})^{d-e-f} (r_{\hat{k}}^{-1-\frac{1}{f}})^f$$
$$= r_{\hat{k}}^{d+1-e-d+e+f-f-1} = 1$$

so that $\prod_{j=1}^{d} \Phi_j(\mathrm{k}) = 1$ for all $k \in N^d$.

We now consider the formal sum

$$\sum_{i=1}^{\infty} \lambda_{i,1}(z) \bigotimes_{d} P_{\alpha,1}(x_{i,1}(z))$$
$$+ \sum_{i=1}^{\infty} \sum_{k=2}^{\infty} \lambda_{i,k}(z) 2^{-kd} r_2^{-d} \bigotimes_{d} (r_2 2^k P_{\alpha,k}(x_{i,k}(z)))$$
$$+ \sum_{i=1}^{\infty} \sum_{\substack{k=(k_1,\dots k_d) \\ |A_k|<d}} \lambda_{i,\hat{k}}(z) 2^{-\Sigma_{j=1}^{d} k_j} S\left(\bigotimes_{d,j} (r_2^{\frac{1}{2d}} \Phi_j(k) 2^{k_j} P_{\alpha,k_j}(x_{i,\hat{k}}(z))) \right)$$

We claim that the elements of E which appear in the tensor part of the above sum form a bounded subset of E. We consider the different cases that may arise and adopt the following notation.

If $j \leqq q$ we may use (2.3) to find $C_{q,j} > 0$ such that $\|\cdot\|_q \leqq C_{q,j} \|\cdot\|_j$ on $P_{\alpha,j}(E)$ and we let $c_q = \sup_{1 \leqq j \leqq q} C_{q,j}$.

In cases 3, 4 and 5, $k \in N$, $k \geqq 2$, and let $y_{k,i}(z) = r_2 2^k P_{\alpha,k}(x_{i,k}(z))$.

In cases 6 to 12, $k = (k_1, \dots, k_d) \in N^d$ and $|A_k| < d$ and we let

$$w_{i,k,j}(z) = d r_2^{\frac{1}{2d}} \Phi_j(k) 2^{k_j} P_{\alpha,k_j}(x_{i,\hat{k}}(z)).$$

We let q denote a positive integer.

Case 1. $q = 1$.

$$\|P_{\alpha,1}(x_{i,1}(z))\|_1 \leqq \|x_{i,1}(z)\|_1 \leqq 1$$

Case 2. $q > 1$.

$$\|P_{\alpha,1}(x_{i,1}(z))\|_q \leqq C_{q,1}\|P_{\alpha,1}(x_{i,1}(z))\|_1 \leqq C_{q,1}$$

Case 3. $q < k$.

$$\begin{aligned}\|y_{k,i}(z)\|_q &\leqq r_2 2^k \|P_{\alpha,k}(x_{i,k}(z))\|_{k-1}\\ &\leqq r_2 2^k 2^{-k} r_k^{-(d+2)}\|P_{\alpha,k}(x_{i,k}(z))\|_k\\ &\leqq r_2 r_k^{-(d+2)} r_k\\ &< r_2^{-d}\end{aligned}$$

Case 4. $q = k$.

$$\begin{aligned}\|y_{k,i}(z)\|_q &= r_2 2^q \|P_{\alpha,q}(x_{i,q}(z))\|_q\\ &\leqq r_2 r_q 2^q\end{aligned}$$

Case 5. $q > k$.

$$\begin{aligned}\|y_{k,i}(z)\|_q &\leqq r_2 2^k C_{q,k}\| P_{\alpha,q}(x_{i,k}(z))\|_k\\ &\leqq r_2 2^q c_q r_q\end{aligned}$$

Case 6. $q < k_j = \hat{k}$. We have $\Phi_j(k) \leqq r_{\hat{k}}^{\frac{d+1-e}{e}}$

$$\begin{aligned}\|w_{i,k,j}(z)\|_q &\leqq d r_2^{\frac{1}{2d}} r_{\hat{k}}^{\frac{d+1-e}{e}} 2^{\hat{k}} \|P_{\alpha,\hat{k}}(x_{i,\hat{k}}(z))\|_{\hat{k}-1}\\ &\leqq d r_2^{\frac{1}{2d}} r_{\hat{k}}^{\frac{d+1-e}{e}} 2^{\hat{k}} 2^{-\hat{k}} r_{\hat{k}}^{-(d+2)} \|P_{\alpha,\hat{k}}(x_{i,\hat{k}}(z))\|_{\hat{k}}\\ &\leqq d r_2^{\frac{1}{2d}} r_{\hat{k}}^{\frac{d+1-e}{e}-(d+2)+1}\\ &\leqq d r_2^{\frac{1}{2d}} r_{\hat{k}}^{-1}\\ &\leqq d r_2^{\frac{1}{2}} r_2^{-1}\\ &\leqq 1 \quad \text{since } r_2 > (2d)^{d^2} > d^2\end{aligned}$$

Case 7. $q < k_j < \hat{k}$. We have $\Phi_j(k) \leqq r_{\hat{k}}^{-1}$

$$
\begin{aligned}
\|w_{i,k,j}(z)\|_q &\leqq dr_2^{\frac{1}{2d}} r_{\hat{k}}^{-1} 2^{k_j} \|P_{\alpha,k_j}(x_{i,\hat{k}}(z))\|_{k_j-1} \\
&\leqq dr_2^{\frac{1}{2d}} r_{\hat{k}}^{-1} 2^{-k_j} r_{k_j}^{-(d+2)} \|P_{\alpha,k_j}(x_{i,\hat{k}}(z))\|_{k_j} \\
&\leqq dr_2^{\frac{1}{2d}} r_{\hat{k}}^{-1} r_{k_j}^{-(d+2)} \|(x_{i,\hat{k}})(z)\|_{\hat{k}} \\
&\leqq dr_2^{\frac{1}{2d}} r_{k_j}^{-(d+2)} \\
&\leqq dr_2^{-(d+1)} \\
&\leqq r_2^{-d}
\end{aligned}
$$

Case 8. If $q = k_j = \hat{k}$. We have $\Phi_j(k) \leqq r_q^{\frac{d+1-e}{e}}$

$$
\begin{aligned}
\|w_{i,j,k}(z)\|_q &\leqq dr_2^{\frac{1}{2d}} r_q^{\frac{d+1-e}{e}} 2^q \|P_{\alpha,q}(x_{i,q}(z))\|_q \\
&\leqq d2^q r_q^{\frac{1}{2d}+d+1}
\end{aligned}
$$

(Note that in this case we cannot have $q = 1$ since this would imply $\hat{k} = 1$ and $k = (1, \ldots, 1)$. Hence $|A_k| = d$ and this is not possible).

Case 9. If $1 = q = k_j < \hat{k}$. In this case $|B_k| \neq 0$ and $\Phi_j(k) = r_{\hat{k}}^{-1-\frac{1}{f}}$

$$
\begin{aligned}
\|w_{i,j,k}(z)\|_1 &\leqq dr_2^{\frac{1}{2d}} r_{\hat{k}}^{-1-\frac{1}{f}} 2 \|P_{\alpha,1}(x_{i,\hat{k}}(z))\|_1 \\
&\leqq 2dr_2^{\frac{1}{2d}} r_{\hat{k}}^{-1-\frac{1}{f}} r_{\hat{k}} \\
&= 2dr_2^{\frac{1}{2d}} r_{\hat{k}}^{-\frac{1}{f}} \\
&\leqq 2dr_2^{\frac{1}{2d}-\frac{1}{d}} \quad \text{since } 1 \leqq f \leqq d \\
&\leqq 2dr_2^{-\frac{1}{2d}} \\
&\leqq 1 \quad \text{since } r_2 \leqq (2d)^{2d}.
\end{aligned}
$$

Case 10. *If* $1 < k_j \leqq q < \hat{k}$. We have $\Phi_j(k) = r_{\hat{k}}^{-1}$.

$$
\begin{aligned}
\|w_{i,j,k}(z)\|_q &\leqq dr_2^{\frac{1}{2d}} r_{\hat{k}}^{-1} 2^{k_j} \|P_{\alpha,k_j}(x_{i,\hat{k}}(z))\|_q \\
&\leqq d2^q r_2^{\frac{1}{2d}} r_{\hat{k}}^{-1} r_{\hat{k}} \\
&= d2^q r_2^{\frac{1}{2d}}
\end{aligned}
$$

Case 11. If $k_j = \hat{k} < q$. We have $\Phi_j(k) \leqq r_{\hat{k}}^{\frac{d+1-e}{e}}$.

$$\begin{aligned}\|w_{i,j,k}(z)\|_q &\leqq d r_2^{\frac{1}{2d}} r_{\hat{k}}^{\frac{d+1-e}{e}} 2^{\hat{k}} \|P_{\alpha,\hat{k}}(x_{i,\hat{k}}(z))\|_q \\ &\leqq d r_2^{\frac{1}{2d}} r_{\hat{k}}^{\frac{d+1-e}{e}} 2^q C_{q,\hat{k}} \|P_{\alpha,\hat{k}}(x_{i,\hat{k}}(z))\|_{\hat{k}} \\ &\leqq d 2^q r_2^{\frac{1}{2d}} r_{\hat{k}}^{\frac{d+1-e}{e}} r_{\hat{k}} C_{q,\hat{k}} \\ &= d 2^q r_2^{\frac{1}{2d}} r_q^{\frac{d+1}{e}} c_q\end{aligned}$$

Case 12. If $k_j < \hat{k} < q$. Then $\Phi_j(k) \leqq r_{\hat{k}}^{-1}$.

$$\begin{aligned}\|w_{i,j,k}(z)\|_q &\leqq d r_2^{\frac{1}{2d}} r_{\hat{k}}^{-1} 2^{k_j} \|P_{\alpha,k_j}(x_{i,\hat{k}}(z))\|_q \\ &\leqq d r_2^{\frac{1}{2d}} r_{\hat{k}}^{-1} 2^q C_{q,\hat{k}} \|P_{\alpha,k_j}(x_{i,\hat{k}}(z))\|_k \\ &\leqq d r_2^{\frac{1}{2d}} 2^q c_q\end{aligned}$$

Hence the set consisting of

$$\{P_{\alpha,1}(x_{i,1}(z))\}_{\substack{i=1\\ z\in B}}^{\infty} \cup \{r_2 2^k P_{\alpha,k}(x_{i,k}(z))\}_{i=1,k\geqq 2,z\in B}^{\infty}$$
$$\cup \{d r_2^{\frac{1}{2d}} \Phi_j(k) 2^{k_j} P_{\alpha,k_j}(x_{i,\hat{k}}(z))\}_{\substack{i=1,k=(k_1,\ldots,k_d)\in N^d\\ z\in B,\,1\leqq j\leqq d,\,|A_k|<d}}^{\infty}$$

is a bounded subsetof E and we denote by B_2 its closed convex balanced hull. By cases 1, 3, 6, 7 and 9 it follows that $B_2 \subset U_1$.

By (2.4)

$$\begin{aligned}&\left\| S\left(\bigotimes_{d,j} \Phi_m{}^j(k) r_2^{\frac{1}{2d}} 2^{k_j} P_{\alpha,k_j}(x_{i,\hat{k}}(z))\right)\right\|_{B_2} \\ &\leqq \frac{d^d}{d!} \prod_{j=1}^{d} \left\| r_2^{\frac{1}{2d}} \Phi_j(k) 2^{k_j} P_{\alpha,k_j}(x_{i,\hat{k}}(z))\right\|_{B_2} \\ &\leqq \frac{d^d}{d!} \cdot \frac{1}{d^d} < 1 \quad \text{(by case (6) to (12)).}\end{aligned}$$

Hence

$$S\left(\bigotimes_{d,j} r_2^{\frac{1}{2d}} \Phi_j(k) 2^{k_j} P_{\alpha,k_j}(x_{i,k}(z))\right)$$
$$= \sum_{i=1}^{\infty} \bigotimes_{d} \theta_{l,k,i}^d$$

where $\sum_{l=1}^{\infty} \|\theta_{l,k,i}\|_{B_2}^d \leqq 1$.

We now have the formal sum of symmetric tensors

$$\sum_{i=1}^{\infty} \lambda_{i,1}(z) \bigotimes_{d} P_{\alpha,1}(x_{i,1}(z)) + \sum_{i=1}^{\infty} \sum_{k=2}^{\infty} \lambda_{i,k}(z) 2^{-kd} r_2^{-d} \bigotimes_{d} (r_2 2^k P_{\alpha,k}(x_{i,k}(z)))$$

$$+ \sum_{i=1}^{\infty} \sum_{\substack{k=(k_1,\ldots,k_d) \\ |A_k|<d}}^{\infty} \lambda_{i,\hat{k}}(z) 2^{-\Sigma_{j=1}^{d} k_j} r_2^{-\frac{1}{2}} \left(\sum_{l=1}^{\infty} \|\theta_{l,k,i}\|_{B_2}^{d} \bigotimes_{d} \left(\frac{\theta_{l,k,i}}{\|\theta_{l,k,i}\|_{B_2}} \right) \right)$$

Now

$$\sum_{i=1}^{\infty} |\lambda_{i,1}(z)| \leqq 1 + \varepsilon, \quad \sum_{i=1}^{\infty} \sum_{k=2}^{\infty} |\lambda_{i,k}(z)| 2^{-kd} r_2^{-d} \leqq \frac{1+\varepsilon}{r_2^d} \leqq \varepsilon$$

and

$$\sum_{i=1}^{\infty} \sum_{\substack{k=(k_1,\ldots,k_d) \\ |A_k|<d}} |\lambda_{i,\hat{k}}(z)| 2^{-\Sigma_{j=1}^{d} k_j} r_2^{-\frac{1}{2}} \left(\sum_{l=1}^{\infty} \|\theta_{l,k,i}\|_{B_2}^{d} \right)$$

$$\leqq \frac{1+\varepsilon}{r_2^{\frac{1}{2}}} \leqq \varepsilon$$

and

$$\|P_{\alpha,1}(x_{i,1}(z))\|_{B_2} \leqq 1, \quad \|r_2 2^k P_{\alpha,k}(x_{i,k}(z))\|_{B_2} \leqq 1$$

so that this formal sum is absolutely convergent to some z_0 in $\widehat{\bigotimes}_{d,\pi,s} E$.

For any $k = (k_1, \ldots, k_d) \in N^d$ we have

$$P_{\alpha,k_1} \otimes P_{\alpha,k_2} \cdots \otimes P_{\alpha,k_d}(z_0) = \sum_{i=1}^{\infty} \lambda_{i,\hat{k}}(z) \bigotimes_{j,d} P_{\alpha,k_j}(x_{i,\hat{k}}(z))$$

$$= P_{\alpha,k_1} \otimes P_{\alpha,k_2} \cdots \otimes P_{\alpha,k_d}(z).$$

Hence $z = z_0$ and

$$z = \sum_{j=1}^{\infty} \beta_j \bigotimes_{d} x_j \quad \text{where } \sum_{j=1}^{\infty} |\beta_j| \leqq 1 + 3\varepsilon$$

and $\|x_j\|_{B_2} \leqq 1$ for all j.

This shows that

$$B \subset (1+3\varepsilon)\bar{\Gamma}\left(\bigotimes_{d,s} B_2 \right) = \bar{\Gamma}\left(\bigotimes_{d,s} (1+3\varepsilon)^{\frac{1}{d}} B_2 \right)$$

Since ε was arbitrary this completes the proof.

§3. Applications to Spaces of Holomorphic Functions.

To apply propositon 11 we first need to improve inequality (1.5). This we do in the following proposition.

PROPOSITION 12. *Let U denote a balanced open subset of a Fréchet space E and let $(V_j)_j$ denote a neighbourhood basis at the origin. If p is a τ_ω continuous semi-norm on $\mathscr{H}(U)$ then there exists a compact subset K of U and a non-decreasing surjective mapping Φ: $N \cup \{0\} \to N$ and $c > 0$ such that*

$$p\left(\frac{\hat{d}^n f(0)}{n!}\right) \leqq c\left\|\frac{\hat{d}^n f(0)}{n!}\right\|_{K+V_{\Phi(n)}}$$

for all $f \in \mathscr{H}(U)$ and all n.

PROOF. By (1.5) there exists a compact balanced subset K of U such that for every neighbourhood V of 0 there exists $c(V) > 0$ such that

$$p\left(\frac{\hat{d}^n f(0)}{n!}\right) \leqq c(V)\left\|\frac{\hat{d}^n f(0)}{n!}\right\|_{K+V}$$

for all $f \in \mathscr{H}(U)$ and all n.

Choose $\lambda > 1$ such that λK is also a compact subset of U. For each positive integer j we can choose a positive integer n_j such that $c(\lambda^{-1}V_j)/\lambda^n \leqq 1$ for all $n \geqq n_j$. Then

$$\begin{aligned}
p\left(\frac{\hat{d}^n f(0)}{n!}\right) &\leqq c(\lambda^{-1}V_j)\left\|\frac{\hat{d}^n f(0)}{n!}\right\|_{K+\frac{1}{\lambda}V_j} \\
&\leqq \frac{c(\lambda^{-1}V_j)}{\lambda^n}\left\|\frac{\hat{d}^n f(0)}{n!}\right\|_{\lambda K+V_j} \\
&\leqq \left\|\frac{\hat{d}^n f(0)}{n!}\right\|_{\lambda K+V_j}
\end{aligned}$$

for all $f \in \mathscr{H}(U)$ and all $n \geqq n_j$.

We may suppose, without loss of generality, that the sequence $(n_j)_j$ is strictly increasing.

Let

$$\Phi(n) = \begin{cases} 1 & \text{for} \quad n < n_1 \\ j & \text{for} \ \ n_j \leqq n < n_{j+1}, j \geqq 1 \end{cases}$$

Then

$$p\left(\frac{\hat{d}^n f(0)}{n!}\right) \leqq \left\| \frac{\hat{d}^n f(0)}{n!} \right\|_{\lambda K + V_{\Phi(n)}}$$

for all $f \in \mathscr{H}(U)$ and $n \geqq n_1$.

For $n < n_1$ we have

$$p\left(\frac{\hat{d}^n f(0)}{n!}\right) \leqq \frac{c(\lambda^{-1} V_1)}{\lambda^n} \left\| \frac{\hat{d}^n f(0)}{n!} \right\|_{\lambda K + V_1}$$

for all $f \in \mathscr{H}(U)$.

If $c = 1 + c(\lambda^{-1} V_1)$ this completes the proof.

THEOREM 13. *Let E denote a Fréchet space with a T-Schauder decomposition and the density condition and let U denote a convex balanced T-invariant open subset of E. Then*

$$(\mathscr{H}(U), \tau_\omega) = (\mathscr{H}(U), \tau_b).$$

PROOF. Let p denote a τ_ω continuous semi-norm on $\mathscr{H}(U)$ and let $(\|\cdot\|_k)_{k \geqq 1}$ denote a T-adaptable set of seminorms on E. We may suppose, by proposition 12 and Lemma 7, that

$$p\left(\sum_{n=0}^{\infty} \frac{\hat{d}^n f(0)}{n!}\right) = \sum_{n=0}^{\infty} p\left(\frac{\hat{d}^n f(0)}{n!}\right)$$

and

$$p\left(\frac{\hat{d}^n f(0)}{n!}\right) \leqq \left\| \frac{\hat{d}^n f(0)}{n!} \right\|_{K + U_{\phi(n)}} \tag{3.1}$$

for all $f \in \mathscr{H}(U)$ and all n where K is an absolutely convex balanced T-invariant compact subset of U, $U_j = \{x \in E; \|x\|_j \leqq 1\}$ and $\phi: N \cup \{0\} \to N$ is a non-decreasing surjective mapping.

We now consider the semi-norm $p_n := p|_{\mathscr{P}(^nE)}$ restricted to $\mathscr{P}(^nE)$ for some fixed n. By proposition 11, $\tau_b = \beta$ on $\mathscr{P}(^nE)$ and since E has the density condition $\beta = \tau_\omega$ on $\mathscr{P}(^nE)$.

Let q_1 denote the semi-norm on E with unit ball $V := K + U_{\phi(n)}$, and let $q_k = \|\cdot\|_{\phi(n)+k-1}$ for $k \geqq 2$. Since $(\|\cdot\|_k)_{k \geqq \phi(n)}$ is a T-adaptable set of semi-norms on E proposition 8 implies that $(q_k)_{k \geqq 1}$ is also a T-adaptable set of seminorms on E.

Let $B = \{\phi \in \mathscr{P}(^nE)'; |\phi| \leqq p_n\}$. We have $p_n(P) = \|P\|_B$ for all $P \in \mathscr{P}(^nE)$.

By (3.1)

$$p_n(P) \leqq \|P\|_{\left(\bigotimes_{s,n} V\right)^{00}}$$

where we are considering P as an element of $\left(\widehat{\bigotimes}_{n,\pi,s} E\right)'$.

By the Hahn-Banach theorem $B \subset \left(\bigotimes_{s,n} V\right)^{00}$. Since E has the density condition $\left(\widehat{\bigotimes}_{n,\pi,s} E\right)$ is distinguished and hence, as B is a bounded subset of $\left(\left(\widehat{\bigotimes}_{n,\pi,s} E\right)'_{\beta}\right)'_{\beta}$ we have, by remark 1.4 of [17] (see also [19, lemma 1]), that there exists a bounded subset A *of* $\overline{\bigotimes_{n,s} V}$ such that $B \subset A^{00}$.

Hence $p_n(P) \leqq \|P\|_A$ for all $P \in \mathscr{P}(^nE)$.

By proposition 11 there exists, for any $\varepsilon > 0$, a bounded subset C of $(1 + \varepsilon)\bar{V}$ such that $A \subset \bar{\Gamma}\left(\bigotimes_{s,n} C\right)$. Hence $p_n(P) \leq \sup_{x \in C} |P(x)|$ for all $P \in \mathscr{P}(^nE)$.

Now let $C = C_n$ and $\varepsilon = \varepsilon_n$.

We have thus shown that for all n there exists a bounded subset C_n of $(1 + \varepsilon_n)(K + U_{\phi(n)})$ such that

$$p(P) \leqq \|P\|_{C_n} \quad \text{for all } P \in \mathscr{P}(^nE).$$

By choosing the ε_n's sufficiently small and by noting that $\phi(n) \to \infty$ as $n \to \infty$ we see that the sequence $(C_n)_n$ converges to the compact set K. By (3.1) we have

$$p\left(\sum_{n=0}^{\infty} \frac{\hat{d}^n f(0)}{n!}\right) \leqq \sum_{n=0}^{\infty} \left\|\frac{\hat{d}^n f(0)}{n!}\right\|_{C_n}$$

and hence p is a τ_β continuous seminorm on $\mathscr{H}(U)$. This completes the proof.

Corollary 14. *If K is a T-invariant compact convex balanced subset of a Fréchet space E with a T-Schauder decomposition and E has the density condition then the τ_ω topology on $\mathscr{H}(K)$ is generated by semi-norms of the form*

$$p\left(\sum_{n=0}^{\infty} \frac{\hat{d}^n f(0)}{n!}\right) = \sum_{n=0}^{\infty} \left\|\frac{\hat{d}^n f(0)}{n!}\right\|_{B_n}$$

where $(B_n)_n$ is a sequence of bounded subsets of E which converges to K as $n \to \infty$.

Theorem 13 and corollary 14 are also true for entire functions and germs at the origin when E is a complemented subspace of a Fréchet space with the density condition and a T-Schauder decomposition.

By proposition 3.6 of [15] we also have the following corollary.

COROLLARY 15. *If U is a balanced open subset of a Fréchet space E with a T-Schauder decomposition and the density condition then $(\mathscr{H}_b(U), \beta)$ is quasinormable.*

(β is the topology of uniform convergence on the bounded subsets of E which lie strictly inside U).

By example 10 we have the following result.

COROLLARY 16. *If U is a balanced open subset of a Banach space then $(\mathscr{H}_b(\mathscr{H}_b(U)), \beta)$ is a quasinormable Fréchet space.*

The method used in the proof of proposition 12 can also be used to prove the following result.

PROPOSITION 17. *If K is a compact balanced subset of a Fréchet space then the τ_0 topology on $\mathscr{H}(K)$ is generated by all seminorms of the form*

$$p\left(\sum_{n=0}^{\infty} \frac{\hat{d}^n f(0)}{n!}\right) = \sum_{n=0}^{\infty} \left\| \frac{\hat{d}^n f(0)}{n!} \right\|_{K_n}$$

where $(K_n)_n$ is a sequence of compact subsets of E which converges to K.

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