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APPROXIMATION BY SOLUTIONS OF ELLIPTIC EQUATIONS ON CLOSED SUBSETS OF EUCLIDEAN SPACE

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0. Introduction.

In the last twenty years the question of how to reduce problems of qualitative approximation by analytic or harmonic functions on unbounded closed sets to the compact case has been often considered and different answers have been given in a variety of particular cases. In this paper we provide a general method of localization which applies to all instances previously dealt with and which considerably simplyfies the available proofs.

As an example of the kind of results we are envisaging we mention Nersesjan's Theorem on uniform approximation by analytic functions of one complex variable. Denoting by α continuous analytic capacity [17] we have

THEOREM 1. (Nersesjan [11]). *Let F be a closed subset of the complex plane. Then the following are equivalent:*

(i) *Each continuous function on F which is analytic on F^0 can be uniformly approximated on F by functions which are analytic on some neighbourhood (depending on the approximating function) of F.*

(ii) $\alpha(D \backslash F) = \alpha(D \backslash F^0)$, *for each disc D.*

For compact F the above statement is just the well known Vitushkin Theorem [17, p. 183] on rational approximation. To settle the case of unbounded closed sets Nersesjan had to find a suitable new localization argument. Later on Hadjiisky [6] discovered another way of localizing the problem which gave a direct proof of (ii) $\Rightarrow$ (i) without appealing to the compact case. This idea was further exploited in [2] to deal with the analytic approximation problem on general closed sets in L^p, Lipschitz and BMO norms. Another interesting reference is [15].

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In Section 1 we give a short proof of Theorem 1 which already contains some of the elements of our general method. To get a better insight into it we prove in Section 2 the harmonic analog of Theorem 1 (see Section 2 for a precise statement). In that proof the reader will find in action all the ingredients of our idea. In Section 3 we present a fairly general context in which our localization method works: we are able to deal with a homogeneous elliptic operator on $\mathbf{R}^n$ with constant complex coefficient, the approximation taking place in the norm of a Banach space satisfying certain conditions.

In Section 4 we study weighted uniform analytic approximation in the plane showing that the conditions required in Section 3 are, in some sense, sharp.

Our notational conventions will be standard. For example, C will denote a constant, independent of the relevant variables under consideration and which might be different in different occurencies. The open ball with center a and radius δ is denoted by $B(a, \delta)$. If B is a ball, kB is the ball with the same center and radius k times the radius of B.

1. Proof of Nersesjan's Theorem.

The important part in Theorem 1 is (ii) $\Rightarrow$ (i). Let us then assume that F is a closed subset of the complex plane and that f is a continuous function on F which is analytic on F^0. Extend f continuously to the whole of $\mathbf{C}$. We wish now to localize the singularities of f by means of Vitushkin's method. To this end take a covering of $\mathbf{C}$ by discs (D_j) of radius 1, which is almost disjoint, in the sense that each point in $\mathbf{C}$ belongs to at most a fixed number of discs D_j. Let (φ_j) be a C^∞ partition of unity subordinated to (D_j). Set $f_j = V_{\varphi_j}(f) \equiv \dfrac{1}{\pi z} * (\varphi_j \bar{\partial} f)$. Then f_j is continuous on $\mathbf{C}$, analytic on F^0 and outside D_j, and [17, p. 150]

$$\|f\| \leqq CN_j \|f\|_{D_j} \tag{1}$$

where C is an absolute constant and $N_j = \|\varphi_j\| + \|\nabla \varphi_j\|$. In this and in the next section we will denote by $\|\ \|_E$ the supremum norm on the set E and we will let $\|\ \|$ stand for $\|\ \|_{\mathbf{C}}$.

Condition (ii) implies that there exists $C > 0$ such that, setting $F_j = F \cap \bar{D}_j$, we have $\alpha(D \setminus F_j^0) \leqq C\alpha(2D \setminus F_j)$ for all discs D. Then Vitushkin's Theorem gives (i) with F replaced by F_j. Therefore, for fixed j, there exists h_j, analytic on a neighbourhood of F_j such that $\|f - h_j\|_{F_j} < \eta = \varepsilon/(2^j N_j)$, where ε is a given positive number. A well known modification argument, which we reproduce below for the reader's convenience, gives that in fact we can assume $\|f - h_j\| < \eta$. To see this set $d_j = f - h_j$. For some open neighbourhood U of F_j on which h_j is analytic we still have $\|d_j\|_{\bar{U}} < \eta$. Extend d_j from $\bar{U}$ to a continuous function on $\mathbf{C}$, still denoted by d_j, satisfying $\|d_j\| < \eta$. Modify h_j outside U in such a way that the identity

$h_j = f - d_j$ holds everywhere. Then h_j is analytic on a neighbourhood of F_j and $\|f - h_j\| < \eta$.

Define $g_j = V_{\varphi_j}(h_j)$, so that for some absolute constant C we have by (1) $\|f_j - g_j\| = \|V_{\varphi_j}(f - h_j)\| \leqq CN_j\|f - h_j\| < C\varepsilon/2^j$. It is not difficult to see that $g = f - \sum_j (f_j - g_j)$ is analytic on some neighbourhood of F. Since $\|f - g\| < C\varepsilon$, g is the desired approximant.

2. Uniform harmonic approximation.

In this section we will prove the following harmonic analog of Nersesjan's Theorem [5], [9].

THEOREM 2. *Let F be a closed subset of $\mathbf{R}^n$. Then the following are equivalent.*

(i) *Each continuous function on F which is harmonic on F^0 can be uniformly approximated on F by functions which are harmonic on neighbourhoods of F.*

(ii) $\operatorname{Cap}(B\backslash F) = \operatorname{Cap}(B\backslash F^0)$, *for each ball B.*

Here Cap stands for the classical Wiener capacity of potential theory. For F compact the result goes back to Deny and Keldysh [3], [8] in slightly different formulations. For $n \geqq 3$ the proof proceeds along the lines of the preceeding section but for $n = 2$ a new difficulty arises owing to the fact that the fundamental solution $\frac{1}{2\pi}\log|z|$ of the Laplacean Δ is unbounded at ∞. We shall therefore concentrate on the proof of (ii) $\Rightarrow$ (i) in the plane. Before starting with the details some remarks are in order concerning Vitushkin's localization operator for Δ in dimension 2.

Let D be a disc of radius δ, $\varphi \in C_0^2(D)$ and set $N(\varphi) = \sum_{j=0}^{2} \delta^j \|\nabla^j \varphi\|$. Given a continuous function f in $\mathbf{C}$ define $V_\varphi f = \frac{1}{2\pi}\log|z| * (\varphi\Delta f)$, so that $\Delta(V_\varphi f) = \varphi\Delta f$ in the distributional sense. A simple computation [1] gives

$$V_\varphi f(z) = \varphi(z)f(z) + \frac{1}{2\pi}(\log|\zeta| * f\Delta\varphi)(z) - \frac{1}{\pi}\left(\frac{1}{\zeta} * f\bar{\partial}\varphi\right)(z) - \frac{1}{\pi}\left(\frac{1}{\bar{\zeta}} * f\partial\varphi\right)(z),$$

and so

$$\|V_\varphi f\|_D \leqq CN(\varphi)\|f\|_D, \tag{2}$$

the constant C depending only on δ. It is important to realize that we cannot replace the left hand side of (2) by $\|V_\varphi f\|$ because $V_\varphi f$ has a logarithmic singularity at ∞, and this fact is the only obstruction to the argument used in Section 1.

We proceed now to the proof of Theorem 2. Let F be a closed subset of the plane and f a continuous function on $\mathbf{C}$ which is harmonic on F^0. Let (D_j) be an almost disjoint covering of $\mathbf{C}$ by open discs of radius 1 and (φ_j) a C^∞ partition of the unity subordinated to (D_j). For fixed j and given $\eta_j > 0$ (to be specified later) choose h_j harmonic on a neighbourhood of $F_j = \bar{D}_j \cap F$ such that $\|f - h_j\|_{F_j} < \eta_j$. This is possible, arguing as in the preceding section, because we know that Theorem 2 holds for the compact sets F_j. Using the modification argument of Section 1 we can furthermore suppose that h_j is continuous on $\mathbf{C}$ and $\|f - h_j\| < \eta_j$. Set $g_j = V_{\varphi_j}(h_j)$. The function $f_j - g_j$ is harmonic outside D_j and has a logarithmic singularity at ∞. Thus, assuming that D_j is centered at the origin,

$$f_j(z) - g_j(z) = a_j \log|z| + H_j(z), \quad |z| > 1,$$

where H_j is harmonic outside D_j and at ∞, and $a_j = \dfrac{1}{2\pi}\displaystyle\int \Delta\varphi_j(f - h_j)\,dx\,dy.$

Hence, for some constant C_j depending only on j, $|a_j| \leqq C_j\|f - h_j\|_{D_j} \leqq C_j\eta_j$. If $D_j \subset F$ then $f_j \equiv 0$, so we can take for granted that D_j contains a disc $D \subset \mathbf{C}\backslash F$. Let ψ be a C^∞ function such that $\psi = 1$ outside D and $\psi = 0$ on $\frac{1}{2}D$. Set $L_j(z) = a_j\psi(z)\log|z - c|$, where c is the center of D. Then L_j is harmonic on a neighbourhood of F, $\|L_j\|_{D_j} \leqq C_j\eta_j$, and $f_j - g_j - L_j$ is harmonic outside D_j and at ∞. Therefore, using (2),

$$\|f_j - g_j - L_j\| \leqq \|f_j - g_j - L_j\|_{D_j} \leqq \|V_{\varphi_j}(f - h_j)\|_{D_j} + \|L_j\|_{D_j} \leqq C_j\eta_j.$$

Choose now η_j so that $C_j\eta_j = \varepsilon/2^j$, where ε has been given in advance. Set $g = f - \sum_j (f_j - g_j - L_j)$. It is easy to check that g is harmonic on a neighbourhood of F. Since $\|f - g\| < \varepsilon$ the proof is complete.

3. The main result.

The goal of this section is to describe a general setting in which our localization method works.

We do not wish to restrict our attention to uniform approximation, so we start by introducing a certain class of Banach spaces in whose norm the approximation will take place.

Following [12] we let V stand for a Banach space, whose norm is denoted by $\|\ \|$, which contains C_0^∞, the set of test functions in $\mathbf{R}^n$, and is contained in $(C_0^\infty)^*$, the space of distributions. We assume that V is a topological C_0^∞-submodule of $(C_0^\infty)^*$, which means that for $f \in V$ and $\varphi \in C_0^\infty$

$$|\langle f, \varphi\rangle| \leqq C(\varphi)\|f\|, \tag{3}$$

$\langle f, \varphi\rangle$ denoting the action of the distribution f on the test function φ, and

$$\|\varphi f\| \leqq C(\varphi)\|f\|, \tag{4}$$

where $C(\varphi)$ is a constant independent of f.

Given a closed subset F of R^n let $I(F)$ be the closure in V of those $f \in V$ whose support (in the sense of distributions) is disjoint from F. The Banach space $V(F) = V/I(F)$, endowed with the quotient norm, should be viewed as the natural version of V on F. We will write $\|f\|_F$ for the norm of the equivalence class in $V(F)$ of the distribution $f \in V$.

We need also to introduce local versions of V and of $V(F)$. Let V_{loc} be the set of distributions f such that $\varphi f \in V$, $\varphi \in C_0^\infty$. There is a natural Frechet topology in V_{loc} given by the seminorms $\|f\|_m = \|\varphi_m f\|_{B_m}$, where $B_m = \{|x| \leqq m\}$ and $\varphi_m \in C_0^\infty$ is a fixed function taking the value 1 on some neighbourhood of B_m. Define $V_{\mathrm{loc}}(F) = V_{\mathrm{loc}}/J(F)$, where $J(F)$ is the closure in V_{loc} of those distributions in V_{loc} whose support is disjoint from F.

We present now some examples (see also Section 4 in which a non-translation invariant example of V is considered).

EXAMPLE 1. $V = L^p(\mathsf{R}^n)$, $1 \leqq p \leqq \infty$. Clearly $L^p(F)$, L^p_{loc} and $L^p_{\mathrm{loc}}(F)$ are the standard spaces denotes by these symbols.

EXAMPLE 2. $V = \mathrm{VMO}(\mathsf{R}^n)$, the space of functions of vanishing mean oscillation. In [7] one finds an intrinsic characterization of $\mathrm{VMO}(F)$ involving only the values taken by functions on F.

EXAMPLE 3. $V = C^m(\mathsf{R}^n)$, m being a non-negative integer. This is the space of functions with bounded continuous derivatives up to order m endowed with any of the standard norms associated to it. For example,

$$\|f\| = \sup_{|\alpha| \leqq m} \|\partial^\alpha f\|_\infty .$$

In this case $C^m(\mathsf{R}^n)_{\mathrm{loc}}$ is just the space of functions with continuous partial derivatives up to order m. Notice that, for $m \geqq 1$ and $F \neq \mathsf{R}^n$, $C^m(F)$ is not a space of functions. With the help of the Whitney extension theorem, it can be identified with a space of jets (see [14, Chapter VI]).

EXAMPLE 4. $V = \lambda^s(\mathsf{R}^n)$, s a non-integer positive real number. Writing $s = m + \sigma$, with m an integer and $0 < \sigma < 1$, we have that $f \in \lambda^s(\mathsf{R}^n)$ if and only if $f \in C^m(\mathsf{R}^n)$, $\sup_{\delta > 0} \omega(\delta)\delta^{-\sigma} < \infty$ and $\omega(\delta)\delta^{-\sigma} \to 0$ as $\delta \to 0$, where $\omega(\delta) = \sup_{|\alpha| = m} \sup_{|x-y| < \delta} |\partial^\alpha f(x) - \partial^\alpha f(y)|$. We set $\|f\| = \|f\|_{C^m(\mathsf{R}^n)} + \sup_{\delta > 0} \omega(\delta)\delta^{-\sigma}$.

For $0 < s < 1$, $\lambda^s_{\mathrm{loc}}(F)$ turns out to be the set of functions on F which locally satisfy a little "o" Lipschitz condition of order s. For $1 < s$, the remark concerning jets made in the previous example still applies.

The approximating functions in our abstract theorem will not be necessarily analytic or harmonic, but instead they will be annihillated by a complex constant coefficients homogeneous elliptic operator L of order r, as in [16].

We describe now the two basic assumptions on V which make our localization argument work.

First, the Vitushkin localization operator associated to L satisfies adequate estimates. Let B be an open ball of radius δ, $\varphi \in C_0^\infty(B)$, and set $V_\varphi f = \Phi * (\varphi L f)$, where Φ is a fundamental solution of L and f a distribution on R^n. We recall that Φ can be taken of the form $\Phi(x) = \Phi_0(x) + P(x) \log |x|$, where $\Phi_0(x)$ is a C^∞ function in $\mathrm{R}^n \setminus \{0\}$, homogeneous of degree $r - n$, and $P(x)$ is a polynomial which is either zero (this is the case if $r < n$) or homogeneous of degree $r - n$. We will require that our Banach space V satisfies the estimate

$$\|V_\varphi f)\|_{\bar{B}} \leqq N(\varphi, B) \|f\|, \tag{5}$$

where $N(\varphi, B)$ is independent of f. The definition of $\|f\|_K$, $K =$ support of φ, immediately gives that (5) can be improved to

$$\|V_\varphi f\|_{\bar{B}} \leqq N(\varphi, B) \|f\|_K. \tag{6}$$

Notice that (6) (or equivalently (5)) means exactly that V_φ sends continuously V_{loc} into V_{loc}.

The proof of (5) for the examples considered above can be found in [1], [2], [10], [12], [13], [16]. In [12] the reader will even find a proof of (5) for a wide class of abstract Banach spaces.

Our second assumption on V is more technical. We require that for some non-negative integer p one has

$$\|\partial^\alpha \Phi\|_{\mathrm{R}^n \setminus B(0,R)} \leqq \alpha!\, \varepsilon(R)^{|\alpha|}, \quad |\alpha| \geqq p, \tag{7}$$

where $\varepsilon(R) \to 0$ as $R \to \infty$.

That (7) holds in the examples 1–4 follows from

$$|\partial^\alpha \Phi(x)| \leqq \alpha!\, C^{|\alpha|} |x|^{-(n-r+|\alpha|)} (\log |x|| + 1), \quad x \neq 0, \tag{8}$$

which is essentially equivalent to the real analyticity of Φ outside the origin. For instance, p is 0 for $L = \bar{\partial}$ and $V = C^m(\mathrm{C})$, and p is 1 for $L = \bar{\partial}$ and $V = L^2(\mathrm{C})$. For $L = \Delta$, $V = C^m(\mathrm{C})$, p is 1 in the plane and p is 0 for all dimensions larger than 2.

Our next task will be to prove that (7) gives a sort of maximum principle for the exterior of a ball and the norm of V. We start by discussing expansions of potentials at ∞.

From (8) it follows that there exists $k > 1$ such that given any $x \neq 0$ we have an expansion $\Phi(z) = \sum_{|\alpha| \geqq 0} \frac{\partial^\alpha \Phi(x)}{\alpha!} (z - x)^\alpha$, in the ball $|z - x| < k^{-1}|x|$, the series being absolutely convergent there. Consequently, given points a and x such that $|x - a| > k\delta$ for some $\delta > 0$, we have an expansion

$$\Phi(x - y) = \sum_{|\alpha| \geqq 0} (-1)^{|\alpha|} \frac{\partial^\alpha \Phi(x - a)}{\alpha!} (y - a)^\alpha, \quad |y - a| < \delta,$$

the series converging in $C^\infty(B(a, \delta))$. Let T be a distribution with compact support contained in $B(a, \delta)$ and set $f = \Phi * T$. Then, for $|x - a| > k\delta$

$$f(x) = \langle T, \Phi(x - y) \rangle = \sum_{|\alpha| \geqq 0} c_\alpha \partial^\alpha \Phi(x - a),$$

where $c_\alpha = (-1)^{|\alpha|}(\alpha!)^{-1}\langle T, (y - a)^\alpha \rangle$, the series converging in $C^\infty(|x - a| > k\delta)$.

We would like to point out here that the above statement is not true for $k = 1$ as was claimed in [10] and [16]. The results proved there are not affected by this missing dilation factor.

LEMMA 1. *Let f be a distribution such that Lf has compact support and $f(x) \to 0$ as $|x| \to \infty$. Then $f = \Phi * Lf$.*

PROOF. $f - (\Phi * Lf)$ is a tempered distribution annihillated by L. Thus $f = \Phi * Lf + P$ for some polynomial P. To show that $P \equiv 0$ set, for $|x| \to \infty$,

$$(\Phi * Lf)(x) = \sum_{|\alpha| \geqq k} c_\alpha \partial^\alpha \Phi(x),$$

where $c_\alpha \neq 0$ for some α with $|\alpha| = k$. One can write [16, p. 161]

$$\sum_{|\alpha| = k} c_\alpha \partial^\alpha \Phi(x) = H(x) + Q(x) \log |x|$$

for some polynomial Q of degree $r - n - k$ and some function H, C^∞ outside the origin and homogeneous of degree $r - n - k$. Clearly

$$f = H + Q \log |x| + g + P, \tag{9}$$

where $g = \sum_{|\alpha| > k} c_\alpha \partial^\alpha \Phi$. Arguing from the homogeneities of the different terms in (9), it is not difficult to conclude that $P \equiv 0$.

LEMMA 2. *Let B be an open ball and f a distribution in V_{loc} such that $Lf = 0$ on $\mathsf{R}^n \backslash B$ and $f(x) = O(|x|^{-d})$ as $x \to \infty$, where $d = \max\{p + n - r, 1\}$, p being the integer appearing in condition* (7). *Then $f \in V$ and $\|f\| \leqq C \|f\|_{3\bar{B}}$.*

PROOF. By Lemma 1 $f = \Phi * Lf$. Thus, for $q = d - n + r \geqq p$,

$$f(x) = \sum_{|\alpha| \geqq q} c_\alpha \partial^\alpha \Phi(x), \quad x \notin k\bar{B}.$$

Choose $\varphi \in C_0^\infty(2B)$, $\varphi = 1$ on B and $\psi \in C_0^\infty(3B)$, $\psi = 1$ on $2B$. Let us suppose first that B is centered at the origin. Then, using the Leibnitz formula,

$$\alpha!\,|c_\alpha| = |\langle Lf, \varphi(x)x^\alpha\rangle| = |\langle f, L(\varphi(x)x^\alpha)\rangle| \leqq \sum_{|\beta|=0}^{r} |\alpha|^r\,|\langle f, L^\beta(\varphi)x^{\alpha-\beta}\rangle| =$$

$$= \sum_{|\beta|=0}^{r} |\alpha|^r\,|\langle(\psi(x)x)^{\alpha-\beta} f, L^\beta(\varphi)\rangle|,$$

where L^β is a differential operator of order $r - |\beta|$ and in the sums above only indexes β with $\alpha - \beta \in \mathbf{Z}^n_+$ are taken. Observe now that (3) and (4) hold with $\|f\|$ replaced by $\|f\|_K$, K being the support of φ. Applying (3) and (4) in this sharper form, we get

$$\alpha!\,|c_\alpha| \leqq A\,|\alpha|^r C_1^{|\alpha|}\,\|f\|_{3\bar{B}},$$

where A depends only on r and

$$C_1 = \max_{|\beta|\leqq r} C(L^\beta\varphi) + \max_{1\leqq j\leqq n} C(\psi(x)x_j),$$

the constants in the right hand side being those appearing in (3) and (4) for the indicated functions.

If R satisfies $B(0, R) \supset kB$, we get

$$\|f\|_{\mathbf{R}^n\setminus B(0,R)} = \left\| \sum_{|\alpha|\geqq q} c_\alpha \partial^\alpha \Phi \right\|_{\mathbf{R}^n\setminus B(0,R)} \leqq$$

$$\leqq \sum_{|\alpha|\geqq q} (\alpha!)^{-1} A\,|\alpha|^r C_1^{|\alpha|}\,\|f\|_{3\bar{B}}\,\alpha!\,\varepsilon(R)^{|\alpha|} \leqq C\,\|f\|_{3\bar{B}},$$

provided R is large enough so that $C_1\varepsilon(R) < 1$.

It is easily proved that

$$\|f\| \leqq C(\|f\|_{2\bar{B}(0,R)} + \|f\|_{\mathbf{R}^n\setminus B(0,R)}).$$

On the other hand

$$\|f\|_{2\bar{B}(0,R)} = \|V_\varphi f\|_{2\bar{B}(0,R)} \leqq C\,\|f\|_{2\bar{B}},$$

because of (6) applied to $2\bar{B}(0, R)$, and so the desired estimate follows.

We are left with the task of removing the assumption that B is centered at the origin. Let $B = B(a, \delta)$ and take $\varphi \in C_0^\infty(3B)$, $\varphi = 1$ on $2B$. Then $f = V_\varphi(f)$ and so we get from (6)

$$\|f\|_{3\bar{B}(0,|a|+\delta)} \leqq C\,\|f\|_{3\bar{B}}. \tag{10}$$

Since Lf vanishes outside $\bar{B}(0, |a| + \delta)$,

$$\|f\| \leqq C\,\|f\|_{3\bar{B}(0,|a|+\delta)}. \tag{11}$$

Combining (10) and (11) one completes the proof of the Lemma.

We are now ready to prove our main result.

THEOREM 3. *Let V be a Banach space satisfying* (3), (4), (5) *and* (7), *F a closed subset of R^n and $f \in V_{\mathrm{loc}}$. Then the following statements are equivalent.*

(i) *Given a positive number ε there exists $g \in V_{\mathrm{loc}}$ such that $Lg = 0$ on some neighbourhood of F and $\|f - g\|_F < \varepsilon$.*

(ii) *Given a ball B and a positive number ε there exists $g \in V_{\mathrm{loc}}$ such that $Lg = 0$ on some neighbourhood of $F \cap \bar{B}$ and $\|f - g\|_{F \cap \bar{B}} < \varepsilon$.*

PROOF. We only need to show that (ii) implies (i). Let (B_j) be an almost disjoint covering of R^n by open balls B_j of radius 1 and let (φ_j) be a partition of unity subordinated to (B_j). Set $N_j = N(\varphi_j, 3\bar{B}_j)$. For fixed j and given $\eta > 0$ (to be specified later) choose $h_j \in V_{\mathrm{loc}}$ such that $Lh_j = 0$ on some neighbourhood of $F_j = F \cap \bar{B}_j$ and $\|f - h_j\|_{F_j} < \eta$.

We claim now that h_j can be modified to $H_j \in V_{\mathrm{loc}}$ so that $LH_j = 0$ on some neighbourhood of F_j and $\|f - H_j\| < \eta$. Using the definition of the norm in $V(F_j)$ we find an open neighbourhood U of F_j on which $Lh_j = 0$ and $\|f - h_j\|_{\bar{U}} < \eta$. Let $g_j \in V$ be such that $f - h_j = g_j$ on U and $\|g_j\| < \eta$. The distribution $H_j = f - g_j$ fulfills all requirements in the claim.

Set $f_j = V_{\varphi_j}(f)$ and $G_j = V_{\varphi_j}(H_j)$. If $B_j \subset F$ then $f_j = 0$ and if $B_j \subset \mathsf{R}^n \backslash F$ then $L(f_j) = 0$ on a neighbourhood of F. Hence, in what follows we will consider only indexes j such that B_j intersects ∂F. For such indexes B_j contains a ball $B = B(a, \delta) \subset \mathsf{R}^n \backslash F$. Let $\psi \in C^\infty(\mathsf{R}^n)$ be 1 outside B and 0 on $\frac{1}{2}B$. Set $\chi_j = \psi(x)\Phi(x - a)$ and $K_j = \sum_{|\alpha| < q} c_\alpha \partial^\alpha \chi_j$, where the coefficients c_α are defined by the expansion $f_j(x) - G_j(x) = \sum_{|\alpha| \geqq 0} c_\alpha \partial^\alpha \Phi(x - a)$ and $q = \max\{p, r - n + 1\}$, p being the integer appearing in (7).

Since $\alpha! \, |c_\alpha| = |\langle f - H_j, L(\varphi_j(x)(x - \alpha)^\alpha)\rangle|$, applying (3) we get the estimate

$$|c_\alpha| \leqq C(\alpha, j) \, \|f - H_j\| \leqq C(\alpha, j)\eta,$$

where $C(\alpha, j)$ is a constant depending only on α and j. Hence

$$\|K_j\|_{3\bar{B}_j} \leqq \sum_{|\alpha| < q} C(\alpha, j)\eta \, \|\partial^\alpha \chi_j\|_{3\bar{B}_j} = C(q, j)\eta,$$

where now $C(q, j)$ stands for a constant depending only on q and j.

On the other hand, the function $f_j - G_j - K_j$ satisfies the hypothesis of Lemma 2. Applying Lemma 2 and (6) we obtain

$$\|f_j - G_j - K_j\| \leqq C_j \, \|f_j - G_j - K_j\|_{3\bar{B}_j} \leqq$$

$$\leqq C_j \, \|V_{\varphi_j}(f - H_j)\|_{3\bar{B}_j} + C(q, j)\eta \leqq C_j N_j \eta + C(q, j)\eta = C(q, j)\eta.$$

Choose η so that $C(q, j)\eta = \varepsilon/2^j$, where ε has been given in advance, and define

$g = f - \sum_j (f_j - G_j - K_j)$. Then $\|f - f\| < \varepsilon$ and $Lg = 0$ on some neighbourhood of F. This shows (i).

REMARK. It is clear that Theorem 3 also gives the corresponding approximation results for classes of functions in the spirit of theorems 1 and 2. We also would like to mention that small modifications of our arguments would prove analogous theorems for Banach spaces V defined on subdomains of R^n.

4. Weighted uniform approximation.

Let ω be a positive radial continuous function on the plane. Let V be the set of continuous functions on $\mathbf{C}$ such that

$$\|f\|_\omega \equiv \sup_{z \in \mathbf{C}} |f(z)|\, \omega(z) < \infty.$$

If $F \subset \mathbf{C}$ is closed then, as it is easily seen, $f \in V(F)$ if and only if $\|f\|_{\omega,F} \equiv \sup_{z \in F} |f(z)|\, \omega(z) < \infty$, and the norm of $V(F)$ is exactly $\| \ \|_{\omega,F}$. Conditions (3), (4) and (5) clearly hold for any ω. Our last result states that condition (7) with Φ replaced by $1/\pi z$ is equivalent to the fact that local analytic approximation implies global analytic approximation.

THEOREM 4. *The following statements are equivalent.*

(i) *Let f be a continuous function on a closed subset F of the plane. If for each disc D and $\varepsilon > 0$ there exists a function g, analytic on some neighbourhood of $F \cap \bar{D}$ such that $\|f - g\|_{\omega, F \cap \bar{D}} < \varepsilon$, then for each $\varepsilon > 0$ there exists a function g, analytic on some neighbourhood of F such that $\|f - g\|_{\omega,F} < \varepsilon$.*

(ii) *Condition* (7), *with Φ replaced by* $1/\pi z$, *holds.*

(iii) *There exists a positive integer d such that if $f \in C(\mathbf{C})$ is analytic on $\mathbf{C} \setminus \bar{D}(0, \delta)$ and $f(z) = O(|z|^{-d})$ as $z \to \infty$ then $\|f\|_\omega \leqq C \|f\|_{\omega, \bar{D}(0, 3\delta)}$.*

(iv) *There exists a positive integer q such that* $\lim_{z \to \infty} \omega(z) |z|^{-q} = 0$.

PROOF. That (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i) follows from Theorem 3. A simple computation shows that (iv) $\Rightarrow$ (ii). Thus we only need to prove that (i) $\Rightarrow$ (iv).

If (iv) is not true then $\limsup_{z \to \infty} \omega(z) |z|^{-q} = \infty$ for all $q > 0$. Set $F = \{z \in \mathbf{C}: |z| \geqq 1\}$. Then the only function in $V(F)$ which is analytic on $\mathring{F}$ is the zero function, as one can easily check using the maximum principle and the fact that ω is radial. Let now f be a continuous function on F, analytic on F^0 and such that it can not be continued analytically on a neighbourhood of F. If g is analytic on a neighbourhood of F and $\|f - g\|_{\omega,F} < 1$ then $f = g$, which is impossible. Therefore f can not be globally approximated on F, but a local approximation is possible because $\| \ \|_\omega$ is locally equivalent to the uniform norm. Thus (i) fails.

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